

UNIVERSITY OF LJUBLJANA
INSTITUTE OF MATHEMATICS, PHYSICS AND MECHANICS
DEPARTMENT OF MATHEMATICS
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SUBDIVISION GRAPHS AND AS
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GRAPHS

Sandi Klavžar Alenka Lipovec

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Partial cubes as subdivision graphs and as generalized Petersen graphs

Sandi Klavžar*

Department of Mathematics, University of Maribor
Koroška cesta 160, 2000 Maribor, Slovenia
email: sandi.klavzar@uni-lj.si

Alenka Lipovec

Department of Education, University of Maribor
Koroška cesta 160, 2000 Maribor, Slovenia
email: alenka.lipovec@uni-mb.si

Abstract

Isometric subgraphs of hypercubes are known as partial cubes. The subdivision graph of a graph G is obtained from G by subdividing every edge of G . It is proved that for a connected graph G its subdivision graph is a partial cube if and only if every block of G is either a cycle or a complete graph. Regular partial cubes are also considered. In particular it is shown that among the generalized Petersen graphs, $P(10, 3)$ and $P(2n, 1)$, $n \geq 2$, are the (regular) partial cubes.

1 Introduction

Partial cubes are isometric subgraphs of hypercubes. They were first introduced in computer science [12] and have later found several other applications, for instance in mathematical chemistry and biology, cf. [5, 9, 15].

Clearly, partial cubes are bipartite. If G is an arbitrary graph, a simple way to modify it to a bipartite graph is to subdivide every edge of G by a single vertex. Such a graph is called a *subdivision graph* of G and denoted $S(G)$. A natural question appears for which graphs G their subdivision graphs are partial cubes.

Subdivision graphs were studied before in different contexts, see, for instance, [1, 3, 19]. A more general construction is to replace edges of G by paths, thus obtaining a *subdivided graph* of G . In the context of partial cubes, subdivided graphs of wheels turned out to be useful in settling (in negative) a conjecture of Chepoi and Tardif asserting that a bipartite graph is a partial cube if and only if all of its intervals are

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convex [2]. Subdivided graphs of wheels were also studied in [6] where it was proved that except in three particular cases they are the so-called l_1 -graphs—a class of graphs that properly contains partial cubes.

The main result of this paper asserts that $S(G)$ is a partial cube if and only if every block of G is either a cycle or a complete graph. We also consider the problem of classifying regular partial cubes and show in particular that $P(10, 3)$ and $P(2n, 1)$, $n \geq 2$, are the only (regular) partial cubes among the generalized Petersen graphs.

All graphs considered in this paper are connected and simple. The vertex set of the n -cube Q_n consists of all n -tuples $b_1 b_2 \dots b_n$ with $b_i \in \{0, 1\}$, where two vertices are adjacent if the corresponding tuples differ in precisely one place. The vertices of Q_n can also be understood as characteristic vectors of subsets of an n -set. Then, two such subsets are adjacent if their symmetric difference consists of a single element.

The *Cartesian product* $G \square H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ in which the vertex (a, x) is adjacent to the vertex (b, y) whenever $ab \in E(G)$ and $x = y$, or $a = b$ and $xy \in E(H)$. Note that the Cartesian product of n copies of K_2 is the n -cube Q_n .

For a graph G , the *distance* $d_G(u, v)$, or briefly $d(u, v)$, between vertices u and v is defined as the number of edges on a shortest u, v -path. A subgraph H of G is called *isometric* if $d_H(u, v) = d_G(u, v)$ for all $u, v \in V(H)$. A subgraph H of G is *convex*, if for any $u, v \in V(H)$, all shortest u, v -paths belong to H .

A maximal connected subgraph without a cutvertex is called a *block*. Thus, every block of a connected graph is either a maximal 2-connected subgraph or a bridge (with its ends), cf. [7].

Two edges $e = xy$ and $f = uv$ of G are in the Djoković-Winkler [8, 20] relation Θ if $d_G(x, u) + d_G(y, v) \neq d_G(x, v) + d_G(y, u)$. Clearly, Θ is reflexive and symmetric. Let Θ^* be the transitive closure of Θ . Winkler [20] proved the following characterization of partial cubes.

Theorem 1.1 *A bipartite graph is a partial cube if and only if $\Theta = \Theta^*$.*

We now list several basic and well-known properties of Θ to be used in the sequel, cf. [13], for Lemma 1.6 see [11].

Lemma 1.2 *Let P be a shortest path in a graph G . Then no two distinct edges of P are in relation Θ .*

Lemma 1.3 *Let G be a bipartite graph and let $e = uv$, $f = xy$ be two edges of G with $e \Theta f$. Then the notation can be chosen such that $d(u, x) = d(v, y) = d(u, y) - 1 = d(v, x) - 1$.*

Lemma 1.4 *Suppose that a walk P connects the endpoints of an edge e but does not contain it. Then P contains an edge f with $e \Theta f$.*

Lemma 1.5 *Let G be a partial cube, C an isometric cycle of G , and e an edge of C . Then the Θ -class of e intersects C in exactly two edges.*

Lemma 1.6 *Let e and f be edges from different blocks of a graph G . Then e is not in relation Θ with f .*

Let G be a connected graph. A *proper cover* of G consists of two isometric subgraphs G_1 and G_2 with nonempty intersection and union G . The *expansion* of G with respect to G_1, G_2 is the graph H obtained by the following procedures:

- (i) Replacement of each vertex $v \in G_1 \cap G_2$ by vertices v_1, v_2 and insertion of the edge v_1v_2 .
- (ii) Insertion of edges between v_1 and the neighbors of v in $G_1 \setminus G_2$ as well as between v_2 and the neighbors of v in $G_2 \setminus G_1$.
- (iii) Insertion of the edges v_1u_1 and v_2u_2 whenever $v, u \in G_1 \cap G_2$ are adjacent in G .

Chepoi [4] proved:

Theorem 1.7 *A graph is a partial cube if and only if it can be obtained from the one vertex graph by a sequence of expansions.*

2 Partial cubes as subdivision graphs

If T is a tree, then $S(T)$ is also a tree and thus a partial cube. Similarly, subdivision graphs of cycles are partial cubes since $S(C_n) = C_{2n}$. Moreover, we have:

Proposition 2.1 *For any $n \geq 1$, $S(K_n)$ is a partial cube.*

Proof. As $S(K_1) = K_1$ and $S(K_2) = P_3$, the assertion is true for $n = 1, 2$. Let $n \geq 2$ and consider the subset representation of Q_n . Let G be the subgraph of Q_n induced by the subsets on at most two elements. Let G_1 be the subgraph of G induced by the subsets on at most one element and G_2 the subgraph of G obtained from G by removing the empty set. Clearly, G_1 is isometric in G and it is also easy to check that G_2 is isometric in G . Thus G_1 and G_2 form a proper cover of G and since G is also a partial cube, the expansion H of G with respect to G_1, G_2 is a partial cube by Theorem 1.7. Finally observe that H is isomorphic to $S(K_{n+1})$. \square

The proof of Proposition 2.1 is illustrated in Fig. 1. The vertices of $S(K_4)$ that are obtained by subdividing edges are shown as filled squares.

In the rest of this section we are going to prove that these are precisely the cases when $S(G)$ is a partial cube. More precisely, we will prove:

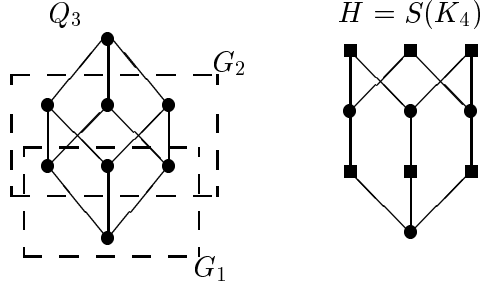


Figure 1: Expanding a subgraph of Q_3 to $S(K_4)$.

Theorem 2.2 *Let G be a connected graph. Then $S(G)$ is a partial cube if and only if every block of G is either a cycle or a complete graph.*

Let u be a vertex and e an edge of a graph G . We will denote the vertex of $S(G)$ corresponding to u by \bar{u} , and by \bar{e} the vertex of $S(G)$ that is obtained by a subdivision of e .

The following straightforward lemma will be implicitly used in the rest of the section.

Lemma 2.3 *Let e and f be edges of a graph G and let u and v be endpoints of e and f , respectively, such that $d(u, v)$ is minimal. Then*

- (i) $d_{S(G)}(\bar{u}, \bar{v}) = 2d_G(u, v)$,
- (ii) $d_{S(G)}(\bar{e}, \bar{f}) = 2d_G(u, v) + 2$,
- (iii) $d_{S(G)}(\bar{u}, \bar{f}) = 2d_G(u, v) + 1$.

Lemma 2.4 *Let G be a 2-connected graph that is not a cycle. Then G contains two isometric cycles $C_1 = u_1u_2 \dots u_ku_{k+1} \dots u_nu_1$ and $C_2 = u_1u_2 \dots u_kv_{k+1} \dots v_mu_1$, where $n \geq m > k \geq 2$ and $v_i \neq v_j$ for $i, j \geq k + 1$.*

Proof. Let C_1 be a shortest cycle of G . Clearly, C_1 is isometric. As G is not a cycle, there exists a vertex $u \in C_1$ of degree at least 3. Let $u' \notin C_1$ be a neighbor of u . Note that such a vertex exists because C_1 is isometric and thus chordless. As u is not a cut vertex, there is a path P between u' and a vertex of C_1 . Let v be the endpoint of P different from u . We may select u and v such that $d(u, v)$ is as small as possible among such pairs. Fixing a pair u, v , let P be as short as possible. We claim that the cycle $C_2 : u \rightarrow u' \rightarrow \dots \rightarrow P \dots \rightarrow v \rightarrow \dots \rightarrow C_1 \dots \rightarrow u$ is an isometric cycle. Indeed, there is no shortcut between a vertex of $C_2 \cap C_1$ and a vertex of P because of the way u, v and P are selected. Likewise, there is no shortcut between two vertices of P . \square

Lemma 2.5 *Let G be a partial cube, and let C_1 and C_2 be cycles of G as in Lemma 2.4. Let $e = uu'$ be an edge of $C_1 \setminus C_2$, let $f = vv'$ be an edge of $C_2 \setminus C_1$, and let h be an edge of $C_1 \cap C_2$ such that $e\Theta h\Theta f$. If $d(u, u_1) < d(u', u_1)$ and $d(v, u_1) < d(v', u_1)$, then $d(u, v) = d(u', v') = d(u, v') - 1 = d(u', v) - 1$.*

Proof. By Lemma 1.3, $d(u, v) = d(u', v')$ and $d(u, v') = d(u', v)$. Set $d(u, v) = n$. Let P be a shortest u, v' -path. We need to prove that $|P| = n + 1$.

Suppose first that $C_1 \cup C_2$ induces an isometric subgraph. Then we may assume that P contains only vertices of $C_1 \cup C_2$. If P contains u' , then since $d(u, v) = d(u', v')$ we get $d(u, v') = n + 1$. In the other case P contains u_1 . But then P does not contain u_k because $f\Theta h$. Now the conclusion is clear.

Suppose now that $C_1 \cup C_2$ does not induce an isometric subgraph. Assume that the conclusion of the lemma is false. Then, since G is bipartite, $d(u, v') = n - 1$. Note that v is not on P for otherwise $d(u, v) \leq n - 2$. Similarly, u' cannot lie on P .

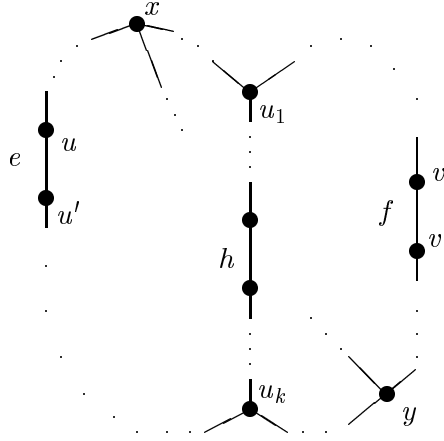


Figure 2: A path between u and v' .

Let x be the last vertex of P on C_1 and let y be the first vertex of P on C_2 , cf. Fig. 2. Note that x lies on a shortest path between u and u_1 on C_1 and that y is on a shortest path between u_k and v' on C_2 . Observe also that $u = x$ or $v' = y$ is possible. Since C_1 and C_2 are isometric but $C_1 \cup C_2$ is not, $x \neq u_1$ and $y \neq u_k$. Let P' be the subpath of P between x and y . Then P' is internally disjoint with $C_1 \cup C_2$ hence $C = x \rightarrow \dots \rightarrow P' \rightarrow \dots \rightarrow y \rightarrow \dots \rightarrow C_2 \rightarrow \dots \rightarrow v' \rightarrow v \rightarrow \dots \rightarrow C_2 \rightarrow \dots \rightarrow u_1 \rightarrow \dots \rightarrow C_1 \rightarrow \dots \rightarrow x$ is a cycle. Therefore, by Lemma 1.4, there exists an edge g on C that is in relation Θ to f . Since $f\Theta h$ we infer from Lemma 1.5 that g is not on C_2 . Moreover, g is not on C_1 since otherwise we would have an equivalence class of Θ with more than two edges on C_1 . It follows that $g \in P'$. Then the path $v \rightarrow v' \rightarrow \dots \rightarrow C_2 \rightarrow \dots \rightarrow y \rightarrow \dots \rightarrow P' \rightarrow \dots \rightarrow$

$x \rightarrow \dots C_1 \dots \rightarrow u$ is of length n . Therefore it is a geodesic that contains two different edges that are in relation Θ . As this is not possible, the proof is complete. \square

Lemma 2.6 *Let G be a 2-connected graph that is not a cycle. If $S(G)$ is a partial cube and C_1 and C_2 cycles of G as in Lemma 2.4, then $C_1 \cup C_2$ induces a K_4 .*

Proof. Let $S(G)$ be a partial cube and let \bar{C}_1 and \bar{C}_2 be the cycles of $S(G)$ that correspond to the cycles C_1 and C_2 of G . Consider the middle two edges of $\bar{C}_1 \cap \bar{C}_2$. More precisely, if k is even, then these are the edges of the path $\bar{u}_{k/2} \rightarrow \bar{e} \rightarrow \bar{u}_{k/2+1}$ and if k is odd then these are the edges of the path $\bar{e} \rightarrow \bar{u}_{(k+1)/2} \rightarrow \bar{f}$. We denote the corresponding vertices of $S(G)$ with \bar{x} , \bar{y} , and \bar{z} .

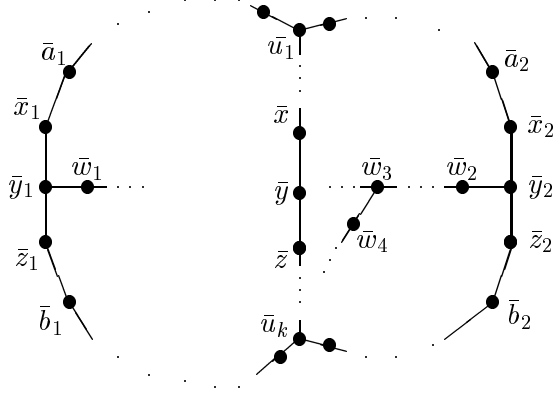


Figure 3: $C_1 \cup C_2$ is not isometric.

By Lemma 1.5 there are edges $\bar{y}_1 \bar{z}_1$ on \bar{C}_1 and $\bar{y}_2 \bar{z}_2$ of \bar{C}_2 such that $\bar{x} \bar{y} \Theta \bar{y}_1 \bar{z}_1$ and $\bar{x} \bar{y} \Theta \bar{y}_2 \bar{z}_2$. Similarly, $\bar{y} \bar{z}$ is in relation Θ to $\bar{x}_1 \bar{y}_1$ on \bar{C}_1 and to $\bar{x}_2 \bar{y}_2$ on \bar{C}_2 , cf. Fig. 3. As $S(G)$ is a partial cube we infer that $\bar{y}_1 \bar{z}_1 \Theta \bar{y}_2 \bar{z}_2$ and $\bar{x}_1 \bar{y}_1 \Theta \bar{x}_2 \bar{y}_2$. Let $d(\bar{x}_1, \bar{x}_2) = n$, where $n \geq 2$. By Lemma 1.3 we have $d(\bar{x}_1, \bar{x}_2) = d(\bar{y}_1, \bar{y}_2) = d(\bar{z}_1, \bar{z}_2)$. Hence Lemma 2.5 implies $d(\bar{x}_1, \bar{y}_2) = d(\bar{y}_1, \bar{x}_2) = d(\bar{y}_1, \bar{z}_2) = d(\bar{z}_1, \bar{y}_2) = n + 1$.

Let P be a shortest \bar{y}_1, \bar{y}_2 -path. By the above distances none of the vertices $\bar{x}_1, \bar{x}_2, \bar{z}_1$, and \bar{z}_2 lies on P . It follows that the degrees of \bar{y}_1 and \bar{y}_2 are at least 3. Hence, by the definition of $S(G)$, the degrees of $\bar{x}_1, \bar{x}_2, \bar{z}_1$, and \bar{z}_2 are 2.

Let \bar{w}_1 be the neighbor of \bar{y}_1 and \bar{w}_2 the neighbor of \bar{y}_2 on P , let \bar{a}_1 be the other neighbor of \bar{x}_1 (on \bar{C}_1), \bar{b}_1 the other neighbor of \bar{z}_1 , \bar{a}_2 the other neighbor of \bar{x}_2 (on \bar{C}_2), and \bar{b}_2 the other neighbor of \bar{z}_2 , cf. Fig. 3.

Suppose that $\bar{w}_2 \bar{y}_2 \Theta \bar{x}_1 \bar{a}_1$, $\bar{w}_2 \bar{y}_2 \Theta \bar{z}_1 \bar{b}_1$, $\bar{w}_1 \bar{y}_1 \Theta \bar{x}_2 \bar{a}_2$, and $\bar{w}_1 \bar{y}_1 \Theta \bar{z}_2 \bar{b}_2$. Then, by the transitivity of Θ , the first two conditions imply $\bar{z}_1 \bar{b}_1 \Theta \bar{a}_1 \bar{x}_1$. As C_1 and C_2 are isometric, this is only possible if $\bar{a}_1 = \bar{x}$ and $\bar{b}_1 = \bar{z}$. Analogously, the second two conditions imply

that $\bar{x} = \bar{a}_2$ and $\bar{z} = \bar{b}_2$ and $\bar{w}_1 = \bar{w}_2$. But then the vertices $\bar{x}_1, \bar{y}_1, \bar{z}_1, \bar{x}, \bar{y}, \bar{z}, \bar{x}_2, \bar{y}_2, \bar{z}_2$, and \bar{w}_1 induce an $S(K_4)$.

Assume now that one of the four conditions of the previous paragraph is not fulfilled. We may without loss of generality assume that $\bar{w}_2\bar{y}_2$ is not in relation Θ with $\bar{z}_1\bar{b}_1$. Now we have $d(\bar{z}_1, \bar{y}_2) = n + 1$, $d(\bar{z}_1, \bar{w}_2) = n$. Moreover, a shortest \bar{z}_1, \bar{z}_2 -path passes \bar{b}_1 , hence $d(\bar{b}_1, \bar{y}_2) \leq n$. In addition, $d(\bar{b}_1, \bar{y}_2) = n$ for otherwise $d(\bar{z}_1, \bar{y}_2) \leq n$. Now, since $\bar{w}_2\bar{y}_2$ is not in relation Θ with $\bar{z}_1\bar{b}_1$ we must have $d(\bar{b}_1, \bar{w}_2) = n - 1$. Let Q be a shortest \bar{b}_1, \bar{w}_2 -path. Let \bar{w}_3 be the last common vertex of P and Q traversing from \bar{w}_2 and set $d(\bar{w}_3, \bar{y}_2) = p$. Let \bar{w}_4 be the neighbor of \bar{w}_3 on Q that is not on P . Then $\bar{w}_3\bar{w}_4$ is in relation Θ with $\bar{y}_1\bar{z}_1$. Indeed, $d(\bar{y}_1, \bar{w}_3) = n - p$, $d(\bar{z}_1, \bar{w}_4) = n - p$, $d(\bar{w}_3, \bar{z}_1) = n - p + 1$, and $d(\bar{w}_4, \bar{y}_1) = n - p + 1$. Transitivity of Θ thus implies that $\bar{w}_3\bar{w}_4\Theta\bar{y}_2\bar{z}_2$. It follows that $d(\bar{w}_3, \bar{y}_2) = d(\bar{w}_4, \bar{z}_2)$. Note that \bar{w}_4 is of degree 2. Therefore, a geodesic between \bar{w}_4 and \bar{z}_2 must pass the neighbor of \bar{w}_4 different from \bar{w}_3 . However, this is not possible as we would get that $d(\bar{z}_1, \bar{z}_2) < n$, a contradiction. \square

Proof. (of Theorem 2.2) Suppose first that each block of G is either a cycle or a complete graph. Then the blocks of $S(G)$ are partial cubes by Proposition 2.1. Now Lemma 1.6 implies that Θ is transitive and as $S(G)$ is bipartite, it is a partial cube by Theorem 1.1.

Conversely, let $S(G)$ be a partial cube. We may without loss of generality assume that G is 2-connected. If G is a cycle, we are done. So we may assume that this is not the case. Then G contains isometric cycles C_1 and C_2 as described in Lemma 2.4. Hence by Lemma 2.6 we have an induced K_4 in G . Let x_1, x_2, x_3 , and x_4 be the vertices of this K_4 . If G has 4 vertices, we are done. Otherwise, let y be another vertex of G adjacent to x_1 . Then, as G is 2-connected, there is a path P between y and another vertex of K_4 , say x_2 . We may select P is such a way that $x_1 \rightarrow y \rightarrow \cdots \rightarrow P \rightarrow \cdots \rightarrow x_2 \rightarrow x_1$ is an isometric cycle. Therefore by Lemma 2.6 y is adjacent with x_2 and x_3 . Similarly we infer that y is also adjacent with x_4 . Induction completes the proof. \square

3 On regular partial cubes

For the (probably) most important subclass of partial cubes, median graphs, Mulder [16] proved that hypercubes are the only regular median graphs. Besides hypercubes, even cycles are regular partial cubes. Moreover, the Cartesian product of two (regular) partial cubes is a (regular) partial cube. In [14] it was asked if in this way one obtains all regular partial cubes. However, this is not the case.

First, an additional example of such a graph is due to Gedeonova [10], see also Fig. 2.4 of [13]. In addition, J. Koolen (personal communication) pointed out that the middle-level graphs are also regular partial cubes. (For a given $(2n + 1)$ -cube its middle-levels graph is the subgraph induced by the middle two levels in its subset representation, cf. [18]. For instance, the middle-levels graph of Q_3 is C_6 .)

The middle-levels graphs can be constructed also in the following way. Consider the $2n$ -cube in its subset representation. Let G_n be the subgraph of Q_{2n} induced by the subsets on $n-1$, n , and $n+1$ elements. Let G' be the subgraph of G_n induced by the subsets on $n-1$ and n elements, and let G'' be the subgraph induced by the subsets on n and $n+1$ elements. Then it is straightforward to verify that G' and G'' form a proper cover of G_n . Thus we may expand G_n with respect to G' and G'' to obtain a partial cube H_n . Observe finally that H_n is the middle-levels graph of Q_{2n+1} .

The above construction gives an alternative argument for the fact that the middle-levels graphs are partial cubes. The construction for Q_4 is shown in Fig. 4. The graph H_2 is isomorphic to the generalized Petersen graph $P(10, 3)$, cf. also [17] where regular subgraphs of hypercubes are studied.

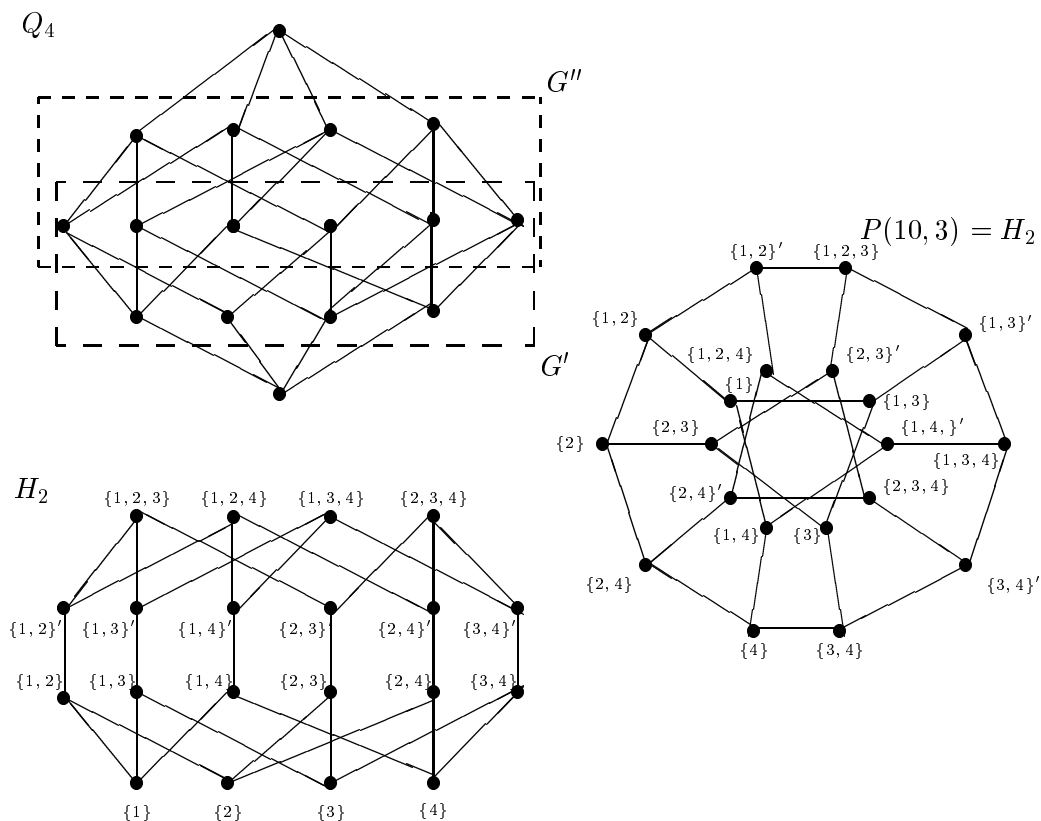


Figure 4: Expanding the three middle levels of Q_4 to $P(10, 3)$.

This observation raises the question whether there are more regular partial cubes among generalized Petersen graphs.

Proposition 3.1 $P(2m, 1)$, $m \geq 2$, and $P(10, 3)$ are the only (regular) partial cubes among the generalized Petersen graphs $P(n, k)$.

Proof. We have seen above that $P(10, 3)$ is a partial cube. Moreover, $P(2m, 1)$ is isomorphic to $C_{2m} \square K_2$, thus it is a (regular) partial cube.

It remains to show that in all the other cases $P(n, k)$, $n > 2k$, is not a partial cube. First note that a bipartite $P(n, k)$ is of the form $P(2k, 2\ell + 1)$. Denote the vertices of the outer cycle of $P(2k, 2\ell + 1)$ with $1, 2, \dots, 2k$ and the corresponding inner vertices by $1', 2', \dots, (2k)'$.

Assume $\ell > 1$. Considering the cycle $1 \rightarrow 2 \rightarrow \dots \rightarrow 2\ell + 2 \rightarrow (2\ell + 2)' \rightarrow 1' \rightarrow 1$ we note that the edge $11'$ is in relation Θ with $(\ell + 2)(\ell + 3)$. Similarly, from the cycle $1 \rightarrow 2 \rightarrow \dots \rightarrow \ell + 3 \rightarrow (\ell + 3)' \rightarrow (2k - \ell + 2)' \rightarrow 2k - \ell + 2 \rightarrow 2k - \ell + 3 \rightarrow \dots \rightarrow 1$ we infer that $(\ell + 2)(\ell + 3)$ is in relation Θ with $1(2k)$. Hence Θ is clearly not transitive.

It remains to consider the case $\ell = 1$. Note first that $11' \Theta 34$ and $34 \Theta 66'$. Continuing in this way along the outer cycle we find that $2k \equiv 0 \pmod{5}$, that is, $n = 10t$. However, in this case $2'5'$ must be in the same Θ -class as $11'$. As this is only possible if $t = 1$, the proof is complete. \square

A classification of regular partial cubes remains a challenging open problem.

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