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Preprint series, Vol. 39 (2001), 739

WALECKI TOURNAMENTS: PART I

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ISSN 1318-4865

January 31, 2001

Ljubljana, January 31, 2001

Walecki Tournaments: Part I

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Abstract. Walecki tournaments were defined by Alspach in 1966. They form a class of regular tournaments that posses a natural Hamilton directed cycle decomposition. It has been conjectured by Kelly in 1964 that every regular tournament possesses such a decomposition. Therefore Walecki tournaments speak in favor of the conjecture. A second interest in Walecki tournaments arises from the mapping between cycles of the complementing circular shift register and isomorphism classes of Walecki tournaments. An upper bound on the number of isomorphism classes of Walecki tournaments was determined by Alspach. It was conjectured that the bound is tight. The problem of enumerating Walecki tournaments has not been solved to date. However, it was published as an open problem in a paper by Alspach in 1989. In an attempt to prove this 34 years old conjecture, we first determine the arc structure of Walecki tournaments for all initial cases and those whose corresponding binary sequences have zero pattern. Subsequent papers deal with more general cases. Techniques used in

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the proofs originate from a diverse range of topics: permutation groups, tournaments, and bijective enumeration, to mention a few.

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1 Introduction

Walecki tournaments were defined by Alspach in 1966, [6]. They are known to be regular and possess a natural Hamilton directed cycle decomposition. In 1964 Kelly conjectured that every regular tournament admits such a decomposition (see Moon [13]). The class of Walecki tournaments speaks in favor of the conjecture. Therefore knowledge of their structure would be of importance.

A second interest in Walecki tournaments is arising from the surjective mapping between cycles of the complementing circular shift register and isomorphism classes of Walecki tournaments. It has been proven by Alspach that an upper bound on the number of isomorphism classes of Walecki tournaments is $\Gamma(n)$. That is,

$$\Gamma(n) = \frac{1}{n} \sum_{d|n,\ d\ odd} \phi(d) 2^{\frac{n}{d}-1}, \tag{1.1}$$

where the sum is taken over the odd divisors of n and ϕ denotes *Euler totient* function from number theory. There are many instances of such sets. Let us refer to them as Γ-classes. For example, all cycles for the complementing circular shift register of length n form a Γ-class (see Subsection 3.1). Alspach conjectured that the bound $\Gamma(n)$ is tight. The problem of enumerating Walecki tournaments has not been solved to date. It has been published as an open problem in a paper by Alspach [8]. An equally important class of sets are sets of cardinality

$$\Sigma(n) = \frac{1}{n} \sum_{d|n,\ d\ odd} \mu(d) 2^{\frac{n}{d}-1}, \tag{1.2}$$

where μ denotes the Möbius function. We refer to them as Σ-*classes*. Some examples of Σ-classes are cycles with $2n$ elements for the complementing circular shift register of length n (see Subsection 3.1).

In an attempt to keep the paper self-contained some background results from topics on tournaments, permutation groups, and algebraic graph theory are included. Section 3 includes the definition of cycles for the complementing circular shift register on binary sequences of length n. Walecki tournaments are defined in Subsection 3.2 and a mapping between the cycles for the complementing circular shift register and Walecki tournaments is presented in Subsection 3.3. Structural results about Walecki tournaments in general are presented in Section 4, while Section 5 determines the arc structure of induced subtournaments for zero pattern. A specific permutation is proven to be an automorphism for Walecki tournaments.

2 Theoretical background

A *tournament* T on n vertices $\{v_{(0)}, v_{(1)}, \ldots, v_{(n-1)}\}$ is an orientation of the complete graph K_n. Let $V(T)$ denote the vertex set of T and let $A(T)$ denote the arc set of T. The *outset* of vertex $v_{(i)}$ is the set of all vertices in T that are dominated by $v_{(i)}$, that is, $N^+(v_{(i)}) = \{v_{(j)} \in V(T) \mid v_{(i)} \longrightarrow v_{(j)}\}$. The *inset* of vertex $v_{(i)}$ is the set of all vertices in T that dominate $v_{(i)}$, that is, $N^-(v_{(i)}) = \{v_{(j)} \in V(T) \mid v_{(j)} \longrightarrow v_{(i)}\}$. We denote by $T\langle X \rangle$ the subtournament of a tournament T on the vertex set $X \subseteq V(T)$. The *score* of vertex $v_{(i)}$, denoted by $s(v_{(i)})$, is the cardinality of the outset of $v_{(i)}$. That is, $s_i = s(v_{(i)}) = |N^+(v_{(i)})|$, for $0 \leq i \leq n-1$. The *score sequence* of T is the ordered n-tuple $(s_0, s_1, \ldots, s_{n-1})$. We assume that the vertices are numbered so that $s_0 \leq s_1 \leq \cdots \leq s_{n-1}$. If $v_{(i)} \longrightarrow v_{(j)}$ and $v_{(j)} \longrightarrow v_{(k)}$ imply $v_{(i)} \longrightarrow v_{(k)}$, the tournament is said to be *transitive*. A tournament is said to be *regular* if it has odd order $2n+1$ and every vertex has in-degree and out-degree equal to n. Tournaments on an even number of vertices can not be regular. However, a tournament is said to be *almost regular* if it has even order, say $2n$, and every vertex has out-degree equal to either n or $n-1$.

Let X be a non-empty set. Let G be a permutation group acting on the set X. Let $x \in X$ and let $\mathcal{O}(x)$ denote the *orbit* of x for G, that is, $\mathcal{O}(x) = \{g(x) \in X \mid g \in G\}$. Let G_x denote the *point stabilizer* of x, that is, $G_x = \{g \in G \mid g(x) = x\}$. We will use the following theorem extensively.

Theorem 2.1 (Orbit-Stabilizer Theorem) *Let G be a permutation group acting on the set X. For $x \in X$, $|G| = |\mathcal{O}(x)|\,|G_x|$.*

Let $H \subset G$, that is, H is a subset of elements of G. We use $\langle H \rangle$ to denote the subgroup of G generated by elements of H.

Let $\mathbb{S}_m$ denote the symmetric group on m elements. If not otherwise stated we will assume that $\mathbb{S}_m$ is acting on the set $\{0, 1, \ldots, m-1\}$. Let $\sigma \in \mathbb{S}_m$. We define the action of σ on the vertex set $\{v_{(0)}, v_{(1)}, \ldots, v_{(m-1)}\}$ of a tournament T by $\sigma(v_{(i)}) = v_{(\sigma(i))}$. Let T_1 and T_2 be tournaments and ϱ be an injective mapping from $V(T_1)$ to $V(T_2)$. We say that ϱ is *dominance-preserving* on T_1 if $u \longrightarrow v \in T_1$ if and only if $\varrho(u) \longrightarrow \varrho(v) \in T_2$ for every pair of distinct vertices belonging to T_1. If ϱ is bijective we say ϱ is an *isomorphism* between T_1 and

T_2. The set of all isomorphisms between tournaments T_1 and T_2 is denoted by $Iso(T_1, T_2)$. Notice, that T_1 and T_2 might be subtournaments of a fixed tournament T.

Let σ be a permutation of the vertices of T. A permutation which is dominance-preserving on all of T is called an *automorphism* of T. The set of all automorphisms of a tournament T forms a group. It is called the *automorphism group* of T and is denoted by $Aut(T)$. A tournament T is said to be *vertex-transitive* if $Aut(T)$ acts transitively on the vertices of T. Clearly, $Aut(T) = Iso(T, T)$.

An *anti-automorphism* of a tournament T is a mapping φ of the vertex set $V(T)$ satisfying $u \longrightarrow v \in T$ if and only if $\varrho(u) \longrightarrow \varrho(v) \notin T$ for every pair of distinct vertices belonging to T. We denote the set of all anti-automorphisms by $AntiAut(T)$. A tournament T is said to be *self-complementary* if it has an anti-automorphism. That is $AntiAut(T) \neq \emptyset$. For example, every transitive tournament on n vertices is self-complementary. Self-complementary tournaments were studied extensively by Alspach [7].

Proposition 2.2 (Alspach, 1970) *For an anti-automorphism ϱ of a tournament T, $\varrho Aut(T) = AntiAut(T) = Aut(\overline{T})\varrho$. Moreover, $Aut(T)$ is a subgroup of $\langle Aut(T), \varrho \rangle$ of index 2.*

Corollary 2.3 (Alspach, 1970) *If T is a vertex-transitive and self-complementary tournament, then there exists an anti-automorphism that fixes an arbitrary vertex of T.*

Let T be a tournament on an odd number of vertices, say $2n+1$, with vertex set $\{v(0), v(1), \ldots, v(2n)\}$. Let $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a set of n distinct integers chosen from $1, 2, \ldots, 2n$ with the property that $\alpha_i + \alpha_j \neq 2n + 1$ for any two $\alpha_i, \alpha_j \in S$. There is an arc from $v(i)$ to $v(j)$ if and only if $j - i \equiv \alpha_k (\text{mod } 2n+1)$ for some $\alpha_k \in S$. Any tournament that is constructed in the above manner is called a *circulant tournament* and S is called the *symbol* of the circulant tournament. We use notation $C(S)$ for a circulant tournament with symbol set S. Figure 1 shows a circulant tournament $C(\{1, 3\})$ on 5 vertices for the symbol set $S = \{1, 3\}$. The next three results discuss the connection between circulant tournaments and their automorphism groups.

Proposition 2.4 (Alspach, 1970) *Circulant tournaments are vertex-transitive.*

Notice, that Proposition 2.4 also implies that circular tournaments are regular.

A permutation $\varrho \in \mathbb{S}_m$ acting on $\{0, 1, \ldots, m-1\}$ is called an *m-cycle* if $\varrho = (i_0, i_1, \ldots, i_{m-1})$, i_j are all distinct for $0 \leq j \leq m-1$, and $\{i_0, i_1, \ldots, i_{m-1}\} = \{0, 1, \ldots, m-1\}$.

Proposition 2.5 (Alspach, 1970) *A tournament T of order m is a circulant tournament if and only if $Aut(T)$ possesses an m-cycle.*

Proposition 2.6 (Alspach, 1970) *A circulant tournament is self-complementary.*

A broad range of techniques is used in the proofs in this paper. For further background on tournaments we refer the reader to Beineke and Reid [10], Moon [13], and for topics on permutation groups to Burnside [11] and Wielandt [14].

3 Walecki tournaments

In Subsection 3.1 we define cycles for the complementing circular shift register on binary sequences of length n. An enumeration result for the cycles of the complementing circular shift register, proven by Alspach [6], has been included in order to keep the paper self-contained. Walecki tournaments are defined in Subsection 3.2. Finally, a mapping between the cycles for the complementing circular shift register and Walecki tournaments is presented in Subsection 3.3.

3.1 Cycles of the complementing circular shift register

Let E_n denote the set of all binary sequences $e = (e_1, e_2, \ldots, e_n)$ with $e_i = 0$ or 1 for all i. When considering particular binary sequences we will use $e_1 e_2 \cdots e_n$ to denote $(e_1, e_2, \ldots, e_n)$. We use standard notation $\overline{e_i}$ to denote $(e_i + 1) \bmod 2$ and $\bar{e}$ to denote $(\overline{e_1}, \overline{e_2}, \ldots, \overline{e_n})$.

Let $R : E_n \longrightarrow E_n$, a *complementing circular shift register* operator, be defined by

$$(R(e))_i = \begin{cases} e_{i+1}, & 1 \leq i \leq n-1 \\ \bar{e}_1, & i = n. \end{cases}$$

For an integer $k \geq 2$ we have $(R^k(e))_i = (R(R^{k-1}(e)))_i$. It is clear that $R^k(e) \in E_n$ for all $k \geq 1$. If $e \in E_n$, the *period* of e is defined to be the smallest positive integer k such that $R^k(e) = e$. That is, $R^k(e) = e$, and $R^j(e) \neq e$ for $2 \leq j \leq k-1$. Some of the basic properties of the operator R are listed. Property 1, Property 2, Property 3, and Property 4 were proven by Alspach [6].

Lemma 3.7 *The following properties hold for a complementing circular shift register $R : E_n \longrightarrow E_n$, sequence $e \in E_n$ with period m, and k a positive integer:*

1. $R^k(e) = R^{k+m}(e)$.

2. $R^k(e), R^{k+1}(e), \ldots, R^{k+m-1}(e)$ *are distinct.*

3. $\{e, R(e), \ldots, R^{m-1}(e)\} = \{R^k(e), R^{k+1}(e), \ldots, R^{k+m-1}(e)\}$.

4. $R^n(e) = \overline{e}$.

5. $R^{n+k}(e) = \overline{R^k(e)}$.

6. $R^{2n}(e) = e$.

7. *If* $R^s(e) = e$ *then* m *divides* s. *In particular,* m *divides* $2n$ *and* $m < n$ *or* $m = 2n$.

Next we state two results obtained by Alspach [6]. Let f and h be sequences in E_{n_1} and E_{n_2}, respectively, and let $e = fh$ denote the sequence of length $n_1 + n_2$ in $E_{n_1+n_2}$.

Lemma 3.8 (Alspach, 1966) *Let* $e \in E_n$. *If a positive integer* k *divides* n *such that* n/k *is odd and if* $e = f\overline{f}f\overline{f}\ldots\overline{f}f$, *for* $f \in E_k$, *then* $R^{2k}(e) = e$. *Moreover,* e *and* f *have the same period.*

Lemma 3.9 (Alspach, 1966) *If* $e \in E_n$ *has period* m, *where* $m < 2n$, *then* $m = 2r$ *where* r *divides* n, n/r *is odd,* $r < n$, *and* $e = f\overline{f}f\overline{f}\ldots\overline{f}f$ *such that* $f \in E_r$ *and the period of* f *is* $2r = m$.

A subset $\{e, R(e), \ldots, R^{m-1}(e)\}$ of E_n is called an *m-cycle* of operator R if e has period m. Define $e \sim_R f$ if and only if $f = R^k(e)$ for $e, f \in E_n$ and some integer k. It is easy to verify that $\sim_R$ is an *equivalence relation* on E_n. Let $[e]_R$, or $[e]$ if no confusion will arise, denote the equivalence class containing $e \in E_n$ under the relation $\sim_R$. If not otherwise stated we will consider the lexicographically smallest element of $[e]_R$ to be the canonical equivalence class representative. This representation is canonical since no two distinct binary sequences have the same lexicographic order. Let $\mathcal{R}(n)$ denote the set of equivalence classes for the equivalence relation $\sim_R$, and let $\mathcal{R}^*(n)$ denote the set of equivalence classes with $2n$ elements. All the cycles of $\mathcal{R}(5)$ are listed in Figure 2.

Let S be the function defined on the positive integers by the following recursive formula. Let $S(1) = 1$ and

$$nS(n) = 2^{n-1} - (r_1 S(r_1) + r_2 S(r_2) + \cdots + r_k S(r_k)) \tag{3.3}$$

where $r_1, r_2, \ldots, r_k$ are all positive divisors of n such that $0 < r_i < n$ and n/r_i is odd. Let G be the function defined on the positive integers by the formula

$$G(n) = S(n) + S(r_1) + S(r_2) + \cdots + S(r_k) \tag{3.4}$$

where $r_1, r_2, \ldots, r_k$ are as above.

Theorem 3.10 (Alspach, 1966) *Under the equivalence relation $\sim_R$ on E_n, $G(n)$ is the number of equivalence classes in $\mathcal{R}(n)$ and $S(n)$ is the number of equivalence classes in $\mathcal{R}^*(n)$.*

We will prove that $S(n) = \Sigma(n)$ and $G(n) = \Gamma(n)$, which suggests a connection between the work of Alspach [6] and Aleš [5].

To prove the two equalities we use a special case of the Möbius inversion formula from number theory, for example, see Lidl and Niederreiter [12], and a classic result from number theory concerning the values of the *Euler* ϕ-function. We omit the proof.

Theorem 3.11 (Aleš, 1999) *For a positive integer n, $S(n) = \Sigma(n)$ and $G(n) = \Gamma(n)$.*

3.2 Definition of Walecki tournaments

In this section we define the main objects of this paper, Walecki tournaments. Let $[v(0), v(1), \ldots, v(2n)]$ be a given undirected Hamilton cycle of order $2n+1$ (see Figure 3). Let $\tau \in \mathbb{S}_{2n+1}$ be a permutation of $\{v(0), v(1), \ldots, v(2n)\}$ defined by

$$\tau(v(i)) = \begin{cases} v(0) & \text{if } i = 0 \\ v(2) & \text{if } i = 1 \\ v(i+2) & \text{if } i \text{ even and } 2 \le i \le 2n-2 \\ v(i-2) & \text{if } i \text{ odd and } 3 \le i \le 2n-1 \\ v(2n-1) & \text{if } i = 2n. \end{cases} \tag{3.5}$$

Figure 4 shows the action of the permutation $\tau \in \mathbb{S}_{2n+1}$ on vertices of the given Hamilton cycle. A Walecki tournament on $2n+1$ vertices is then defined by assigning to each of the n undirected Hamilton cycles

$$\begin{aligned} H_1 &= [v(0), v(1), \ldots, v(2n)], \\ H_2 &= [\tau(v(0)), \tau(v(1)), \ldots, \tau(v(2n))], \\ &\vdots \\ H_n &= [\tau^{n-1}(v(0)), \tau^{n-1}(v(1)), \ldots, \tau^{n-1}(v(2n))], \end{aligned}$$

one of the two possible orientations. For example, Figure 5 shows Walecki tournaments $W(000)$ and $W(010)$ on 7 vertices. It is easy to see that union of n Hamilton directed cycles indeed forms a regular tournament.

We label the vertices of the Walecki tournaments as follows: $2n$ vertices on the circumference of a circle are labeled as $v(1), v(2), v(4), \ldots, v(2n-2)$, $v(2n)$, $v(2n-1), \ldots, v(5), v(3)$ with the central vertex labeled $v(0)$. The permutation τ corresponds to the clockwise rotation of the vertices on the circumference of the circle with $v(0)$ as a fixed point in the center.

In the construction of Walecki tournaments of order $2n+1$, we use n *undirected* Hamilton cycles each of which can be assigned one of the two possible orientations. Let $e \in E_n$ be a binary sequence of length n. Define components of e as follows,

$$e_i = \begin{cases} 0 & \text{if } v_{(0)} \longrightarrow \tau^{i-1}(v_{(1)}), \\ 1 & \text{if } v_{(0)} \longleftarrow \tau^{i-1}(v_{(1)}), \end{cases}$$

for $1 \leq i \leq n$. This establishes a one-to-one correspondence between all 2^n possible orientations of n Hamilton cycles and the elements of E_n.

The permutation $\tau \in \mathbb{S}_{2n+1}$ defined in equation (3.5) written in cyclic notation is $\tau = (v_{(1)}\ v_{(2)}\ v_{(4)}\ v_{(6)} \cdots v_{(2n-2)}\ v_{(2n)}\ v_{(2n-1)}\ v_{(2n-3)} \cdots v_{(5)}\ v_{(3)})(v_{(0)})$. It is easy to see that its inverse τ^{-1} is given by

$$\tau^{-1}(v_{(i)}) = \begin{cases} v_{(0)} & \text{if } i = 0 \\ v_{(1)} & \text{if } i = 2 \\ v_{(i-2)} & \text{if } i \text{ even and } 4 \leq i \leq 2n \\ v_{(i+2)} & \text{if } i \text{ odd and } 1 \leq i \leq 2n-3 \\ v_{(2n)} & \text{if } i = 2n-1. \end{cases} \tag{3.6}$$

Let η denote the permutation $\tau^n \in \mathbb{S}_{2n+1}$. That is, $\eta = (v_{(1)}\ v_{(2n)})(v_{(2)}\ v_{(2n-1)}) \cdots (v_{(n)}\ v_{(n+1)})$. Moreover,

$$\eta(v_{(i)}) = \begin{cases} v_{(0)} & \text{if } i = 0 \\ v_{(2n-i+1)} & \text{if } 1 \leq i \leq 2n. \end{cases} \tag{3.7}$$

Notice that $\tau(v_{(0)}) = \eta(v_{(0)}) = v_{(0)}$. Figure 6 shows the action of the permutation $\eta \in \mathbb{S}_{2n+1}$ on vertices of the Walecki tournament $W(e)$. The n *directed* cycles of $W(e)$ are

$$\vec{H}_k = [\tau^{k-1}(\eta^{e_k}(v_{(0)})), \tau^{k-1}(\eta^{e_k}(v_{(1)})), \ldots, \tau^{k-1}(\eta^{e_k}(v_{(2n)}))], \tag{3.8}$$

for $1 \leq k \leq n$.

In order to simplify notation we introduce second labeling of the vertices of Walecki tournaments. Let $V(T) = \{t_{(0)}, t_{(1)}, \ldots, t_{(2n)}\}$ be a labeling of the vertex set of Walecki tournament T, where

$$t_{(i)} = \begin{cases} v_{(0)} & \text{if } i = 0, \\ \tau^{i-1}(v_{(1)}) & \text{if } 1 \leq i \leq 2n. \end{cases}$$

It is easy to verify that the action of τ on $\{t_{(0)}, t_{(1)}, \ldots, t_{(2n)}\}$ is given by

$$\tau(t_{(i)}) = \begin{cases} t_{(0)} & \text{if } i = 0 \\ t_{(1)} & \text{if } i = 2n \\ t_{(i+1)} & \text{if } 1 \leq i \leq 2n-1. \end{cases} \tag{3.9}$$

The following result follows directly from the definition of the permutation τ.

Proposition 3.12 *For the permutation $\tau \in \mathbb{S}_{2n+1}$ defined in Equation (3.5),*

$$t_{(k+1)} = \begin{cases} v_{(2k)} & \text{if } 1 \leq k \leq n, \\ v_{(1-2k)} & \text{if } -n+1 \leq k \leq 0. \end{cases}$$

Notice that a careful selection of labeling simplifies notation for the particular problem. We will use both labelings throughout the paper to make the technical nature of the problem easier to understand.

3.3 Bijective enumeration of Walecki tournaments

Let $\mathcal{W}(n)$ denote the set of all non-isomorphic Walecki tournaments of order $2n+1$. We define a mapping $\Psi : \mathcal{R}(n) \longrightarrow \mathcal{W}(n)$ between the cycles of the complementing circular shift register on binary sequences of length n and non-isomorphic Walecki tournaments of order $2n+1$ by $\Psi : [e]_R \longrightarrow W(e)$. For example, $\Psi(000) = W(000)$. A result of Alspach implies Ψ is surjective, (see [6]).

Theorem 3.13 (Alspach, 1966) *Let n be a positive integer and let $e \in E_n$. If k is an integer such that $1 \leq k \leq 2n$, then $W(e) \cong W(R^k(e))$.*

Corollary 3.14 (Alspach, 1966) *There are at most $\Gamma(n)$ distinct Walecki tournaments of order $2n+1$ for n a positive integer.*

Alspach conjectured the following.

Conjecture 3.15 (Alspach, 1966) *There are exactly* $\Gamma(n)$ *distinct Walecki tournaments of order* $2n+1$.

This conjecture gave us the motivation to study arc structure of Walecki tournaments.

4 Arc structure of Walecki tournaments

Walecki tournaments are known to be regular. They posses a rich collection of induced subtournaments ranging from transitive to regular as we shall see later. In some instances outsets of $v(0)$ induce regular or almost regular subtournaments. In other cases outsets of $v(0)$ induce subtournaments whose scores differ for at most 2. Specific permutations are proven to be automorphisms for various instances of Walecki tournaments. Various symmetries in the defining binary sequence $e \in E_n$ result in symmetries in the corresponding Walecki tournament which in turn result in non-trivial automorphism groups.

We will first prove various results which give insight into the arc structure of an arbitrary Walecki tournament. We remind the reader of the definition of the permutation $\eta = \tau^n \in \mathbb{S}_{2n+1}$ in Subsection 3.2.

Proposition 4.16 *Walecki tournaments are self-complementary and*

$$AntiAut(W(e)) = Aut(W(e))\eta.$$

PROOF. Let $e \in E_n$. Clearly, $\overline{W(e)} = W(\overline{e})$. Theorem 3.13 furthermore implies that $W(\overline{e}) \cong W(e)$ with isomorphism τ^n. Therefore, $\tau^n = \eta$ is an anti-automorphism of $W(e)$. This proves the result. $\square$

Proposition 4.17 *Let T be a tournament. For $v \in V(T)$ and $g \in Aut(T)_v$, $g(N^+(v)) = N^+(v)$ and $g(N^-(v)) = N^-(v)$.*

Proof. If $u \in N^+(v)$, then $v \longrightarrow u$ and $v = g(v) \longrightarrow g(u)$. Hence, $g(u) \in N^+(v)$ and $g(N^+(v)) \subseteq N^+(v)$. Now, g is a permutation of $V(T)$ which implies $|g(N^+(v))| = |N^+(v)|$. Therefore, $g(N^+(v)) = N^+(v)$. Moreover, $V(T) = N^-(v) \cup \{v\} \cup N^+(v)$ which implies $|g(N^-(v))| = |N^-(v)|$, as required. $\square$

Proposition 4.18 *Let T be a Walecki tournament and let k be an integer such that $1 \leq k \leq 2n$. If $t_{(0)} \longrightarrow t_{(k)} \in \vec{H}_k$, then $t_{(n+k)} \longrightarrow t_{(0)} \in \vec{H}_k$.*

Proof. The result follows from the definition of the Hamilton directed cycle $\vec{H}_k$ used in the construction of Walecki tournaments (see Figure 7). $\square$

Proposition 4.19 *Let $W(e)$ be a Walecki tournament and let i and j be integers such that $0 \leq i, j \leq 2n-1$. If $t_{(i)} \longrightarrow t_{(j)}$, then $t_{(n+i)} \longleftarrow t_{(n+j)}$.*

Proof. The result is an immediate consequence of the fact that $\eta = \tau^n$ is an anti-automorphism of $W(e)$. $\square$

The following result is used in many proofs about structure of Walecki tournaments. It uses binary sequence $e \in E_n$ to determine the direction of a particular arc in $W(e)$. The arcs are grouped according to the Hamilton directed cycle they belong to.

Lemma 4.20 *Let $e \in E_n$ and let $W(e)$ be a Walecki tournament. Let i and j be integers such that $1 \leq i < j \leq 2n$. In the case when $j-i$ is even let $k = i+1+(j-i)/2$. If $1 \leq k \leq n$, then*

$$t_{(j+e_k)} \longrightarrow t_{(i)} \longrightarrow t_{(j+\bar{e}_k)} \tag{4.10}$$

and

$$t_{(j+n+e_k)} \longleftarrow t_{(i+n)} \longleftarrow t_{(j+n+\bar{e}_k)}. \tag{4.11}$$

If $n+1 \leq k \leq 2n$, then

$$t_{(j-n+e_{k-n})} \longrightarrow t_{(i-n)} \longrightarrow t_{(j-n+\bar{e}_{k-n})} \tag{4.12}$$

and

$$t_{(j+e_{k-n})} \longleftarrow t_{(i)} \longleftarrow t_{(j+\bar{e}_{k-n})}. \tag{4.13}$$

In the case when $j-i$ is odd let $\ell = i+1+(j-i-1)/2$. If $1 \leq \ell \leq n$, then

$$t_{(j-e_\ell)} \longleftarrow t_{(i)} \longleftarrow t_{(j-\bar{e}_\ell)} \tag{4.14}$$

and

$$t_{(j+n-e_\ell)} \longrightarrow t_{(i+n)} \longrightarrow t_{(j+n-\bar{e}_\ell)}. \tag{4.15}$$

If $n+1 \leq \ell \leq 2n$, then

$$t_{(j-n-e_{\ell-n})} \longleftarrow t_{(i-n)} \longleftarrow t_{(j-n-\bar{e}_{\ell-n})} \tag{4.16}$$

and

$$t_{(j-e_{\ell-n})} \longrightarrow t_{(i)} \longrightarrow t_{(j-\bar{e}_{\ell-n})}. \tag{4.17}$$

Proof. Let i and j be as in the conditions of the proposition. We first consider the case when $j-i$ is even. Let $k = i+1+(j-i)/2$. The structure of the Hamilton directed cycle $\overrightarrow{H}_k$ implies that if $e_k = 0$, then $t_{(j)} \longrightarrow t_{(i)}$ and $t_{(i)} \longrightarrow t_{(j+1)}$ (see Figure 7). On the other hand, if $e_k = 1$, then $t_{(j+1)} \longrightarrow t_{(i)}$ and $t_{(i)} \longrightarrow t_{(j)}$. Thus, (4.10) follows. Moreover, Proposition 4.19 implies (4.11). To prove (4.12) substitute $j-n$ for j in (4.10) and to prove (4.13) substitute $i-n$ for i in (4.11).

The remaining cases are proven similarly. □

We proceed with structural results for Walecki tournaments whose defining binary sequence has a specific pattern.

5 Zero pattern

We say that an equivalence class $[e]_R$ has *zero pattern* if the lexicographically smallest sequence of $[e]_R$ is $(0, 0, \ldots, 0)$. Tournaments corresponding to the zero pattern sequences have a surprising automorphism group as we shall see. In the case of odd n,

$$\sigma = \big(t_{(1)}\ t_{(2)} \cdots t_{(n)}\big)\big(t_{(2n)}\ t_{(2n-1)} \cdots t_{(n+1)}\big)\big(t_{(0)}\big) \in \mathbb{S}_{2n+1}$$

is in fact an automorphism of $W((0, 0, \ldots, 0))$ as proven in Theorem 5.23. We will prove in Theorem 5.24 and Theorem 5.29 that Walecki tournaments of order $2n+1$ with zero pattern possess transitive subtournaments of order n that are induced by the outset of vertex $v_{(1)}$. When n is odd they also contain circulant subtournaments of order n induced by the outset of vertex $v_{(0)}$ as proven in Proposition 5.25. This furthermore implies that these subtournaments are regular (see Proposition 5.26). For example see Figure 5 which shows Walecki tournament $W(000)$.

5.1 Automorphism σ

The next few results are needed in the proof of the main theorem in Aleš [2] for the case when n is odd, $n \geq 3$, and $e = (0, 0, \ldots, 0) \in E_n$.

Proposition 5.21 *Let $e \in E_n$ and $n \geq 3$. Consider the Walecki tournament $W(e)$. If $e_i = e_{i+1}$ and $1 \leq i \leq n-1$, then τ is dominance-preserving on $\vec{H}_i$ and τ^{-1} is dominance-preserving on $\vec{H}_{i+1}$*

PROOF. The result follows from the definition of the Hamilton directed cycles $\vec{H}_1, \vec{H}_2, \ldots, \vec{H}_n$ comprising $W(e)$. $\square$

Notice, that permutation τ is not an automorphism of $W(e)$.

Lemma 5.22 *Let $e \in E_n$ and $n \geq 3$. Consider the Hamilton directed cycle $\vec{H}_i$, $1 \leq i \leq n$, in the Walecki tournament $W(e)$. Let $u \longrightarrow w$ be any arc of $\vec{H}_i$ of the form $u = t_{(i-j)}$ and $w = t_{(i+j+1)}$ or $u = t_{(i+j)}$ and $w = t_{(i-j)}$. Define ϱ by letting $\varrho = \tau^{-1}$ on $t_{(i-j)}$, $1 \leq j \leq n-2$, and $\varrho = \tau$ on $t_{(i+j)}$, $1 \leq j \leq n-2$. Then ϱ is dominance-preserving on the arc $u \longrightarrow w$.*

PROOF. The arc joining $\varrho(u)$ and $\varrho(w)$ also lies on $\vec{H}_i$. Moreover, $\varrho(u) \longrightarrow \varrho(w)$ because $u \longrightarrow w$ and from the way $\vec{H}_i$ is constructed. $\square$

Let n be odd and let the permutation $\sigma \in \mathbb{S}_{2n+1}$ be defined by $\sigma = (t_{(1)}\, t_{(2)} \cdots t_{(n)})(t_{(2n)}\, t_{(2n-1)} \cdots t_{(n+1)})(t_{(0)})$, (see Figure 8). That is,

$$\sigma(t_{(i)}) = \begin{cases} t_{(0)} & \text{if } i = 0, \\ t_{(i+1)} & \text{if } 1 \leq i \leq n-1, \\ t_{(1)} & \text{if } i = n, \\ t_{(i-1)} & \text{if } n+2 \leq i \leq 2n, \\ t_{(2n)} & \text{if } i = n+1. \end{cases} \tag{5.18}$$

It follows that

$$\sigma(v_{(i)}) = \begin{cases} v_{(0)} & \text{if } i = 0, \\ v_{(2)} & \text{if } i = 1, \\ v_{(i+2)} & \text{if } 2 \leq i \leq 2n-3 \\ v_{(1)} & \text{if } i = 2n-2, \\ v_{(2n)} & \text{if } i = 2n-1, \\ v_{(3)} & \text{if } i = 2n. \end{cases} \tag{5.19}$$

Theorem 5.23 *If $e = (0, 0, \ldots, 0)$, a binary sequence of length n, for n odd, and $n \geq 3$, then σ is an automorphism of $W(e)$.*

PROOF. We want to show that σ is dominance-preserving on all of $W(e)$. By definition, σ fixes $t_{(0)}$, cyclically permutes the vertices of the outset of $t_{(0)}$, and cyclically permutes the vertices of the inset of $t_{(0)}$. Thus, σ is dominance-preserving on the arcs incident with $t_{(0)}$. Figure 8 shows the action of $\sigma \in \mathbb{S}_{2n+1}$ on vertices of the Walecki tournament $W(e)$, for $e = (0, 0, \ldots, 0) \in E_n$, n odd, and $n \geq 3$.

Note that σ restricted to $V^+ = N^+(t_{(0)}) - \{t_{(n)}\} = \{t_{(1)}, t_{(2)}, t_{(3)}, \ldots, t_{(n)}\}$ has the same action as τ. It then follows from Proposition 5.21 that σ is dominance-preserving on any arc both of whose vertices lie in V^+ because such an arc is not in $\vec{H}_n$. Similarly, σ restricted to $V^- = N^-(t_{(0)}) - \{t_{(n+1)}\} = \{t_{(n+2)}, t_{(n+3)}, \ldots, t_{(2n)}\}$ has the same action as τ^{-1}. Again it follows from Proposition 5.21 that σ is dominance-preserving on any arc both of whose vertices lie in V^- because such an arc is not in $\vec{H}_1$. (Figure 4 shows the action of the permutation τ on vertices of the Walecki tournament $W(e)$.)

By Lemma 5.22, σ is dominance-preserving on any arc with one end vertex in V^+ and the other end vertex in V^- because σ acts like τ on V^+ and τ^{-1} on V^-. It remains to show that σ is dominance-preserving on any arc incident with $t_{(n)} = v_{(2n-2)}$ or $t_{(n+1)} = v_{(2n)}$. Since $e_n = 0$ we have $v_{(2n-2)} \longrightarrow v_{(2n)} \in \vec{H}_n$.

Furthermore, $\sigma(v_{(2n-2)}) = v_{(1)} = \tau^0(v_{(1)}) = \tau^{n-1}(v_{(2n-1)})$ and $\sigma(v_{(2n)}) = v_{(3)} = \tau^{-1}(v_{(1)}) = \tau^{n-1}(v_{(2n)})$ imply $\sigma(v_{(2n-2)}) \longrightarrow \sigma(v_{(2n)}) \in \overrightarrow{H}_n$.

We divide the proof for the arcs incident with either $v_{(2n-2)}$ or $v_{(2n)}$ into two cases. Let $u \in V^- \cup V^+$. First we consider arcs $u \longrightarrow v_{(2n-2)}$ and $u \longrightarrow v_{(2n)}$. We use Proposition 3.12 and Lemma 4.20 extensively. We remind the reader that arcs $\tau^{k-1}(v_{(i)}) \longrightarrow \tau^{k-1}(v_{(i+1)})$, $1 \leq i \leq 2n-1$, belong to the Hamilton directed cycle $\overrightarrow{H}_k$. In the figures accompanying this proof we use arrows ⋯⋯➤ to denote the action of σ and arrows ——➤ to denote arcs of the tournament.

Case 1.1. Let $u \in V^-$. If k is an integer such that $1 \leq k \leq (n-1)/2$, then vertices $t_{(n+2k+1)}$ and $t_{(n+2k)}$ belong to V^-, and arcs $t_{(n+2k+1)} = \tau^{k-1}(v_{(2(n-k)-1)}) \longrightarrow t_{(n)} = \tau^{k-1}(v_{(2(n-k))})$ and $t_{(n+2k)} = \tau^{k-1}(v_{(2(n-k)+1)}) \longrightarrow t_{(n+1)} = \tau^{k-1}(v_{(2(n-k+1))})$ belong to the Hamilton directed cycle $\overrightarrow{H}_k$ (see Figure 9). Now, σ is dominance-preserving on these two arcs since $\sigma(t_{(n+2k+1)}) = t_{(n+2k)} = \tau^{k+(n-1)/2}(v_{(n+2k-1)})$ and $\sigma(t_{(n)}) = t_{(1)} = \tau^{k+(n-1)/2}(v_{(n+2k)})$ imply $\sigma(v_{(2(n-2k)+1)}) \longrightarrow \sigma(v_{(2n-2)}) \in \overrightarrow{H}_{k+(n+1)/2}$. Also $\sigma(t_{(n+2k)}) = t_{(n+2k-1)} = \tau^{k+(n-3)/2}(v_{(n+2k-1)})$ and $\sigma(t_{(n+1)}) = t_{(2n)} = \tau^{k+(n-3)/2}(v_{(n+2k)})$ imply $\sigma(v_{(2(n-2k+1)+1)}) \longrightarrow \sigma(v_{(2n)}) \in \overrightarrow{H}_{k+(n-1)/2}$.

Case 1.2. Let $u \in V^+$. If k is an integer such that $(n+1)/2 \leq k \leq n-1$, then vertices $t_{(2k-n+1)}$ and $t_{(2k-n+2)}$ belong to V^+, and arcs $t_{(2k-n+1)} = \tau^{k-1}(v_{(2(n-k)-1)}) \longrightarrow t_{(n)} = \tau^{k-1}(v_{(2(n-k))})$ and $t_{(2k-n+2)} = \tau^{k-1}(v_{(2(n-k)+1)}) \longrightarrow t_{(n+1)} = \tau^{k-1}(v_{(2(n-k+1))})$ belong to the Hamilton directed cycle $\overrightarrow{H}_k$ (see Figure 10). Similarly as before, σ is dominance-preserving on these two arcs since $\sigma(t_{(2k-n+1)}) = t_{(2k-n+2)} = \tau^{k-(n-1)/2}(v_{(2k-n+1)})$ and $\sigma(t_{(n)}) = t_{(1)} = \tau^{k-(n-1)/2}(v_{(2k-n+2)})$ imply $\sigma(v_{(2(2k-n))}) \longrightarrow \sigma(v_{(2n-2)}) \in \overrightarrow{H}_{k-(n-3)/2}$. Moreover, $\sigma(t_{(2k-n+2)}) = t_{(2k-n+1)} = \tau^{k-(n+1)/2}(v_{(2k-n+1)})$ and $\sigma(t_{(n+1)}) = t_{(2n)} = \tau^{k-(n+1)/2}(v_{(2k-n+2)})$ imply $\sigma(v_{(2(2k-n-1))}) \longrightarrow \sigma(v_{(2n)}) \in \vec{H}_{k-(n-1)/2}$.

Similarly, we consider arcs $v_{(2n-2)} \longrightarrow u$ and $v_{(2n)} \longrightarrow u$ for $u \in V^- \cup V^+$. We state all cases but leave proofs to the reader.

Case 2.1. Let $u \in V^-$. If k is an integer such that $2 \leq k \leq (n-1)/2$, then vertices $v_{(2(n-2k+1)+1)}$ and $v_{(2(n-2k+2)+1)}$ belong to V^-. We have to consider vertices $v_{(3)}$ and $v_{(2n-1)} \in V^-$ as a special case.

Case 2.2. Let $u \in V^+$. If k is an integer such that $(n+3)/2 \leq k \leq n-1$, then vertices $v_{(2(2k-n-1))}$ and $v_{(2(2k-n+2))}$ belong to V^+. We consider vertex $v_{(1)} \in V^+$as a special case.

We have proven that $u \longrightarrow w$ implies $\sigma(u) \longrightarrow \sigma(w)$ for every arc $u \longrightarrow w$ in $W(e)$. Therefore, $\sigma \in Aut(W(e))$. This completes the proof. □

The importance of σ in the theory of automorphism groups of Walecki tournaments was previously unknown. However, once zero pattern sequences were determined as a potential source of Walecki tournaments with non-trivial automorphism groups, permutation σ became a natural candidate for their generator.

5.2 Regular and transitive subtournaments

In the following result we prove transitivity of subtournaments of Walecki tournament with zero pattern. The linear orderings of subsets of vertices that induce transitive subtournaments are given in the proof.

Theorem 5.24 *Let $T = W(e)$ for $e = (0, 0, \dots, 0) \in E_n$, n odd, and $n \geq 3$. For $t_{(i)} \in N^+(t_{(0)})$ and $t_{(j)} \in N^-(t_{(0)})$ the tournaments $T\langle N^+(t_{(i)})\rangle$, $T\langle N^-(t_{(i)}) - \{t_{(0)}\}\rangle$, $T\langle N^+(t_{(j)}) - \{t_{(0)}\}\rangle$, and $T\langle N^-(t_{(j)})\rangle$ are transitive subtournaments of T.*

Proof. Proposition 4.16 tells us that $W(e) \cong \overline{W(e)}$. Since $\sigma \in Aut(T)$, it suffices to prove the lemma for the vertex $t_{(1)} \in N^+(t_{(0)})$.

Let us consider the outset of vertex $t_{(1)}$. The arcs $v_{(2i+1)} \longrightarrow v_{(2i+2)}$ lie in $\overrightarrow{H}_1$ for $0 \leq i \leq n-1$ so that $t_{(1)} = \tau^i(v_{(2i+1)}) \longrightarrow \tau^i(v_{(2i+2)}) = \tau^{2i+1}(v_{(1)}) \in \overrightarrow{H}_{i+1}$. Hence

$$N^+(t_{(1)}) = \{t_{(2i+2)} \mid 0 \leq i \leq n-1\}. \tag{5.20}$$

We prove that the vertices of $N^+(t_{(1)})$ in the order $t_{(2n)}, t_{(2)}, t_{(2n-2)}, t_{(4)}, \dots, t_{(2n-2i)}, t_{(2i+2)}, \dots, t_{(n+3)}, t_{(n-1)}, t_{(n+1)}$ determine the score sequence $(s_j)_{j=0}^{n-1}$, where $s_j = j$ for $0 \leq j \leq n-1$. That is,

$$\begin{aligned} s_{2i} &= s(t_{(2n-2i)}) = 2i \quad \text{for } 0 \leq i \leq \tfrac{n-3}{2}, \\ s_{2i+1} &= s(t_{(2i+2)}) = 2i+1 \quad \text{for } 0 \leq i \leq \tfrac{n-3}{2}, \\ s_{n-1} &= s(t_{(n+1)}(1)) = n-1. \end{aligned}$$

We do so by proving that all arcs in the subtournament $T\langle N^+(t_{(1)})\rangle$ point from right to left in the ordering of the vertices given above. Figure 11 shows seven different types of arcs considered. We divide the proof into several cases and show details for some of them. In all of them the index i is an integer such that $0 \leq i \leq \frac{n-3}{2}$.

Case 1.1. The arcs of type $t_{(n+1)} \longrightarrow t_{(2i+2)}$ belong to cycles $\overrightarrow{H}_{(n+3)/2+i}$, because $t_{(n+1)} = \tau^{(n+1)/2+i}(v_{(n-2i-1)})$ and $t_{(2i+2)} = \tau^{(n+1)/2+i}(v_{(n-2i)})$. We omit proofs of $t_{(n+1)} \longrightarrow t_{(2n-2i)} \in \overrightarrow{H}_{(n+1)/2-i}$ and $t_{(2i)} \longrightarrow t_{(2n-2i)} \in \overrightarrow{H}_1$. In the remaining cases the index j is in the range $i \leq j \leq \frac{n-3}{2}$.

CASE 1.2. We have $t_{(2j+2)} \longrightarrow t_{(2i+2)} \in \vec{H}_{i+j+2}$, because $t_{(2j+2)} = \tau^{i+j+1}(v_{(2(j-i))})$ and $t_{(2i+2)} = \tau^{i+j+1}(v_{(2(j-i)+1)})$. We omit proofs of $t_{(2j+2)} \longrightarrow t_{(2n-2i)} \in \vec{H}_{j-i+1}$, $t_{(2n-2j)} \longrightarrow t_{(2i+2)} \in \vec{H}_{n+i-j+1}$, and $t_{(2n-2j)} \longrightarrow t_{(2n-2i)} \in \vec{H}_{n-i-j}$. We have proven that the scores of vertices in the subtournament $T\langle N^+(t_{(1)})\rangle$ are $s_j = j$ for $0 \le j \le n-1$. Thus, the subtournament $T\langle N^+(t_{(1)})\rangle$ is transitive.

Next we consider the set of vertices $N^-(t_{(1)}) - \{t_{(0)}\}$. Since $N^+(t_{(1)}) = t_{(2i+2)} \mid 1 \le i \le n-1\}$, it follows that $N^-(t_{(1)}) - \{t_{(0)}\} = \{t_{(2i+1)} \mid 1 \le i \le n-1\}$. We will prove that the labeling of the vertices of $N^-(v_{(1)}) - \{v_{(0)}\}$ in the order $t_{(2n-1)}, t_{(3)}, t_{(2n-3)}, t_{(5)}, \ldots, t_{(2n-2i+1)}, t_{(2i+1)}, \ldots, t_{(n+2)}, t_{(n)}$ determines the score sequence $(s_j)_{j=0}^{n-1}$, where $s_j = j$ for $0 \le j \le n-2$. That is,

$$\begin{array}{rcl} s_{2i-2} & = & s(t_{(2n-2i+1)}) = 2i-2 \text{ for } 1 \le i \le \frac{n-1}{2}, \\ s_{2i-1} & = & s(t_{(2i+1)}) = 2i-1 \text{ for } 1 \le i \le \frac{n-1}{2}. \end{array}$$

Similarly as in the previous case we prove that all arcs in the subtournament $T\langle N^-(t_{(1)}) - \{t_{(0)}\}\rangle$ point from right to left in the ordering of the vertices given above. Thus, the subtournament $T\langle N^-(t_{(1)}) - \{t_{(0)}\}\rangle$ is transitive. □

The following result relates Walecki tournaments with zero pattern for n odd and circulant tournaments.

Proposition 5.25 *Let $T = W(e)$ for $e = (0,0,\ldots,0) \in E_n$, n odd and $n \ge 3$. $T\langle N^+(t_{(0)})\rangle$ and $T\langle N^-(t_{(0)})\rangle$ are circulant tournaments. Moreover, their automorphism groups contain n-cycles $(t_{(1)}\ t_{(2)}\ t_{(3)}\ \cdots t_{(n-1)}\ t_{(n)})$ and $(t_{(2n)}\ t_{(2n-1)}\ t_{(2n-2)} \cdots\ t_{(n+2)}\ t_{(n+1)})$, respectively.*

PROOF. The permutation $\sigma \in \mathbb{S}_{2n+1}$ is an element of $Aut(T)$. It induces an n-cycle $(t_{(1)}\ t_{(2)}\ t_{(3)}\ \cdots\ t_{(n-1)}\ t_{(n)})$ on the vertex set $N^+(t_{(0)})$. Proposition 2.5 furthermore implies that $T\langle N^+(t_{(0)})\rangle$ is a circulant tournament. Similarly for $T\langle N^-(t_{(0)})\rangle$. □

Proposition 5.26 *Let* $T = W(e)$ *for* $e = (0, 0, \ldots, 0) \in E_n$ *and* n *odd. The subtournaments* $T\langle N^+(t_{(0)})\rangle$ *and* $T\langle N^-(t_{(0)})\rangle$ *are regular.*

PROOF. Circulant tournaments are vertex-transitive by Proposition 2.4. Since n is odd, $T\langle N^+(t_{(0)})\rangle$ must be regular. Then $T\langle N^-(t_{(0)})\rangle$ is regular since $T \cong \overline{T}$. □

Proposition 5.27 *Let* $T = W(e)$ *for* $e = (0, 0, \ldots, 0) \in E_n$ *and* n *odd. The subtournaments* $T\langle N^+(t_{(0)})\rangle$ *and* $T\langle N^-(t_{(0)})\rangle$ *are self-complementary with an anti-automorphism* $\varphi = (t_{(1)}t_{(n)})(t_{(2)}t_{(n-1)})\cdots(t_{((n-1)/2)}t_{((n+3)/2)})(t_{((n+1)/2)}) \in \mathbb{S}_n$, *of order* 2.

PROOF. The result follows by the proof of Proposition 2.6. □

5.3 Zero pattern for n even

There are some important differences in the zero pattern case between n odd and n even. When n is even σ is not an automorphism because an automorphism group of a tournament has to have an odd order.

Theorem 5.28 *Let $T = W(e)$ for $e = (0, 0, \ldots, 0) \in E_n$, n even, and $n \geq 4$. The subtournaments $T\langle N^+(t_{(0)})\rangle$ and $T\langle N^-(t_{(0)})\rangle$ are almost regular.*

PROOF. It follows from the definition of Walecki tournaments that

$$N^+(t_{(0)}) = \{t_{(i)} \mid 1 \leq i \leq n\}. \tag{5.21}$$

Equation (5.20) also holds for n even. Hence,

$$N^+(t_{(1)}) = \{t_{(2i)} \mid 1 \leq i \leq n\}. \tag{5.22}$$

It follows from (5.21) and (5.22) that $N^+(t_{(0)}) \cap N^+(t_{(1)}) = \{t_{(2i)} \mid 1 \leq i \leq n/2\}$. Therefore, $|N^+(t_{(0)}) \cap N^+(t_{(1)})| = n/2$. It is easy to verify that $N^+(t_{(2)}) = \{t_{(2i+1)} \mid 1 \leq i \leq n-1\} \cup \{t_{(2n)}\}$, which implies $N^+(t_{(0)}) \cap N^+(t_{(2)}) = \{t_{(2i+1)} \mid 1 \leq i \leq n/2 - 1\}$. Furthermore, $|N^+(t_{(0)}) \cap N^+(t_{(2)})| = n/2 - 1$.

The scores of the remaining vertices in $N^+(t_{(0)})$ can be obtained similarly. We omit the proofs. They alternate between $n/2$ and $n/2 - 1$ which proves that $T\langle N^+(t_{(0)})\rangle$ is almost regular. Since T is self-complementary this completes the proof. □

Theorem 5.29 *Let $T = W(e)$ for $e = (0, 0, \ldots, 0) \in E_n$, n even, and $n \geq 4$. For $t^+ \in N^+(t_{(0)})$ and $t^- \in N^-(t_{(0)})$ the tournaments $T\langle N^+(t^+)\rangle$, $T\langle N^-(t^+) - \{t_{(0)}\}\rangle$, $T\langle N^+(t^-) - \{t_{(0)}\}\rangle$, and $T\langle N^-(t^-)\rangle$ are transitive subtournaments of T.*

PROOF. Proposition 4.16 tells us that $T \cong \overline{T}$. Hence, it suffices to prove the theorem for vertices in $N^+(t_{(0)})$. The proof of transitivity of $T\langle N^+(t_{(1)})\rangle$ and

$T\langle N^-(t_{(1)}) - \{t_{(0)}\}\rangle$ is similar to the proof of Theorem 5.24, the difference being that n is even. This changes the proof in two ways.

First, the vertices of $N^+(t_{(1)})$ in the order $t_{(2n)}, t_{(2)}, t_{(2n-2)}, t_{(4)}, \ldots, t_{(2n-2i)}, t_{(2i+2)}, \ldots, t_{(n+2)}, t_{(n)}$ determine the score sequence $(0, 1, 2, \ldots, n-1)$.

Furthermore, since n is even $\sigma \notin Aut(T)$. Therefore, one has to prove that the subtournaments $T\langle N^+(t^+)\rangle$ and $T\langle N^-(t^+) - \{t_{(0)}\}\rangle$ are transitive for all $t^+ \in N^+(t_{(0)})$. However, the proofs are similar to the initial case and we omit them. $\square$

This concludes the exploration of arc structure of Walecki tournaments with zero pattern.

6 Closing remarks

As a step towards proving the Conjecture 3.15 of Alspach we determine the automorphism groups of Walecki tournaments for all initial cases and those with zero pattern (see Aleš [2]). Computational results are also presented (see Aleš [2]). We consider the arc structure of Walecki tournaments whose corresponding binary sequence is periodic in [3] and [4].

Acknowledgment. I would like to express my gratitude to Dr. Brian Alspach for his patient guidance during my research work. Special thanks are also due to Dr. Tomaž Pisanski for many helpful comments.

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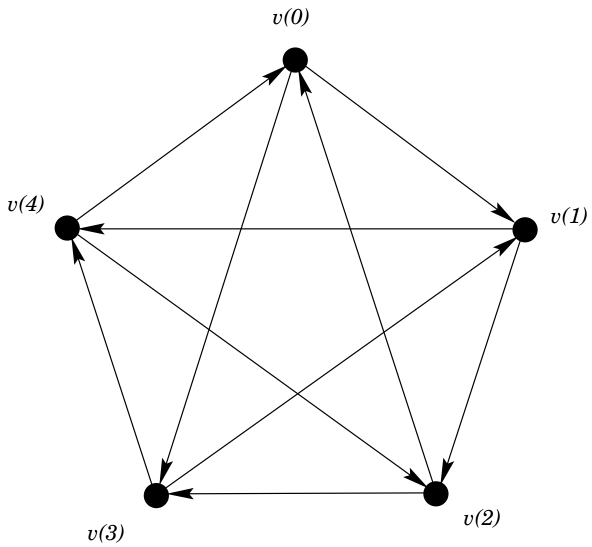

Figure 1: A circulant tournament T_5 on 5 vertices.

$$\begin{bmatrix} 0&0&0&0&0\\ 0&0&0&0&1\\ 0&0&0&1&1\\ 0&0&1&1&1\\ 0&1&1&1&1\\ 1&1&1&1&1\\ 1&1&1&1&0\\ 1&1&1&0&0\\ 1&1&0&0&0\\ 1&0&0&0&0 \end{bmatrix} \begin{bmatrix} 0&0&0&1&0\\ 0&0&1&0&1\\ 0&1&0&1&1\\ 1&0&1&1&1\\ 0&1&1&1&0\\ 1&1&1&0&1\\ 1&1&0&1&0\\ 1&0&1&0&0\\ 0&1&0&0&0\\ 1&0&0&0&1 \end{bmatrix} \begin{bmatrix} 0&0&1&0&0\\ 0&1&0&0&1\\ 1&0&0&1&1\\ 0&0&1&1&0\\ 0&1&1&0&1\\ 1&1&0&1&1\\ 1&0&1&1&0\\ 0&1&1&0&0\\ 1&1&0&0&1\\ 1&0&0&1&0 \end{bmatrix} \begin{bmatrix} 0&1&0&1&0\\ 1&0&1&0&1 \end{bmatrix}$$

Figure 2: The four cycles of the complementing circular shift register in $\mathcal{R}(5)$.

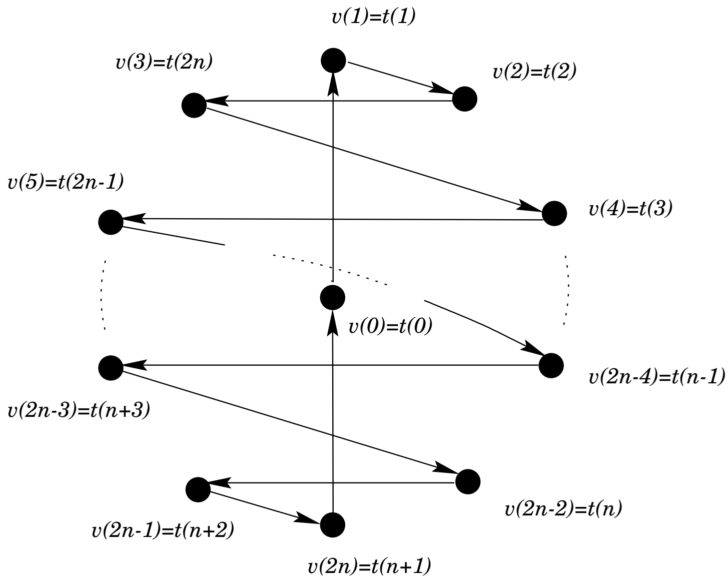

Figure 3: The directed Hamilton cycle $\overrightarrow{H}_1 = [v(0), v(1), \ldots, v(2n)]$.

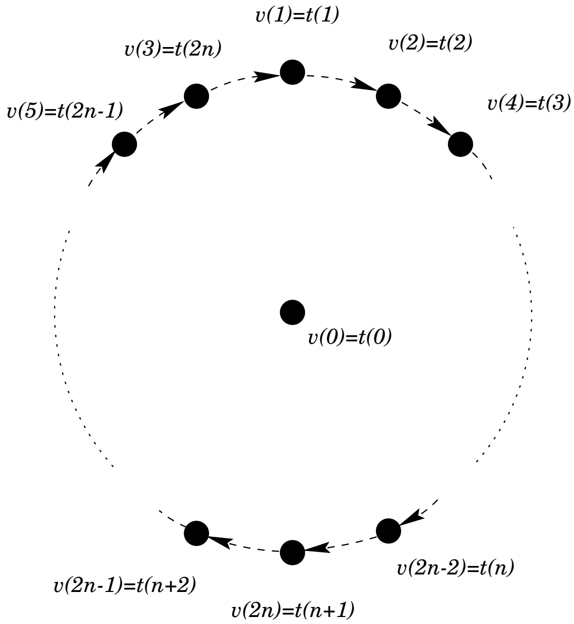

Figure 4: The diagram shows the action of the permutation $\tau \in \mathbb{S}_{2n+1}$ on vertices of the Walecki tournament $W(e)$. Vertex $v(0)$ is fixed by τ.

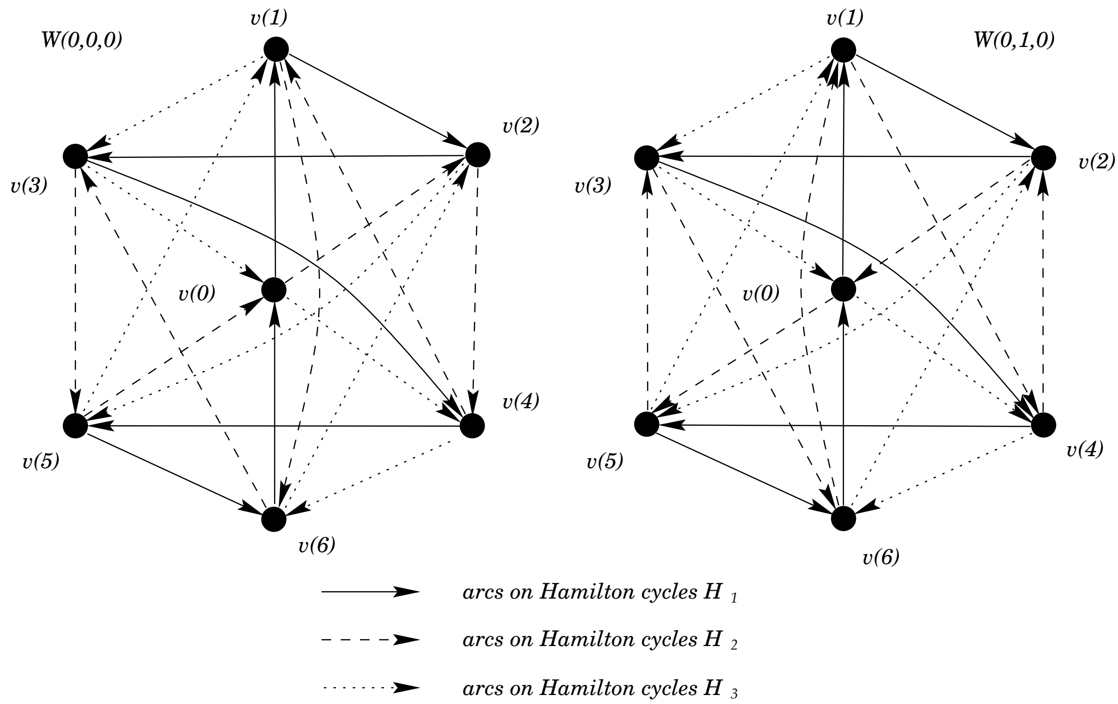

Figure 5: Walecki tournaments $W(000)$ and $W(010)$ on 7 vertices.

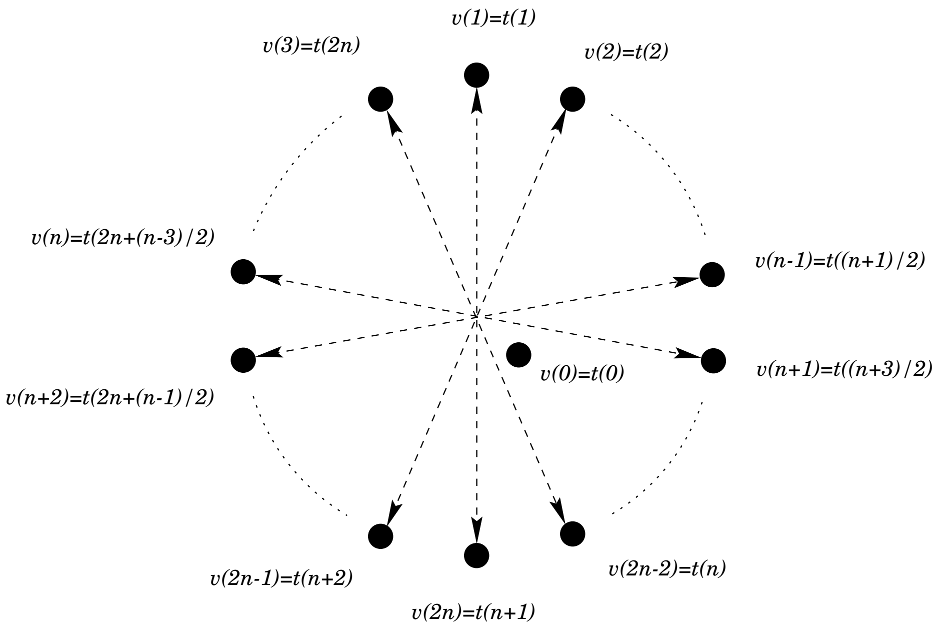

Figure 6: The diagram shows the action of the permutation $\eta = \tau^n \in \mathbb{S}_{2n+1}$ on vertices of the Walecki tournament $W(e)$, for $e \in E_n$, n odd, and $n \geq 1$. Vertex $v(0)$ is fixed by the permutation η. Because the permutation η is an involution we represent its action by two-way arrows.

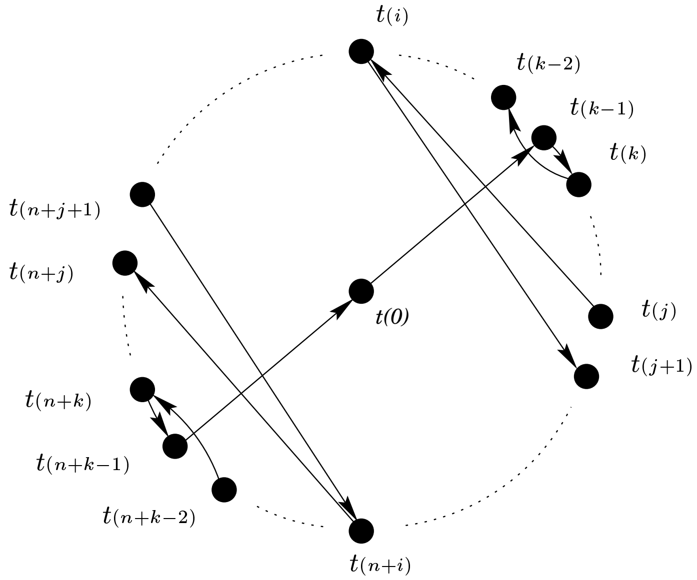

Figure 7: The diagram shows the case when $j-i$ is even and $e_k = 0$ from the proof of Lemma 4.20.

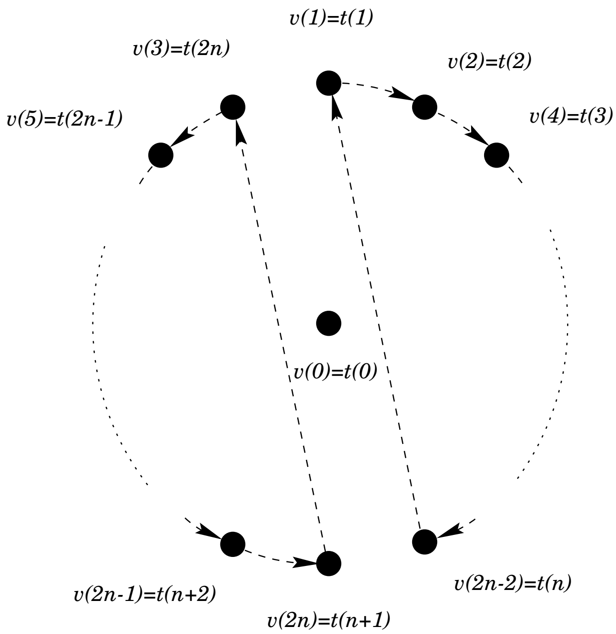

Figure 8: The action of $\sigma \in \mathbb{S}_{2n+1}$.

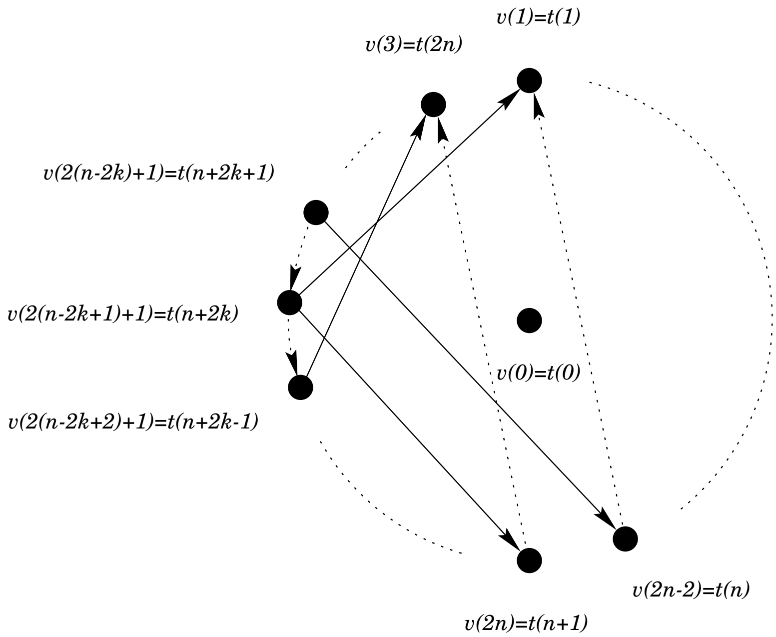

Figure 9: The diagram shows the action of the permutation $\sigma \in \mathbb{S}_{2n+1}$ on arcs from Case 1.1 for the proof of Theorem 5.23.

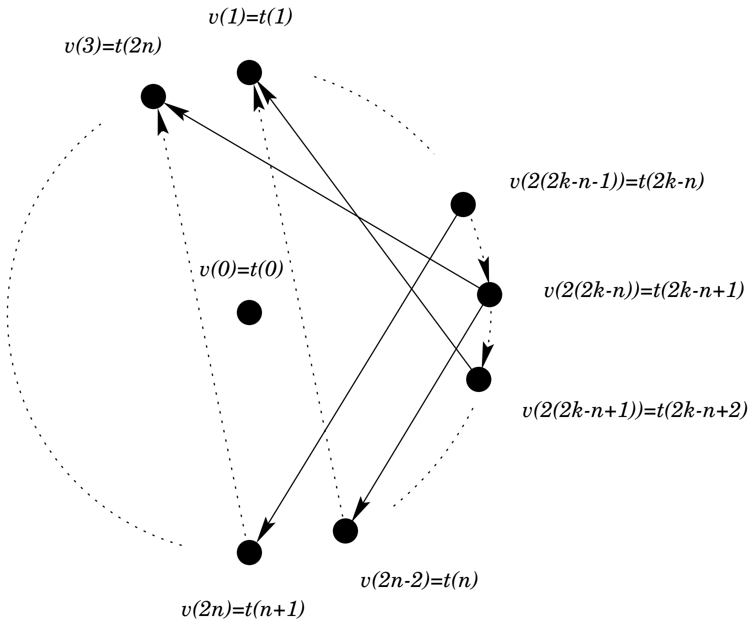

Figure 10: The diagram shows the action of the permutation $\sigma \in \mathbb{S}_{2n+1}$ on arcs from Case 1.2 for the proof of Theorem 5.23.

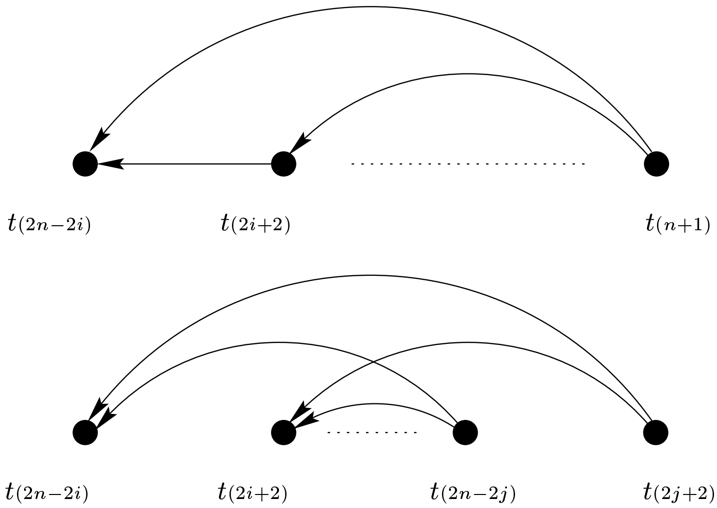

Figure 11: The diagram shows seven different types of arcs from Case 1.1 and Case 1.2 of the proof of transitivity of the tournament $T\langle N^+(v_{(1)})\rangle$ from Theorem 5.24.