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WALECKI TOURNAMENTS:
PART II

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Walecki Tournaments: Part II

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Abstract. Walecki tournaments were defined by Alspach in 1966. They form a class of very interesting tournaments since they speak in favor of Kelly's conjecture: every regular tournament possesses a Hamilton directed cycle decomposition. Enumeration of Walecki tournaments was presented as an open problem in a paper by Alspach in 1989. In an attempt to prove this 34 years old conjecture we determine the automorphism groups of Walecki tournaments for all initial cases and those whose corresponding binary sequences have zero pattern.

1 Introduction

Walecki tournaments were defined by Alspach in 1966, [6]. The interest in their combinatorial structure arises from the fact that they support Kelly's conjecture (see Moon [13]). Namely, they possess a natural Hamilton directed cycle

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decomposition. Secondly, the problem of enumerating Walecki tournaments has not been solved to date. It has been published as an open problem in a paper by Alspach [7].

In an attempt to keep the paper self-contained, background results from topics on permutation groups and algebraic graph theory are included.

As a step towards proving the conjecture of Alspach [7] we determine the automorphism groups of Walecki tournaments for all initial cases and those with zero pattern. The method applied is based on a mapping between cycles of the complementing circular shift register and Walecki tournaments. The topic of this paper continues the work of Aleš [2]. This enumeration problem led us to the study of automorphism groups of Walecki tournaments.

2 Theoretical background

For terminology, definitions, and background results related to topics on tournaments and algebraic graph theory, we refer the reader to Aleš [2]. Group theory results can be found, for example, in Isaacs, (see [9]). Let G denote a group and let H be its subgroup. We use notation $H \subseteq G$. If H is a normal subgroup we write $N \triangleleft G$. We will make use of the following result about normal subgroups of a given group.

Theorem 2.1 *Let $H \subseteq G$, where G is finite and index of H in G is p . If p is the smallest prime divisor of $|G|$ then $H \triangleleft G$.*

Let N and H be groups. The *semi-direct product* $N \rtimes H$ of N by H is defined on element set $(n, h) \in N \times H$ with multiplication defined as follows: $(n_1, h_1)(n_2, h_2) = (n_1(h_1 n_2 h_1^{-1}), n_2 h_2)$. It is easy to see that $N \rtimes H$ is a group.

The following theorem characterizes groups that are in fact semi-direct products of two of its subgroups (see Birkhoff and MacLane [8]).

Theorem 2.2 *Let $H, N \subseteq G$. If $N \triangleleft G$, $NH = G$, and $N \cap H = id$, then $G = N \otimes H$.*

Moon proved that a finite group G is isomorphic to the automorphism group $Aut(T)$ of some tournament T if and only if the order of G is odd (see [12]). We state the necessity of the condition.

Proposition 2.3 (Moon, 1963) *The order of the automorphism group $Aut(T)$ of a tournament T can not be even.*

The automorphism group $Aut(T)$ is a subgroup of the symmetric group $\mathbb{S}_{|V(T)|}$. We prove in Section 3 that for some Walecki tournaments on $2n + 1$ vertices, $n \geq 4$, the upper bound on the order of the automorphism groups is n .

3 Automorphism groups of Walecki tournaments

The arc structure of Walecki tournaments with zero pattern was studied in [1, 2]. It will play a major role in determining the automorphism groups of Walecki tournaments with zero pattern. Firstly we will outline the techniques used in the proof of the main result (Theorem 3.4).

Let G denote $Aut(W(e))$ for $e \in E_n$. The main idea of the proof is to apply the orbit-stabilizer theorem twice on vertices $v(0)$ and $v(1)$ in a Walecki tournament (see [14], [2] Theorem 2.1), implying

$$|G| = |\mathcal{O}(v(0))| |G_{v(0)}| = |\mathcal{O}(v(0))| |\mathcal{O}(v(1))| |G_{v(0), v(1)}|, \quad (3.1)$$

where $\mathcal{O}(v_{(0)})$ denotes the orbit of vertex $v_{(0)}$ for the automorphism group G and $\mathcal{O}(v_{(1)})$ denotes the orbit of vertex $v_{(1)}$ for the subgroup $G_{v_{(0)}}$ of G .

Automorphism groups of some of the Walecki tournaments for initial values of n , $n \leq 4$, contain \mathbb{Z}_{2n+1} as a subgroup. For these instances of Walecki tournaments we prove that $|\mathcal{O}(v_{(0)})| = 2n + 1$. Strategy for the general cases will be to prove that $v_{(0)}$ must be a fixed vertex for any automorphism of $W(e)$, that is, $|\mathcal{O}(v_{(0)})| = 1$. Furthermore, we will prove that if another vertex, say $v_{(1)}$, of a Walecki tournament $W(e)$ is fixed by an automorphism $\varrho \in G$, then all the vertices of $W(e)$ are fixed by ϱ . This would imply $|G_{v_{(0)},v_{(1)}}| = 1$.

Various symmetries in the defining binary sequence e result in symmetries in the corresponding Walecki tournament. In these instances we will prove that $G_{v_{(0)}}$ is isomorphic to a nontrivial subgroup of \mathbb{Z}_n . The orbit-stabilizer theorem and additional arc structure between orbits for the automorphisms imply that G is isomorphic to a subgroup of \mathbb{Z}_n . Techniques used to prove $|\mathcal{O}(v_{(0)})| = 1$ and $|G_{v_{(0)},v_{(1)}}| = 1$ rely on an understanding of the arc structure of Walecki tournaments. Walecki tournaments are known to be regular (see [2, 6]). Walecki tournaments possess a rich collection of subtournaments that range from transitive to regular. In some instances outsets of $v_{(0)}$ induce regular or almost regular subtournaments. However, that is not the case for outsets of vertices in $V(W(e)) - \{v_{(0)}\}$ which in turn implies $|\mathcal{O}(v_{(0)})| = 1$.

The following theorem determines automorphism groups of Walecki tournaments for all initial cases and Walecki tournaments with zero pattern.

Theorem 3.4 *Let $n \in \mathbb{Z}^+$, let $e \in E_n$.*

$$Aut(W(e)) = \begin{cases} \mathbb{Z}_3 & \text{if } e = (0) \in E_1, \\ \mathbb{Z}_5 & \text{if } e = (0, 0) \in E_2, \\ \mathbb{Z}_3 & \text{if } e = (0, 0, 0) \in E_3, \\ \mathbb{Z}_7 \otimes \mathbb{Z}_3 & \text{if } e = (0, 1, 0) \in E_3, \\ \mathbb{Z}_1 & \text{if } e = (0, 0, 0, 0) \in E_4, \\ \mathbb{Z}_3 & \text{if } e = (0, 1, 0, 1) \in E_4, \\ \mathbb{Z}_n & \text{if } e = (0, 0, \dots, 0) \in E_n, n \text{ odd}, n \geq 5, \\ \mathbb{Z}_1 & \text{if } e = (0, 0, \dots, 0) \in E_n, n \text{ even}, n \geq 6, \end{cases}$$

where \mathbb{Z}_n denotes the cyclic group of order n and \otimes denotes a semi-direct product of two groups.

PROOF. Let $n \in \mathbb{Z}^+$. We divide the proof into cases.

CASE 1. Let $e = (0) \in E_1$. There is only one equivalence class in E_1 for the complementing circular shift register R and we use the representative (0) . The corresponding Walecki tournament is a directed 3-cycle. It is easy to see that the permutation $\sigma = (0\ 1\ 2)$ is an automorphism of $W(0)$ (see Figure 3). Therefore, $\mathbb{Z}_3 \cong \langle (0\ 1\ 2) \rangle \subseteq Aut(W(0)) \subseteq \mathbb{S}_3$. Since $Aut(W(0))$ contains no even order permutations of \mathbb{S}_3 , we have $Aut(W(0)) \cong \mathbb{Z}_3$, as required.

CASE 2. Let $e = (0, 0) \in E_2$. Similar to the previous case E_2 contains only one equivalence class, say $[00]$. Let G denote the automorphism group of $W(00)$, that is, $G = Aut(W(00))$. It is easy to see that the permutation $\sigma_1 = (0\ 1\ 2\ 3\ 4)$ is an automorphism of $W(00)$ (see Figure 3). Hence, $\mathbb{Z}_5 \cong \langle (0\ 1\ 2\ 3\ 4) \rangle \subseteq G$. Let X denote the vertex set of $W(00)$. Orbit Stabilizer Theorem implies $|G| = |\mathcal{O}(v(0))| |G_{v(0)}|$, where $\mathcal{O}(v(0))$ denotes the orbit of vertex $v(0)$ for the group G . Since $\sigma_1 \in G$, $\mathcal{O}(v(0)) = \{v(0), v(1), \dots, v(4)\} = X$. Hence, $|\mathcal{O}(v(0))| = 5$. Let $\varrho \in G_{v(0)}$. Since $\varrho(0) = 0$, the outset and inset of $v(0)$ have to be fixed setwise. That is, $\varrho(N^+(v(0))) = N^+(v(0))$ and $\varrho(N^-(v(0))) = N^-(v(0))$. Clearly, $N^+(v(0)) = \{v(1), v(2)\}$ and $|N^+(v(0))| = 2$. Since the order

of the automorphism group of the subtournament $T\langle N^+(v(0)) \rangle$ has to be odd and $Aut(T\langle N^+(v(0)) \rangle) \subseteq \mathbb{S}_2$, it follows that ϱ restricted to the subset of vertices $N^+(v(0))$ must be the identity. That is, $\varrho|_{N^+(v(0))} = id$. Similarly we prove that $\varrho|_{N^-(v(0))} = id$. Hence, $G_{v(0)} = id$ and $|G_{v(0)}| = 1$. Therefore, $|G| = 5$. Since $\langle (0\ 1\ 2\ 3\ 4) \rangle \subseteq G$, we deduce that $G \cong \langle (0\ 1\ 2\ 3\ 4) \rangle \cong \mathbb{Z}_5$. This finishes the proof of Case 2.

There are two equivalence classes in E_3 . The equivalence class [000] contains 6 elements and the equivalence class [010] contains 2 elements. Figure 1 shows non-isomorphic Walecki tournaments on 7 vertices. We prove $Aut(W(000)) \cong \mathbb{Z}_3$ in Case 6 as a part of more general argument. First we prove that $Aut(W(0, 1, 0)) \cong \mathbb{Z}_7 \otimes \mathbb{Z}_3$.

CASE 3. Let $e = (0, 1, 0) \in E_3$. Let G denote the automorphism group of $W(010)$. That is $G = Aut(W(010))$. Let $X = V(W(010))$. Using Figure 1, one can easily check that $\sigma_1 = (1\ 5\ 4)(2\ 3\ 6)$ is an automorphism of $W(010)$. Figure 4 shows another drawing of $W(010)$. Notice that pattern coded arcs do not correspond to directed Hamilton cycles from the construction of $W(010)$, but rather indicate that $W(010)$ is in fact isomorphic to a circulant tournament $C(\{1, 2, 4\})$ on 7 vertices. It is clear that $\sigma_2 = (0\ 1\ 4\ 2\ 5\ 3\ 6)$ is an automorphism of $W(010)$. That is, $\langle \sigma_1, \sigma_2 \rangle \subseteq G$. By double use of Orbit Stabilizer Theorem we have

$$|G| = |\mathcal{O}(v(0))| |G_{v(0)}| = |\mathcal{O}(v(0))| |\mathcal{O}(v(1))| |G_{v(0), v(1)}|, \quad (3.2)$$

where $\mathcal{O}(v(1))$ denotes the orbit of vertex $v(1)$ for the subgroup $G_{v(0)}$ of G . Since $\sigma_2 \in G$, $\mathcal{O}(v(0)) = \{v(0), v(1), \dots, v(6)\} = X$. Hence, $|\mathcal{O}(v(0))| = 7$.

Let us consider the orbit of vertex $v(1)$ for the subgroup $G_{v(0)}$ of G . Orbits

in X for $\langle \sigma_1 \rangle$ are $O_0 = \{v(0)\}$, $O_1 = \{v(1), v(4), v(5)\}$, and $O_2 = \{v(2), v(3), v(6)\}$. Since $N^+(v(0)) = O_1$, it follows that $\mathcal{O}(v(1)) = O_1$ and $|\mathcal{O}(v(1))| = 3$.

Last we prove that $G_{v(0),v(1)} = id$. Let $\varrho \in G_{v(0),v(1)}$. Since $\varrho(v(0)) = v(0)$, the outset and inset of $v(0)$ have to be fixed setwise (see [2] Proposition 4.14). That is, $\varrho(N^+(v(0))) = N^+(v(0))$ and $\varrho(N^-(v(0))) = N^-(v(0))$. Similarly, since $\varrho(v(1)) = v(1)$, the outset and inset of $v(1)$ have to be fixed setwise. Therefore, $N^+(v(0)) \cap N^+(v(1))$, $N^+(v(0)) \cap N^-(v(1))$, $N^-(v(0)) \cap N^-(v(1))$, and $N^-(v(0)) \cap N^+(v(1))$ have to be fixed setwise. It is easy to verify that $N^+(v(0)) \cap N^+(v(1)) = \{v(4)\}$, $N^+(v(0)) \cap N^-(v(1)) = \{v(5)\}$, and $N^-(v(0)) \cap N^-(v(1)) = \{v(6)\}$, which implies that the singleton sets have to be fixed pointwise. Since $N^-(v(0)) \cap N^+(v(1)) = \{v(2), v(3)\}$ the automorphism group of $T(\{v(2), v(3)\})$ is a subgroup of \mathbb{S}_2 . Proposition 2.3 furthermore implies that $Aut(T(\{v(2), v(3)\})) = id$. Therefore, $G_{v(0),v(1)} = id$. Figure 5 shows the partition of the vertices of T with respect to the outlets and insets of vertices $v(0)$ and $v(1)$, respectively.

Equation (3.2) together with $|\mathcal{O}(v(0))| = 7$, $|\mathcal{O}(v(1))| = 3$, and $|G_{v(0),v(1)}| = 1$, imply $|G| = 21$.

If G is cyclic then it would have to contain either a 21-cycle or a product of two disjoint cycles, a 7-cycle and a 3-cycle. In the first case Walecki tournament would have to have 21 vertices and in the later case it would have to have 10 vertices, a contradiction. Therefore, G is not cyclic. Now, $\langle \sigma_1 \rangle \cong \mathbb{Z}_3$ and $\langle \sigma_2 \rangle \cong \mathbb{Z}_7$. Moreover, $\langle \sigma_2 \rangle \subseteq G$ and index of $\langle \sigma_2 \rangle$ in G equals 3. Since 3 is smallest prime divisor of 21 Theorem 2.1 implies $\langle \sigma_2 \rangle \triangleleft G$.

If $\langle \sigma_1 \rangle \triangleleft G$ then σ_1 and σ_2 commute. Furthermore, σ_1 and σ_2 are disjoint

cycles implying that G is cyclic, a contradiction. Conditions of Theorem 2.2 hold for $\langle \sigma_2 \rangle$ and $\langle \sigma_1 \rangle$. It follows that $G \cong \langle \sigma_2 \rangle \otimes \langle \sigma_1 \rangle \cong \mathbb{Z}_7 \otimes \mathbb{Z}_3$. This finishes the proof of Case 3.

There are two equivalence classes in E_4 , $[0000]$ and $[0101]$. They both contain 8 elements. Figure 6 shows non-isomorphic Walecki tournaments on 9 vertices. We prove that $\text{Aut}(W(0000)) \cong \mathbb{Z}_4$ in CASE 6 as a part of a general proof for zero pattern when n is even. We proceed by proving that $\text{Aut}(W(0101)) \cong \mathbb{Z}_3$.

CASE 4. $e = (0, 1, 0, 1) \in E_4$. Let T denote the Walecki tournament $W(0101)$ and let $X = V(T)$. Let G denote the automorphism group of T . One can easily check that $\sigma_1 = (1\ 3\ 7)(2\ 6\ 8)$ is an automorphism of $W(0101)$. Thus, $\langle \sigma_1 \rangle \subseteq G$. By double use of Orbit Stabilizer Theorem we have

$$|G| = |\mathcal{O}(v(0))| |G_{v(0)}| = |\mathcal{O}(v(0))| |\mathcal{O}(v(1))| |G_{v(0),v(1)}|, \quad (3.3)$$

where $\mathcal{O}(v(1))$ denotes the orbit of vertex $v(1)$ for the subgroup $G_{v(0)}$ of G .

We first consider the cardinality of $\mathcal{O}(v(0))$. The outset of vertex $v(0)$ is $N^+(v(0)) = \{v(4), v(1), v(3), v(7)\}$ and the score sequence of the subtournament $T\langle N^+(v(0)) \rangle$ is

$$(s(v(4)), s(v(1)), s(v(3)), s(v(7))) = (0, 2, 2, 2).$$

However, the score sequence of the subtournament $T\langle N^+(v(1)) \rangle$ is

$$(s(v(2)), s(v(5)), s(v(4)), s(v(7))) = (1, 1, 2, 2).$$

Therefore, $v(0)$ cannot be mapped to vertex $v(1)$ by an element of G since $s(v(4)) = 0$ in $T\langle N^+(v(0)) \rangle$ and there is no vertex in $T\langle N^+(v(1)) \rangle$ with score 0. Because $O_1 = \{v(1), v(3), v(7)\}$ is an orbit of $v(1)$ for $\langle \sigma_1 \rangle$, it follows that $v(0)$ can

be mapped to neither vertex $v(3)$ nor vertex $v(7)$. Similarly, the score sequence of subtournament $T\langle N^+(v(4)) \rangle$ is $(s(v(2)), s(v(6)), s(v(8)), s(v(5))) = (1, 1, 1, 3)$. Therefore, $v(0)$ cannot be mapped to vertex $v(4)$ by an element of G since there is no vertex in $T\langle N^+(v(4)) \rangle$ with score 0. We have proven that $v(0)$ cannot be mapped to a vertex from $N^+(v(0))$ by an element of G . Proposition 2.2 from [2] implies $T \cong \overline{T}$ with the graph anti-automorphism τ^4 . Therefore, $v(0)$ cannot be mapped to a vertex from $N^-(v(0))$ by elements of G . We have proven that $v(0)$ must be fixed under the action of G . Thus, $|\mathcal{O}(v(0))| = 1$.

Next we consider the cardinality of $\mathcal{O}(v(1))$. For any ϱ in $G_{v(0)}$, $\varrho(N^+(v(0))) = N^+(v(0))$. Hence, $\varrho(v(1)) \in N^+(v(0))$ and $|\mathcal{O}(v(1))| \leq |N^+(v(0))| = 4$. Clearly, $3 = |\mathcal{O}_1| \leq |\mathcal{O}(v(1))|$ implying that $3 \leq |\mathcal{O}(v(1))| \leq 4$. All elements of G have to have odd order. Thus, $|\mathcal{O}(v(1))| \neq 4$. It follows that $|\mathcal{O}(v(1))| = 3$.

Last we prove that $G_{v(0),v(1)} = id$. Let $\varrho \in G_{v(0),v(1)}$. Since $\varrho(v(0)) = v(0)$, the outset and inset of $v(0)$ have to be fixed setwise. Similarly, since $\varrho(v(1)) = v(1)$, the outset and inset of $v(1)$ have to be fixed setwise. That is, $\varrho(N^+(v(1))) = N^+(v(1))$ and $\varrho(N^-(v(1))) = N^-(v(1))$. Therefore, the sets $N^+(v(0)) \cap N^+(v(1))$, $N^+(v(0)) \cap N^-(v(1))$, $N^-(v(0)) \cap N^-(v(1))$, and $N^-(v(0)) \cap N^+(v(1))$ have to be fixed setwise. It is easy to verify that $N^+(v(0)) \cap N^-(v(1)) = \{v(3)\}$, which implies that $v(3)$ has to be fixed by ϱ . Also, $N^+(v(0)) \cap N^+(v(1)) = \{v(4), v(7)\}$, $N^-(v(0)) \cap N^+(v(1)) = \{v(2), v(5)\}$, and $N^-(v(0)) \cap N^-(v(1)) = \{v(6), v(8)\}$ imply that automorphism groups $Aut(T\langle \{v(4), v(7)\} \rangle)$, $Aut(T\langle \{v(2), v(5)\} \rangle)$, and $Aut(T\langle \{v(6), v(8)\} \rangle)$ are subgroups of \mathbb{S}_2 . Proposition 2.3 furthermore implies that all three groups are trivial. Therefore, ϱ must fix X pointwise. Therefore, $G_{v(0),v(1)} = id$. Figure 7 shows the partition of the vertices of T with respect

to the outsets and insets of vertices $v_{(0)}$ and $v_{(1)}$.

Equation (3.3) together with $|\mathcal{O}(v_{(0)})| = 1$, $|\mathcal{O}(v_{(1)})| = 3$, and $|G_{v_{(0)},v_{(1)}}| = 1$, imply $|G| = 3$. From $\langle \sigma_1 \rangle \subseteq G$, and $\langle \sigma_1 \rangle \cong \mathbb{Z}_3$ it follows that $G \cong \mathbb{Z}_3$. This completes the proof of Case 4.

We proceed with general cases which extend ideas of the proofs for small values of n .

CASE 5. Let us assume that n is odd, $n \geq 3$, and $e = (0, 0, \dots, 0)$. Let T denote the Walecki tournament $W(e)$ and let G denote its automorphism group $\text{Aut}(T)$. As in the previous case we use Orbit Stabilizer Theorem two times to get

$$|G| = |\mathcal{O}(v_{(0)})| |G_{v_{(0)}}| = |\mathcal{O}(v_{(0)})| |\mathcal{O}(v_{(1)})| |G_{v_{(0)},v_{(1)}}|, \quad (3.4)$$

where $\mathcal{O}(v_{(1)})$ denotes the orbit of vertex $v_{(1)}$ for the subgroup $G_{v_{(0)}}$ of G .

We first consider the cardinality of $\mathcal{O}(v_{(0)})$. $T\langle N^+(v_{(0)}) \rangle$ is a regular tournament (see [2] Proposition 5.23). Therefore, it is not transitive. On the other hand $T\langle N^+(v_{(i)}) \rangle$ is transitive for $v_{(i)} \in N^+(v_{(0)})$ (see [2] Theorem 5.21). Thus, $v_{(0)}$ cannot be mapped to a vertex from $N^+(v_{(0)})$ by elements of G . Proposition 2.2 from [2] implies $T \cong \overline{T}$ with the graph anti-automorphism τ^n . Therefore, $v_{(0)}$ cannot be mapped to a vertex from $N^-(v_{(0)})$ by elements of G . We have proven that $v_{(0)}$ must be fixed under the action of G , and thus

$$|\mathcal{O}(v_{(0)})| = 1. \quad (3.5)$$

The fact that $v_{(0)}$ cannot be mapped to any vertex in $N^-(v_{(0)})$ can also be proven directly for $n \geq 5$. Let us consider $T\langle N^+(v_{(0)}) - v_{(i)} \rangle$ for $v_{(i)} \in N^+(v_{(0)})$. $T\langle N^+(v_{(0)}) \rangle$ is regular of degree $(n-1)/2$. Thus, $T\langle N^+(v_{(0)}) - v_{(i)} \rangle$ has $(n-1)/2$

vertices of degree $(n-1)/2$ and $(n-1)/2$ vertices of degree $(n-3)/2$. Therefore, if $n \geq 5$, the subtournament $T\langle N^+(v_{(0)}) - \{v_{(i)}\} \rangle$ is not transitive. However, $T\langle N^+(v_{(j)}) - \{v_{(0)}\} \rangle$ is transitive for $v_{(j)} \in N^-(v_{(0)})$. Therefore, $v_{(0)}$ cannot be mapped to $v_{(j)} \in N^-(v_{(0)})$ by elements of G if $n \geq 5$. We cannot use the same argument for $n = 3$ since the subtournament $T\langle N^+(v_{(0)}) - v_{(i)} \rangle$, for $v_{(i)} \in N^+(v_{(0)})$, is a tournament on two vertices and is therefore transitive.

Next we determine $|\mathcal{O}(v_{(1)})|$. Since $v_{(0)}$ is a fixed point for any element ϱ in G , $\varrho(N^+(v_{(0)})) = N^+(v_{(0)})$. Hence, $\varrho(v_{(1)}) \in N^+(v_{(0)})$ and

$$|\mathcal{O}(v_{(1)})| \leq |N^+(v_{(0)})| = n. \quad (3.6)$$

We proved that the permutation $\sigma \in \mathbb{S}_{2n+1}$ defined by $\sigma = (1 \ 2 \ 4 \cdots 2n - 4 \ 2n - 2)(3 \ 5 \cdots 2n - 3 \ 2n - 1 \ 2n)$, is an element in G (see [2] Theorem 5.20). Since $\sigma(v_{(0)}) = v_{(0)}$, $\sigma \in G_{v_{(0)}}$. Hence, $\langle \sigma \rangle \subseteq G_{v_{(0)}}$. The orbit of $v_{(1)}$ for σ is $N^+(v_{(0)})$ which implies

$$|\mathcal{O}(v_{(1)})| \geq n. \quad (3.7)$$

Equation (3.6) and Equation (3.7) imply

$$|\mathcal{O}(v_{(1)})| = n. \quad (3.8)$$

Last we prove that $G_{v_{(0)},v_{(1)}} = id$. The subtournaments $T\langle N^+(v_{(1)}) \rangle$ and $T\langle N^-(v_{(1)}) - \{v_{(0)}\} \rangle$ are transitive, implying that any automorphism $\varrho \in G_{v_{(0)},v_{(1)}}$ fixes all other vertices. Figure 8 shows the partition of the vertices of T with respect to the outsets and insets of vertices $v_{(0)}$ and $v_{(1)}$. Therefore, $G_{v_{(0)},v_{(1)}} = id$, that is,

$$|G_{v_{(0)},v_{(1)}}| = 1. \quad (3.9)$$

Equations (3.4), (3.5), (3.8), and (3.9) imply that $|G| = n$. Now, $\langle \sigma \rangle \subseteq G_{v(0)} \subseteq G$ and since $\langle \sigma \rangle \cong \mathbb{Z}_n$ we have $G \cong \mathbb{Z}_n$.

CASE 6. Let us assume n is even, $n \geq 4$, and $e = (0, 0, \dots, 0) \in E_n$. Let T denote the Walecki tournament $W(e)$ and let G denote its automorphism group $\text{Aut}(T)$. As in the previous case we use Orbit Stabilizer Theorem two times to get

$$|G| = |\mathcal{O}(v(0))| |G_{v(0)}| = |\mathcal{O}(v(0))| |\mathcal{O}(v(1))| |G_{v(0),v(1)}|. \quad (3.10)$$

We first consider the cardinality of $\mathcal{O}(v(0))$. The subtournament $T\langle N^+(v(0)) \rangle$ is almost regular (see [2] Theorem 5.25). Therefore, it is not transitive. On the other hand, $T\langle N^+(v(i)) \rangle$ is transitive for $v(i) \in N^+(v(0))$ (see [2] Theorem 5.26). Thus, $v(0)$ cannot be mapped to a vertex from $N^+(v(0))$ by elements of G . Proposition 2.2 from [2] implies $T \cong \overline{T}$ via the graph anti-automorphism τ^n . Therefore, $v(0)$ cannot be mapped to a vertex from $N^-(v(0))$ by elements of G . We have proven that $v(0)$ must be fixed under the action of G , and thus

$$|\mathcal{O}(v(0))| = 1. \quad (3.11)$$

Next we determine $|\mathcal{O}(v(1))|$. Since $v(0)$ is a fixed point for any element ϱ in G , $\varrho(N^+(v(0))) = N^+(v(0))$. Hence, $\varrho(v(1)) \in N^+(v(0))$. As seen in the proof of Theorem 5.25 from [2] $|N^+(v(0)) \cap N^+(v(1))| = n/2$ and $|N^+(v(0)) \cap N^-(v(1))| = n/2 - 1$. Furthermore, $T\langle N^+(v(1)) \rangle$ and $T\langle N^-(v(1)) \rangle$ are both transitive which implies that $T\langle N^+(v(0)) \cap N^+(v(1)) \rangle$ and $T\langle N^+(v(0)) \cap N^-(v(1)) \rangle$ are transitive. Moreover, $v(1)$ is dominated by $N^+(v(0)) \cap N^-(v(1))$ implying that $T\langle (N^+(v(0)) \cap N^-(v(1))) \cup \{v(1)\} \rangle$ is transitive. Let $X = N^+(v(0)) \cap N^+(v(1))$ and $Y = (N^+(v(0)) \cap N^-(v(1))) \cup \{v(1)\}$. Now, vertices of X have score $n/2 - 1$ in

$T\langle N^+(v(0)) \rangle$. Similarly, vertices of Y have score $n/2$ in $T\langle N^+(v(0)) \rangle$. Therefore, X and Y have to be fixed setwise. Hence,

$$|\mathcal{O}(v(1))| = 1. \quad (3.12)$$

Last we prove that $G_{v(0),v(1)} = id$. Subtournaments $T\langle N^+(v(1)) \rangle$ and $T\langle N^-(v(1)) - \{v(0)\} \rangle$ are transitive and thus any automorphism fixing both $v(0)$ and $v(1)$ fixes all other vertices. Therefore, $G_{v(0),v(1)} = id$ which implies

$$|G_{v(0),v(1)}| = 1. \quad (3.13)$$

Equations (3.10), (3.11), (3.12), and (3.13) imply that $|G| = 1$ and $G \cong \mathbb{Z}_1$.

This completes the proof. \square

The following result is an immediate consequence of Theorem 3.4.

Corollary 3.5 *Walecki tournaments $W(0)$, $W(0,0)$, and $W(0,1,0)$ are vertex-transitive.*

In the subsequent paper we discuss the arc structure of Walecki tournaments with odd and even patterns (see Aleš [3]). Theoretical research on Walecki tournaments with zero pattern was very much computer directed. The results of Theorem 3.4 were confirmed for n up to 270.

3.1 Computation results

Automorphism groups for all initial cases and zero pattern Walecki tournaments on up to 275 vertices were computed with algorithm NAUTY (No AUTomorphisms, Yes ?). Dr. Brendan McKay has generously opened the leading graph isomorphism program NAUTY to the academic community. It proved to be an

indispensible tool in this computer directed proof (see McKay [10, 11]). The results for all non-trivial automorphism groups of Walecki tournaments for n up to 15 are shown in Table 4.

4 Research problems

In closing remarks we discuss the direction of further research. An obvious goal is to determine the arc structure of Walecki tournaments with a general defining binary sequence. We do so in two subsequent papers (see Aleš [3, 4]).

Secondly, we would like to characterize the automorphism groups for the Walecki tournaments. Binary sequences from the same equivalence class for the complementing circular shift register determine isomorphic Walecki tournaments. Let us discuss the converse. If two Walecki tournaments $W(e)$ and $W(e')$ are isomorphic, then their automorphism groups are isomorphic. We first consider Walecki tournaments for small values of n . There exist only two Walecki tournaments for $n = 1$. The corresponding sequences belong to the same equivalence class since there is only one equivalence class in E_1 . Similarly, for $n = 2$ we have only one equivalence class in E_2 . For $n = 3$ we have two equivalence classes but automorphism groups of the corresponding two Walecki tournaments, one for each equivalence class, are not isomorphic. Similarly, for $n = 4$ there exist two equivalence classes with non-isomorphic automorphism groups of the corresponding two Walecki tournaments. Therefore, the mapping between cycles of the complementing circular shift register and Walecki tournaments is injective for small values of n , $1 \leq n \leq 4$. In order to prove this in general we need a complete characterization of automorphism groups of Walecki

tournaments. This would aid us in showing that vertex $v(0)$ must be fixed for an isomorphism between any two Walecki tournaments, except for the vertex transitive cases stated in Corollary 3.5.

Conjecture 4.6 *Let n be a positive integer and let e and e' be binary sequences of length n . If the Walecki tournaments $W(e)$ and $W(e')$ are isomorphic, then sequences e and e' belong to the same equivalence class for the complementing circular shift register R .*

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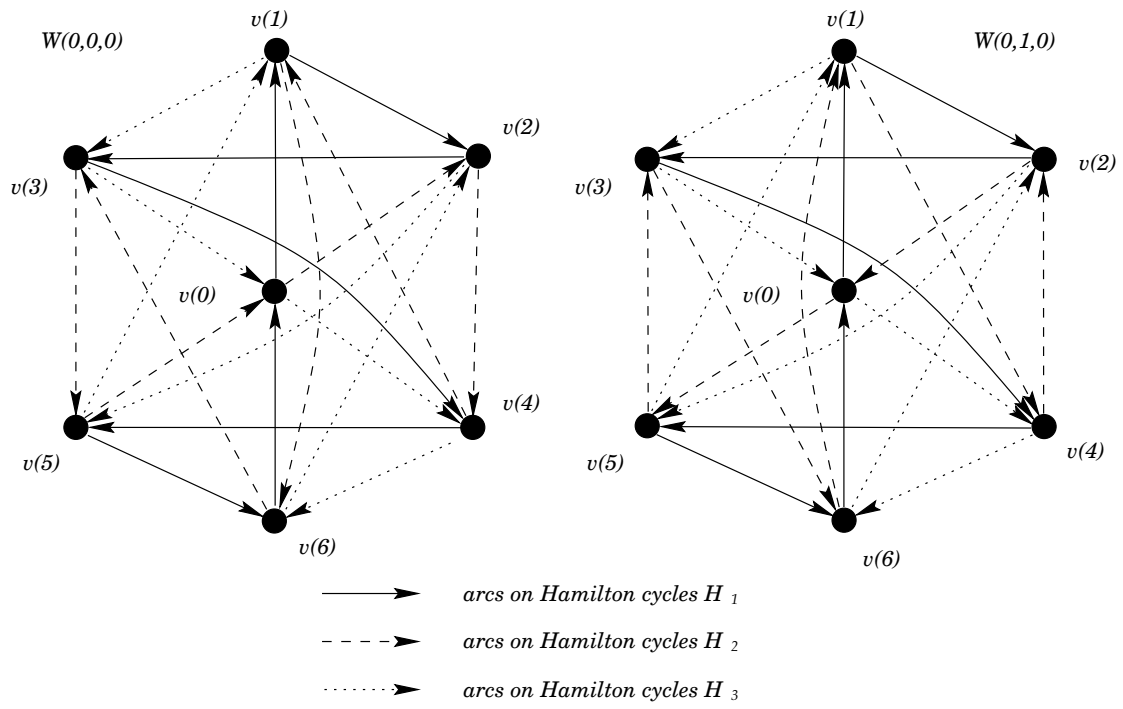


Figure 1: Walecki tournaments $W(000)$ and $W(010)$ on 7 vertices.

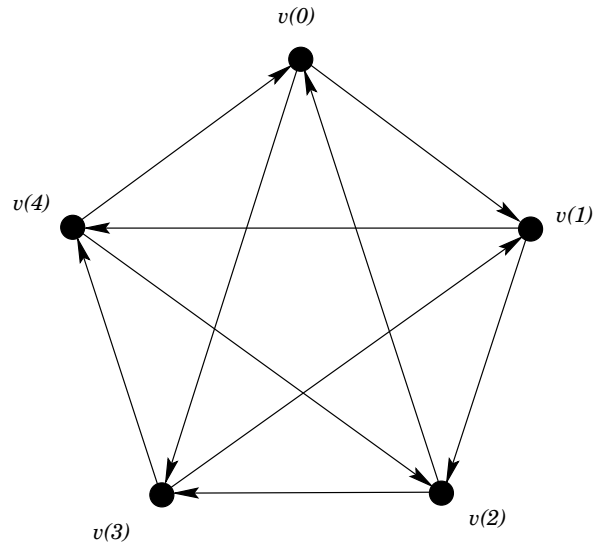


Figure 2: A circulant tournament T_5 on 5 vertices.

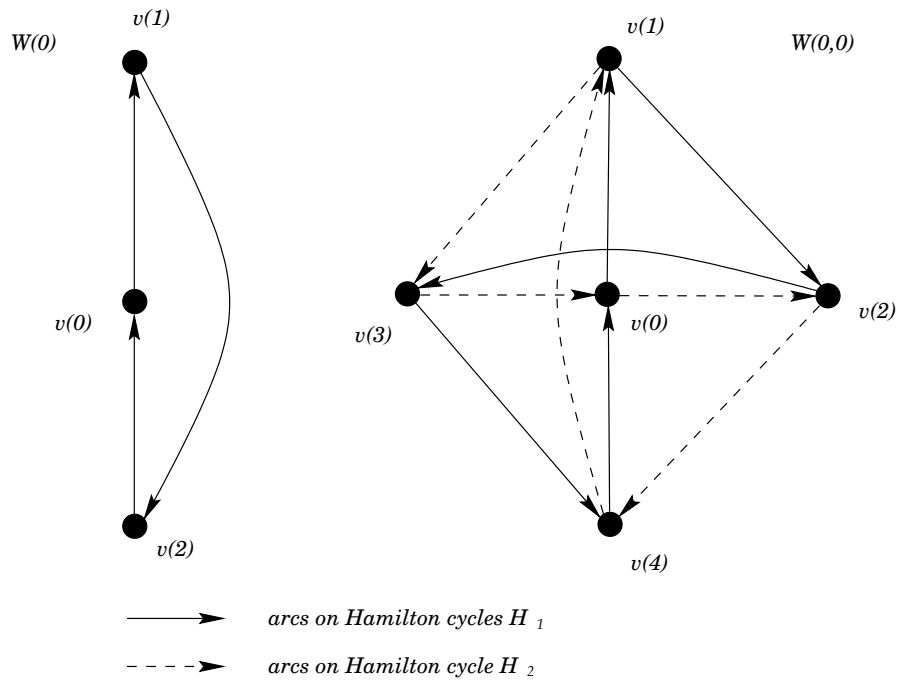


Figure 3: Walecki tournaments $W(0)$ and $W(0,0)$.

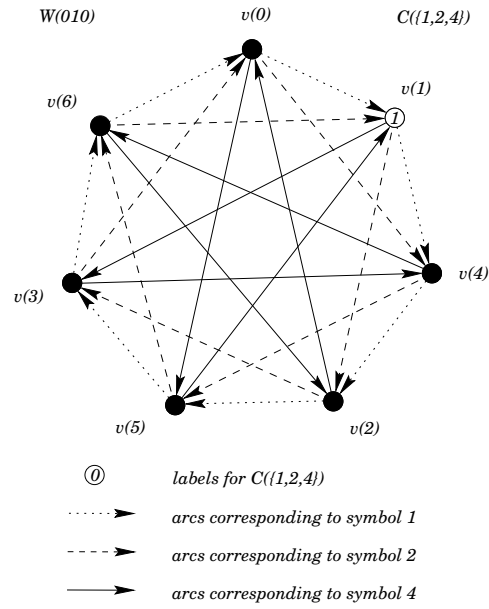


Figure 4: Walecki tournament $W(010)$ which is isomorphic to a circulant tournament $C(\{1, 2, 4\})$.

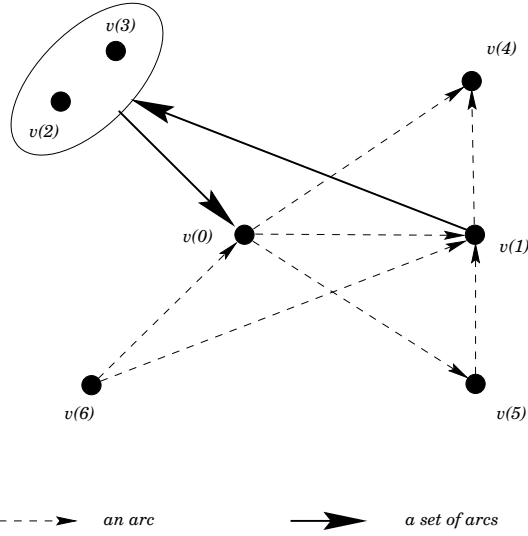


Figure 5: The partition of the vertices of the Walecki tournament $W(010)$ with respect to the outsets and insets of vertices $v(0)$ and $v(1)$. Only the arcs essential for the proof of CASE 3 of Theorem 3.4 are drawn.

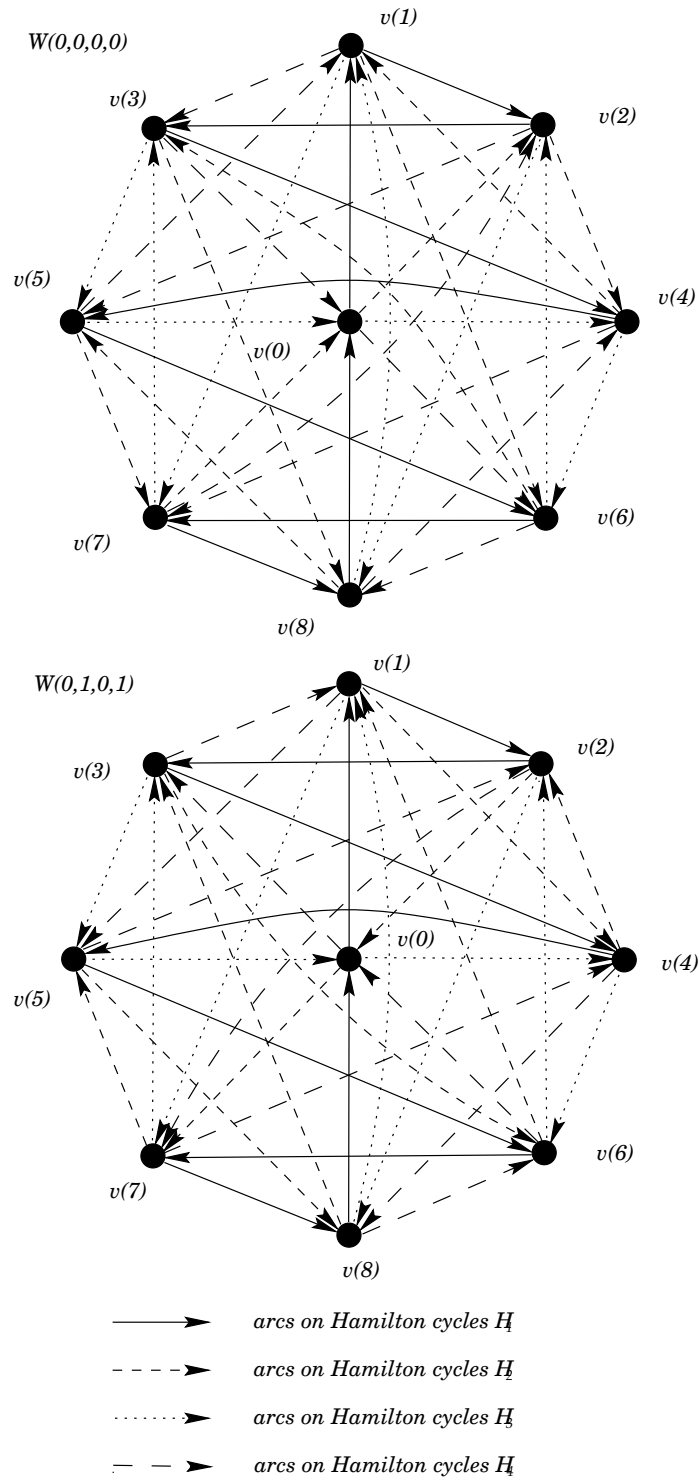


Figure 6: Walecki tournaments $W_{22}(0000)$ and $W(0101)$ on 9 vertices.

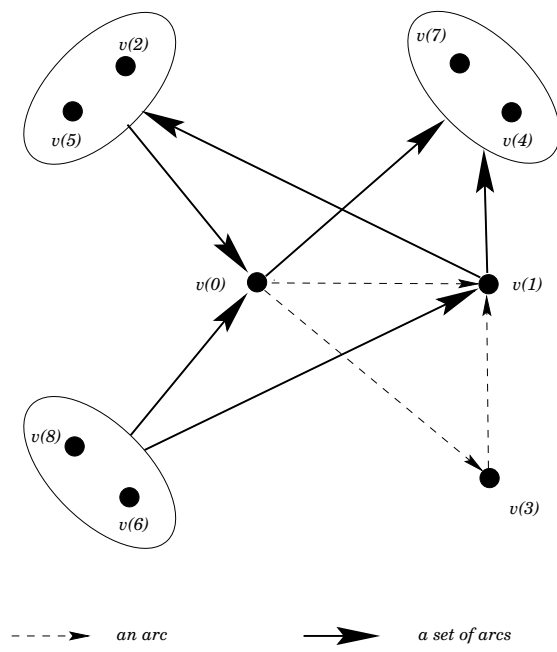


Figure 7: The partition of the vertices of the Walecki tournament $W(0101)$, with respect to the outsets and insets of vertices $v(0)$ and $v(1)$. Only the arcs essential for the proof of CASE 4 of Theorem 3.4 are drawn.

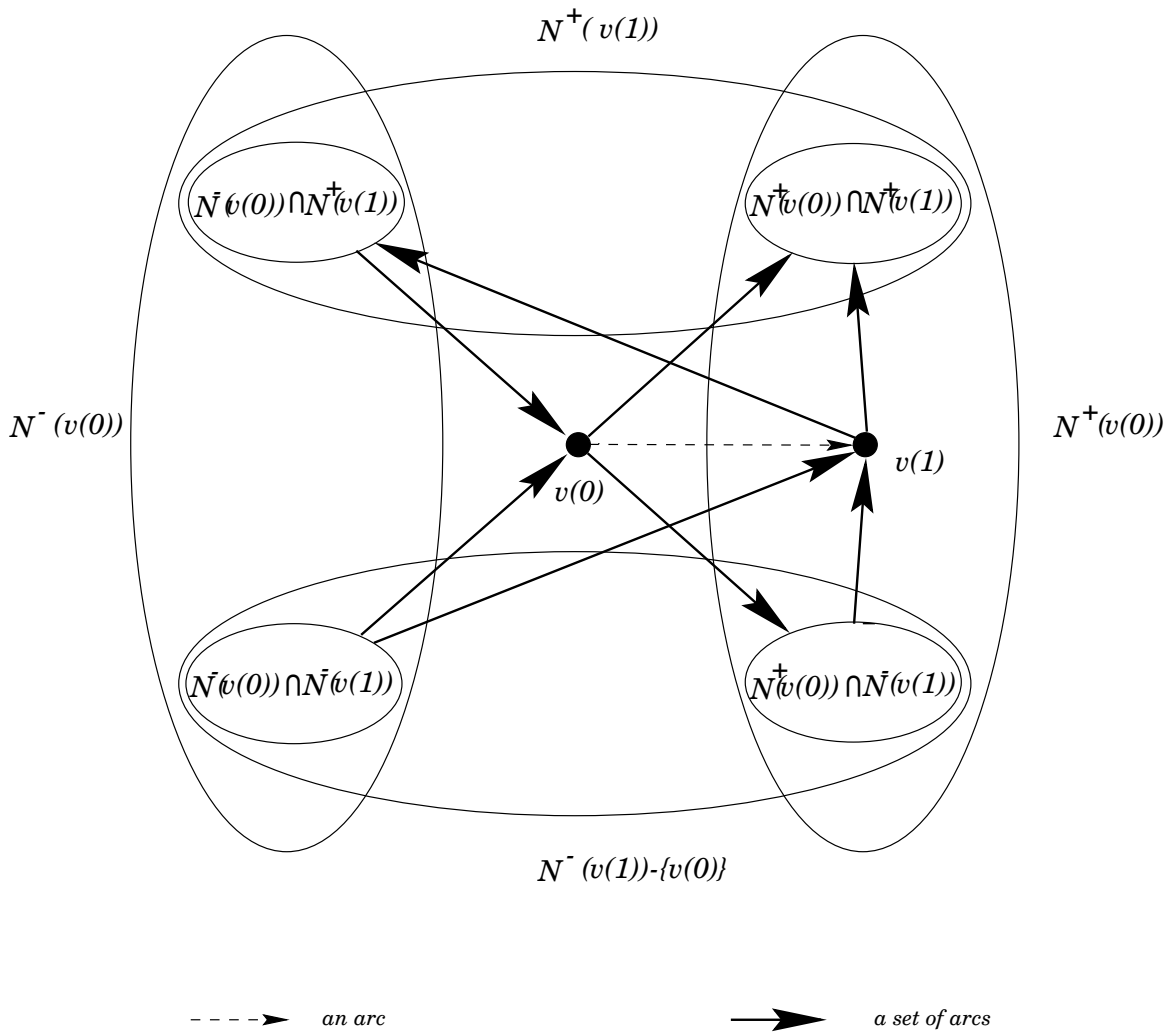


Figure 8: The partition of the vertices of the Walecki tournament $W(e)$, for $e = (0, 0, \dots, 0) \in E_n$, n odd, and $n \geq 3$, with respect to the outsets and insets of vertices $v(0)$ and $v(1)$. Only the arcs essential for the proof of CASE 5 of Theorem 3.4 are drawn.

n	$e \in E_n$	$ e]_R $	$Aut(W(e))$
1	0	2	\mathbb{Z}_3
2	00	4	\mathbb{Z}_5
3	000	6	\mathbb{Z}_3
	0 <u>1</u> 0	2	$\mathbb{Z}_7 \otimes \mathbb{Z}_3$
4	0 <u>1</u> 0 <u>1</u>	8	\mathbb{Z}_3
5	00000	10	\mathbb{Z}_5
6	0 <u>1</u> 0 <u>1</u> 0 <u>1</u>	12	\mathbb{Z}_3
	00 <u>1</u> <u>1</u> 00	4	\mathbb{Z}_3
7	0000000	14	\mathbb{Z}_7
	0 <u>1</u> 0 <u>1</u> 0 <u>1</u> 0	2	\mathbb{Z}_7
9	000000000	18	\mathbb{Z}_9
	000 <u>1</u> <u>1</u> 000	6	\mathbb{Z}_3
	0 <u>1</u> 0 <u>1</u> 0 <u>1</u> 0 <u>1</u> 0	2	\mathbb{Z}_9
10	0 <u>1</u> 0 <u>1</u> 0 <u>1</u> 0 <u>1</u> 0 <u>1</u>	20	\mathbb{Z}_5
	00 <u>1</u> <u>1</u> 00 <u>1</u> <u>1</u> 00	4	\mathbb{Z}_5
11	00000000000	22	\mathbb{Z}_{11}
	0 <u>1</u> 0 <u>1</u> 0 <u>1</u> 0 <u>1</u> 0 <u>1</u> 0	2	\mathbb{Z}_{11}
12	00 <u>1</u> <u>1</u> 00 <u>1</u> <u>1</u> 00 <u>1</u> <u>1</u>	24	\mathbb{Z}_3
	0000 <u>1</u> <u>1</u> <u>1</u> <u>1</u> 0000	8	\mathbb{Z}_3
	0 <u>1</u> <u>1</u> 00 <u>1</u> <u>1</u> 00 <u>1</u> <u>1</u> 0	24	\mathbb{Z}_3
	010 <u>1</u> <u>1</u> 0 <u>1</u> 00 <u>1</u> 0 <u>1</u>	8	\mathbb{Z}_3
13	0000000000000	26	\mathbb{Z}_{13}
	0 <u>1</u> 0 <u>1</u> 0 <u>1</u> 0 <u>1</u> 0 <u>1</u> 0 <u>1</u> 0	2	\mathbb{Z}_{13}
14	0 <u>1</u> 0 <u>1</u> 0 <u>1</u> 0 <u>1</u> 0 <u>1</u> 0 <u>1</u> 0 <u>1</u>	28	\mathbb{Z}_7
	00 <u>1</u> <u>1</u> 00 <u>1</u> <u>1</u> 00 <u>1</u> <u>1</u> 00	4	\mathbb{Z}_7
15	000000000000000	30	\mathbb{Z}_{15}
	00000 <u>1</u> <u>1</u> <u>1</u> <u>1</u> <u>1</u> 00000	10	\mathbb{Z}_3
	000 <u>1</u> 0 <u>1</u> <u>1</u> <u>1</u> 0 <u>1</u> 000 <u>1</u> 0	10	\mathbb{Z}_3
	000 <u>1</u> <u>1</u> <u>1</u> 000 <u>1</u> <u>1</u> <u>1</u> 000	6	\mathbb{Z}_5
	00 <u>1</u> 00 <u>1</u> <u>1</u> 0 <u>1</u> <u>1</u> 00 <u>1</u> 00	10	\mathbb{Z}_3
	0 <u>1</u> 0 <u>1</u> 0 <u>1</u> 0 <u>1</u> 0 <u>1</u> 0 <u>1</u> 0	2	\mathbb{Z}_{15}

Table 1: Non-trivial automorphism groups of Walecki tournaments $W(e)$ on $2n + 1$ vertices, for $1 \leq n \leq 15$ and $e \in E_n$.