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WALECKI TOURNAMENTS: PART III

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Abstract.

Walecki tournaments were defined by Alspach in 1966. They speak in favor of Kelly's conjecture from 1964. Namely, they are regular tournaments which admit Hamilton directed cycle decomposition. The enumeration of Walecki tournaments was presented as an open problem in a paper by Alspach in 1989. These two problems led us to study the arc structure of Walecki tournaments. They possess a broad range of subtournaments isomorphic to some Walecki tournament. A specific permutation is proven to be an automorphism of a Walecki tournament with a given pattern. Subtournaments induced by the outsets of vertex $t_{(0)}$ are proven to be either regular or almost regular.

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1 Introduction

Walecki tournaments were defined by Alspach in 1966 (see [6]). They speak in favor of Kelly's conjecture from 1964 (see Moon [8]). Namely, they are regular tournaments which admit Hamilton directed cycle decomposition. The enumeration of Walecki tournaments was presented as an open problem in a paper by Alspach in 1989 (see [7]). These two problems led us to study the rich arc structure of Walecki tournaments. They possess a broad range of subtournaments isomorphic to some Walecki tournament. Also, subtournaments induced by the outsets of vertex $t(0)$ are proven to be either regular or almost regular. A specific permutation is proven to be an automorphism of a Walecki tournament with a given pattern.

For relevant topics on tournaments, and Walecki tournaments in particular, we refer the reader to two preceding papers (see Aleš [2, 3]). Additional theoretical background on tournaments can be found in a book by Moon [8].

2 Periodic patterns

In this paper we examine the arc structure of Walecki tournaments with the defining binary sequence having a specific pattern. We say that an equivalence class $[e]_R$ has *zero pattern* if the lexicographically smallest sequence of $[e]_R$ is $(0, 0, \ldots, 0)$ (see Aleš [2], Section 5). Furthermore, a sequence $f\overline{f}\ldots\overline{f}f \in E_n$, for $f \in E_r$ and $n/r > 1$ has *odd pattern*. Notice, that n/r is odd. If there exists an element in the equivalence class $[e]_R$ of odd pattern, then all elements in $[e]_R$ have odd pattern. We say that such an equivalence class has odd pattern.

Furthermore, a sequence $f\overline{f}\ldots f\overline{f}$, for $f \in E_r$ with $n/2r$ odd, and $n/2r > 1$

has *even pattern*. Let e be such a sequence. Not all sequences of the equivalence class $[e]_R$ have even pattern. For example, sequences $(0,0,1,0)$ and $(0,1,0,1)$ both belong to the same equivalence class. However, only the latter has even pattern. We say that an equivalence class $[e]_R$ has even pattern if there exists a sequence in $[e]_R$ with even pattern.

To simplify terminology we will refer to an equivalence class with a given pattern as a "sequence" with that pattern. We call a sequence *periodic* if it has either zero, odd, or even pattern. All other sequences are called *aperiodic*. We will furthermore simplify terminology by referring to a Walecki tournament whose corresponding binary sequence has odd pattern, for example, as a Walecki tournament with odd pattern.

Proposition 2.1 *For a positive integer n, there are exactly $\Gamma(n) - \Sigma(n)$ odd pattern equivalence classes in E_n.*

PROOF. Lemma 3.7 from [2] implies that an equivalence class for the complementing circular shift register contains at most $2n$ binary sequences of length n. Lemma 3.8 and Lemma 3.9 from [2] imply that all but odd pattern equivalence classes have $2n$ sequences. There are exactly $S(n)$ equivalence classes with $2n$ sequences and $G(n)$ of all equivalence classes as proven in Theorem 3.10. Therefore, there are $G(n)-S(n)$ equivalence classes with less than $2n$ sequences. Now, Theorem 3.11 completes the proof. □

3 Odd pattern

The following result relates the structure of Walecki tournament $W(e)$ and Walecki tournament $W(f)$, where e has either odd pattern $e = f\overline{f}\dots\overline{f}f \in E_n$ or even pattern $e = f\overline{f}\dots f\overline{f} \in E_n$, and sequence $f \in E_r$ is either a zero pattern or aperiodic. Let T denote the Walecki tournament $W(e)$, $e \in E_n$, and let n be divisible by r. Let $m = 2r$. We introduce a partition of $V(W(e)) - \{t_{(0)}\}$ into m-sets $M_1, M_2, \dots, M_{n/r}$, where $M_i = \{t_{((i-1)m+j)} \mid \text{ for } 1 \le j \le m\}$, for $1 \le i \le n/r$.

Theorem 3.2 *Let $n \ge 1$, $f \in E_r$, and let r divide n. If $e = f\overline{f}\dots\overline{f}f \in E_n$ or $e = f\overline{f}\dots f\overline{f} \in E_n$, then $W(e)\langle\{t_{(0)}\} \cup M_1\rangle \cong W(f)$.*

PROOF. Let $1 \le k \le r$. In the following τ is first considered as a permutation in $\mathbb{S}_{2n+1}$ used in the definition of $W(e)$. Secondly, it is considered as a permutation in $\mathbb{S}_{2r+1}$ used in the definition of $W(f)$, and is denoted by $\overline{\tau}$. Similarly, $t_{(i)}$ denotes a vertex of $W(e)$ and $\overline{t}_{(i)}$ denotes a vertex of $W(f)$.

We define a function $\psi : \{t_{(0)}\} \cup M_1 \longrightarrow V(W(f))$ by $\psi(t_{(0)}) = \overline{t}_{(0)}$ and $\psi(t_{(i)}) = \overline{t}_{(i)}$, for $1 \le i \le 2r$. Clearly, ψ is a bijection. We will show that the Hamilton directed cycle $\overrightarrow{H}'_k$ in $W(f)$ is a union of ψ-images of directed paths belonging to Hamilton directed cycles $\overrightarrow{H}_k$ and $\overrightarrow{H}_{r+k}$ in $W(e)$. Throughout the proof we use Proposition 3.12 from [2] extensively.

Let $\overrightarrow{P}_k$ denote the directed path $[t_{(0)}, t_{(k)}, \dots, t_{(2k)}]$ on $\overrightarrow{H}_k$ and let $\overrightarrow{P}_{r+k}$ denote the directed path $[t_{(0)}, t_{(r+k)}, \dots, t_{(2k)}]$ on $\overrightarrow{H}_{r+k}$. A ψ image of a directed path $\overrightarrow{P}$ is a directed path comprised of the ψ images of vertices of $\overrightarrow{P}$. The ψ

images of $\vec{P}_k$ and $\vec{P}_{r+k}$ are

$$\begin{aligned}
\psi(\vec{P}_k) &= \psi([v(0), v(\tau^{k-1}(1)), v(\tau^{k-1}(2)), v(\tau^{k-1}(3)), v(\tau^{k-1}(4)), \dots \\
&\qquad \dots, v(\tau^{k-1}(2k-3)), v(\tau^{k-1}(2k-2)), v(\tau^{k-1}(2k-1)), v(\tau^{k-1}(2k))]) \quad = \\
&= \psi([t(0), t(k), t(k+1), t(k-1), t(k+2), \dots \\
&\qquad \dots, t(2), t(2k-1), t(1), t(2k)]) \quad = \\
&= [v(\overline{\tau}(0)), v(\overline{\tau}^{k-1}(1)), v(\overline{\tau}^{k-1}(2)), v(\overline{\tau}^{k-1}(3)), v(\overline{\tau}^{k-1}(4)), \dots \\
&\qquad \dots, v(\overline{\tau}^{k-1}(2k-3)), v(\overline{\tau}^{k-1}(2k-2)), v(\overline{\tau}^{k-1}(2k-1)), v(\overline{\tau}^{k-1}(2k))] \quad = \quad \vec{P}'_k
\end{aligned}$$

and

$$\begin{aligned}
\psi(\vec{P}_{r+k}) &= \psi([v(0), v(\tau^{r+k-1}(1)), v(\tau^{r+k-1}(2)), v(\tau^{r+k-1}(3)), \dots \\
&\qquad \dots, v(\tau^{r+k-1}(2r-2k)), v(\tau^{r+k-1}(2r-2k+1))]) \quad = \\
&= \psi([t(0), t(r+k), t(r+k+1), t(r+k-1), \dots \\
&\qquad \dots, t(2r), t(2k)]) \quad = \\
&= [v(\overline{\tau}(0)), v(\overline{\tau}^{k-1}(2r)), v(\overline{\tau}^{k-1}(2r-1)), v(\overline{\tau}^{k-1}(2r-2)), \dots \\
&\qquad \dots, v(\overline{\tau}^{k-1}(2k+1)), v(\overline{\tau}^{k-1}(2k))] \quad = \quad \vec{P}'_{r+k} \, .
\end{aligned}$$

The definition of Walecki tournaments implies that ψ is dominance preserving on paths $\vec{P}_k$ and $\vec{P}_{r+k}$. The pattern of $e = f\overline{f}\dots\overline{f}f$ implies that if $e_k = 0$, then $e_{r+k} = 1$ and $\vec{H}'_k = \vec{P}'_k \cup \overleftarrow{P}'_{r+k}$, where $\overleftarrow{P}'_{r+k}$ denotes the path $\vec{P}'_{r+k}$ with all of its arcs reversed (see Figure 1). If $e_k = 1$, then $e_{r+k} = 0$ and $\vec{H}'_k = \vec{P}'_{r+k} \cup \overleftarrow{P}'_k$. Therefore, $W(e)\langle\{t(0)\} \cup M_1\rangle \cong W(f)$. $\square$

Figure 2 shows Walecki tournament $W(000111000)$ with $W(000111000)$ $\langle\{t(0) \cup M_1\}\rangle \cong W(000)$.

The main topic of this paper is the arc structure of Walecki tournaments with odd pattern. We consider the case when $e \in E_n$ has period $m < 2n$. Lemma 3.9 from [2] implies that $m = 2r$, n/r is odd, $e = f\overline{f}\dots\overline{f}f \in E_n$, and $f \in E_r$. The special form of e implies various symmetries in the corresponding Walecki tournament.

Lemma 3.3 *Let $n \geq 5$ and let $e \in E_n$ with period $m = 2r < 2n$. If k is an integer such that $1 \leq k \leq r$, then $t(2ri+rf_k+k) \in N^+(t(0))$ and $t(2ri+r\overline{f}_k+k) \in$*

$N^-(t_{(0)})$, *for* $0 \le i \le n/r - 1$.

PROOF. Let k be an integer such that $1 \le k \le r$. Since $e = f\overline{f} \dots \overline{f}f$ it follows that $e_{2ri+k} = f_k$ for $0 \le i \le (n/r - 1)/2$. Therefore, if $f_k = 0$, then $t_{(2ri+k)} \in N^+(t_{(0)})$ and if $f_k = 1$, then $t_{(2ri+k)} \in N^-(t_{(0)})$ for $0 \le i \le (n/r - 1)/2$.

On the other hand, $e_{2ri+r+k} = \overline{f}_k$ for $0 \le i \le (n/r - 1)/2 - 1$. Now, if $f_k = 0$, then $t_{(2ri+r+k)} \in N^-(t_{(0)})$ and if $f_k = 1$, then $t_{(2ri+r+k)} \in N^+(t_{(0)})$. Use of Proposition 4.18 from [2] proves the remaining cases. □

Lemma 3.4 *Let* $n \ge 5$ *and let* $e \in E_n$ *with period* $m = 2r < 2n$. *If* k *is an integer such that* $1 \le k \le r$, *then* $t_{(\overline{f}_k+2(2ri+k)-1)} \in N^+(t_{(1)})$ *and* $t_{(f_k+2(2ri+k)-1)} \in N^-(t_{(1)})$. *for* $0 \le i \le (n/r - 1)/2$. *Moreover,* $t_{(f_k+2(2ri+r+k)-1)} \in N^+(t_{(1)})$ *and* $t_{(\overline{f}_k+2(2ri+r+k)-1)} \in N^-(t_{(1)})$. *for* $0 \le i \le (n/r - 1)/2 - 1$.

PROOF. Let n be odd and let $e \in E_n$ have period $m < 2n$. Lemma 3.9 from [2] implies that $m = 2r$, n/r is odd, $e = f\overline{f} \dots \overline{f}f \in E_n$, and $f \in E_r$. The special form of e tells us that $e_{2ri+k} = f_k$ for $0 \le i \le (n/r - 1)/2$. The structure of the Hamilton directed cycle $\vec{H}_k$ implies that if $f_k = 0$, then $t_{(2(2ri+k)-1)} \in N^-(t_{(1)})$ and $t_{(2(2ri+k))} \in N^+(t_{(1)})$. Furthermore, if $f_k = 1$, then $t_{(2(2ri+k)-1)} \in N^+(t_{(1)})$ and $t_{(2(2ri+k))} \in N^-(t_{(1)})$. Therefore, $t_{(2(2ri+k)+\overline{f}_k-1)} \in N^+(t_{(1)})$ and $t_{(2(2ri+k)+f_k-1)} \in N^-(t_{(1)})$.

As in the previous paragraph the form of e implies that $e_{2ri+r+k} = \overline{f}_k$ for $0 \le i \le (n/r - 1)/2 - 1$. We prove this case similarly as above which completes the proof. □

The odd pattern of the sequence $e \in E_n$ and Theorem 3.2 imply the existence of n/r Walecki subtournaments $T\langle\{t_{(0)}\} \cup M_i\rangle$, for $1 \leq i \leq n/r$, of tournament T. Next we determine certain subtournaments of $W(e)$ which are isomorphic to some Walecki tournament with odd pattern. This is a generalization of Theorem 3.2. The vertices that induce the subtournament can be simply chosen on the circumference in clockwise order starting at the vertex $t_{(1)}$.

Theorem 3.5 *Let $n \geq 5$ and let $e = f\overline{f}\ldots\overline{f}f \in E_n$ with period $m = 2r < 2n$, and $f \in E_r$. If ℓ is an odd integer such that $1 \leq \ell \leq n/r - 2$ then, $W(e)\langle\{t_{(0)}\} \cup M_1 \cup M_2 \cup \cdots \cup M_\ell\rangle \cong W(\underbrace{f\overline{f}\ldots\overline{f}f}_{\ell})$.*

Proof. If $\ell = 1$, the conclusion is true by Theorem 3.2. Let ℓ be an odd integer such that $3 \leq \ell \leq n/r - 2$. Let $e' = f\overline{f}\ldots\overline{f}f \in E_{\ell r}$. In the following τ is first considered as a permutation in $\mathbb{S}_{2n+1}$ used in the definition of $W(e)$. Secondly, it is considered as a permutation in $\mathbb{S}_{2\ell r+1}$ and is denoted by $\overline{\tau}$. Similarly, $t_{(i)}$ denotes a vertex of $W(e)$ and $\overline{t}_{(i)}$ denotes a vertex of $W(e')$.

We define a function $\psi : \{t_{(0)}\} \cup M_1 \cup M_2 \cup \cdots \cup M_\ell \longrightarrow V(W(e'))$ by $\psi(t_{(0)}) = \overline{t}_{(0)}$ and $\psi(t_{(i)}) = \overline{t}_{(i)}$, for $1 \leq i \leq 2\ell r$. Clearly, ψ is a bijection. We will show that the Hamilton directed cycle $\vec{H}'_k$ in $W(e')$ is a union of ψ-images of directed paths belonging to Hamilton directed cycles $\vec{H}_k$ and $\vec{H}_{\ell r+k}$ in $W(e)$ in the case when $\ell r + k \leq n$. Otherwise, $\vec{H}'_k$ is a union of ψ-images of directed paths $\vec{H}_k$ and $\vec{H}_{\ell r+k-n}$.

Case 1. Let us first consider the case when $\ell r + k \leq n$. Let $\vec{P}_k$ denote the directed path $[t_{(0)}, t_{(k)}, \ldots, t_{(2k)}]$ on $\vec{H}_k$ and let $\vec{P}_{\ell r+k}$ denote the directed path $[t_{(0)}, t_{(\ell r+k)}, \ldots, t_{(2k)}]$ on $\vec{H}_{\ell r+k}$. Similarly as in the proof of Theorem 3.2 we can prove $\psi(\vec{P}_k) = \vec{P}'_k$ and $\psi(\vec{P}_{\ell r+k}) = \overleftarrow{P}'_{\ell r+k}$. The definition of Walecki tournaments implies that ψ is dominance preserving on paths $\overrightarrow{P}_k$ and $\overrightarrow{P}_{\ell r+k}$. By assumption, the difference between $\ell r + k$ and k is an odd multiple of r. Furthermore, odd pattern of e implies that if $e_k = 0$ then $e_{\ell r+k} = 1$ and $\overrightarrow{H}'_k = \overrightarrow{P}'_k \cup \overleftarrow{P}'_{\ell r+k}$ (see Figure 3). If $e_k = 1$, then $e_{\ell r+k} = 0$ and $\overrightarrow{H}'_k = \overrightarrow{P}'_{\ell r+k} \cup \overleftarrow{P}'_k$.

Case 2. Let $\ell r + k > n$. Note that $n - \ell r < k \le \ell r$. Let the directed path $\overrightarrow{P}_k$ be defined as in the previous paragraph and let $\overrightarrow{P}_{\ell r+k-n}$ denote the directed path $[t(2k), \ldots, t(\ell r+k), t(0)]$ on $\overrightarrow{H}_{\ell r+k-n}$. Similarly as before $\psi(\overrightarrow{P}_{\ell r+k-n}) = \overrightarrow{P}_{\ell r+k-n}$. The definition of Walecki tournaments implies that ψ is dominance preserving on the path $\overrightarrow{P}_{\ell r+k}$. Now, n/r is odd and, by assumption, ℓ is also odd which implies that the difference $\ell r - n = (\ell - n/r)r$ is an even multiple of r. Furthermore, the odd pattern of e implies that if $e_k = 0$ then $e_{\ell r+k-n} = 0$ and $\overrightarrow{H}'_k = \overrightarrow{P}'_k \cup \overrightarrow{P}'_{\ell r+k-n}$. If $e_k = 1$, then $e_{\ell r+k-n} = 1$ and $\overrightarrow{H}'_k = \overleftarrow{P}'_{\ell r+k-n} \cup \overleftarrow{P}'_k$. This completes the proof. $\Box$

The significance of the permutation τ for the isomorphisms between Walecki tournaments can be recognized immediately from Theorem 3.13 from [2]. However, the importance of τ^m for the automorphism groups of Walecki tournaments with odd pattern was previously unknown. The subtournaments $T\langle\{t(0)\} \cup M_i\rangle$, for $1 \le i \le n/r$, are isomorphic to $W(f)$ which suggests that the permutation τ^m, which is a product of $m = 2r$ disjoint cycles of length n/r, might be an automorphism of $W(e)$.

Proposition 3.6 *Let $n \ge 5$. If $e \in E_n$ has period $m < 2n$, then $\tau^m \in Aut(W(e))$.*

PROOF. Lemma 3.9 from [2] implies that $m = 2r$, n/r is odd, $e = f\overline{f}\dots\overline{f}f \in E_n$, and $f \in E_r$. Furthermore, Lemma 3.8 from [2] implies that e and f have the same period $2r$. Hence, $R^m(e) = R^{2r}(e) = e$. The result follows by Theorem 3.13 from [2]. $\square$

4 Transitive subtournaments and multiple fan structure

A second partition of the vertices of $V(W(e)) - \{t_{(0)}\}$ is now introduced in such a way that each partition set contains exactly one element from each of the m-sets $M_1, M_2, \dots, M_{n/r}$. The partition is simply defined by the orbits $O_1, O_2, \dots, O_m$, for the permutation τ^m. The orbits have length n/r and can be expressed as follows:

$$O_\ell = \{t_{(im+\ell)} \mid 0 \le i \le n/r - 1\}, \tag{4.1}$$

for $1 \le \ell \le m$. One can easily prove that if $f_k = 0$, then $O_k \subseteq N^+(t_{(0)})$ and $O_{r+k} \subseteq N^-(t_{(0)})$, and if $f_k = 1$, then $O_k \subseteq N^-(t_{(0)})$ and $O_{r+k} \subseteq N^+(t_{(0)})$. In order to simplify the notation we introduce the permutation $\zeta \in \mathbb{S}_{2r}$ acting on the set $\{1, 2, \dots, 2r\}$ defined by $\zeta = (1\ r+1)(2\ r+2)\cdots(r\ 2r)$. That is, $\zeta(i) = i + r$ for $1 \le i \le r$, and $\zeta(i) = i - r$ for $r + 1 \le i \le 2r$. Note that for $1 \le k \le r$,

$$O_{\zeta^{f_k}(k)} = \{t_{(im+\zeta^{f_k}(k))} \mid \text{ for } 0 \le i \le n/r - 1\}. \tag{4.2}$$

It follows from the observations above that $O_{\zeta^{f_k(k)}} \subseteq N^+(t_{(0)})$ and $O_{\zeta^{\overline{f}_k(k)}} \subseteq N^-(t_{(0)})$. Therefore,

$$N^+(t_{(0)}) = \bigcup_{k=1}^{r} O_{\zeta^{f_k(k)}} \tag{4.3}$$

and

$$N^-(t_{(0)}) = \bigcup_{k=1}^{r} O_{\zeta^{\overline{f}_k(k)}}. \tag{4.4}$$

Orbits and m-sets are "orthogonal" in the sense that each orbit contains exactly one vertex of each m-set, and vice-versa.

The arc structure of subtournaments induced by the orbits O_ℓ, for the permutation τ^m, is determined by the value of e_ℓ, for $1 \leq \ell \leq m$, as shown in the following theorem. We first introduce the arc structure between two subsets of vertices of O_ℓ. Let the two *layers* $Y'_\ell, Y''_\ell \subseteq O_\ell$ be defined by

$$Y'_\ell = \{t_{(4ri+\ell)} \mid 0 \leq i \leq (n/r - 1)/2\}$$

and

$$Y''_\ell = \{t_{(4rj+2r+\ell)} \mid 0 \leq j \leq (n/r - 3)/2\}.$$

Notice that Y'_ℓ and Y''_ℓ partition O_ℓ.

We say that the arcs between Y'_ℓ and Y''_ℓ have a *multiple fan structure rooted* at Y'_ℓ (see Figure 4), if $t_{(4ri+\ell)} \longrightarrow t_{(4rj+2r+\ell)}$ whenever $4ri < 4rj + 2r$, and $t_{(4ri+\ell)} \longleftarrow t_{(4rj+2r+\ell)}$ whenever $4ri > 4rj + 2r$, for $0 \leq i \leq (n/r - 1)/2$ and $0 \leq j \leq (n/r - 3)/2$. We call Y'_ℓ the *root* of the multiple fan structure. If all the arcs are reversed we say that the multiple fan structure is rooted at Y''_ℓ.

Theorem 4.7 *Let $n \geq 1$ and let T denote the Walecki tournament $W(e)$ for $e \in E_n$ and $n \geq 5$. If e has period $m < 2n$, then the orbits $O_1, O_2, \ldots, O_m$ for* *the permutation τ^m induce regular tournaments $T\langle O_1\rangle, T\langle O_2\rangle, \ldots, T\langle O_m\rangle$. If ℓ is an integer such that $1 \leq \ell \leq m$, the subtournaments $T\langle O_\ell \cap N^+(t_{(1)})\rangle$ and $T\langle O_\ell \cap N^-(t_{(1)})\rangle$ are transitive and the directions of all their arcs are determined by e_ℓ. Furthermore, the multiple fan structure of arcs between $O_\ell \cap N^+(t_{(1)})$ and $O_\ell \cap N^-(t_{(1)})$ is determined by e_ℓ.*

PROOF. Under the conditions for the sequence $e \in E_n$, Proposition 3.6 implies that $O_1, O_2, \ldots, O_m$ are orbits for the permutation $\tau^m \in Aut(T)$ which proves that the subtournaments $T\langle O_1\rangle, T\langle O_2\rangle, \ldots, T\langle O_m\rangle$ are regular. To prove the rest of the theorem we first consider the subtournaments $T\langle O_1 \cap N^+(t_{(1)})\rangle$ and $T\langle O_1 \cap N^-(t_{(1)})\rangle$. We make use of Lemma 3.3 and Lemma 3.4.

Let $m = 2r$. Lemma 3.8 from [2] implies that $e = f\overline{f}\ldots\overline{f}f$ where $f \in E_r$. Vertices of an arbitrary orbit were determined in (4.1) and (4.2). We may assume $f_1 = 0$ for if not we may work with $\overline{W(e)}$ instead. We first consider the orbit containing vertex $t_{(1)}$. Since $f_1 = 0$ we have $O_{\varsigma^{f_1}(1)} = O_1 \subseteq N^+(t_{(0)})$ and $O_1 = \{t_{(2ri+1)} \mid \text{ for } 0 \leq i \leq n/r - 1\}$. Let $Y^+ = O_1 \cap N^+(t_{(1)})$ and $Y^- = O_1 \cap N^-(t_{(1)})$. Since $e_{r+1} = \overline{f}_1 = 1$, we have $t_{(2r+1)} \in Y^+$. Similarly, $e_{2r+1} = f_1 = 0$ implies $t_{(4r+1)} \in Y^-$. Furthermore, $\tau^m \in Aut(T)$ implies $Y^+ = \{t_{(4ri+2r+1)} \mid \text{ for } 0 \leq i \leq (n/r-3)/2\}$ and $Y^- = \{t_{(4ri+1)} \mid \text{ for } 1 \leq i \leq (n/r-1)/2\}$.

Let us consider Y^+. We will prove that in the labeling of the vertices of Y^+ in the order $t_{(2r+1)}, t_{(6r+1)}, \ldots, t_{(2n-4r+1)}$. all arcs point from right to left in the subtournament $T\langle Y^+\rangle$. Let $0 \leq i \leq (n/r-3)/2$. The equalities $\tau^{4rj+2r}(1) = \tau^{2r(j+i+1)}(\tau^{2r(j-i)}(1)) = \tau^{2r(j+i+1)}(4r(j-i))$ and $\tau^{4ri+2r}(1) = \tau^{2r(j+i+1)}(\tau^{2r(i-j)}(1)) = \tau^{2r(j+i+1)}(4r(j-i)+1)$ imply that the arc $t_{(4rj+2r+1)}$

$\longrightarrow t_{(4ri+2r+1)}$ belongs to the Hamilton directed cycle $\overrightarrow{H}_{2r(j+i+1)+1}$. This proves that the subtournament $T\langle Y^+\rangle$ is transitive. Moreover, $\tau^m(Y^+) = Y^-$ implies that the subtournament $T\langle Y^-\rangle$ is also transitive.

Next we consider orbits in $N^+(t_{(0)})$ that do not contain vertex $t_{(1)}$. Equation (4.3) implies that $O_{\zeta^{f_k}(k)} \subseteq N^+(t_{(0)})$ for $2 \leq k \leq r$. Let us first assume that $f_k = 0$. Now, using (4.2) we have

$$O_{\zeta^{f_k}(k)} = O_k = \{t_{(2ri+k)} \mid 0 \leq i \leq n/r - 1\}. \tag{4.5}$$

We further divide the proof depending on the parity of k.

CASE 1. Let k be odd. Clearly, $k+1$ is even and $e_{(k+1)/2} = f_{(k+1)/2}$ and $e_{r+(k+1)/2} = \overline{f}_{(k+1)/2}$ determine the membership of $t_{(k)}$ and $t_{(2r+k)}$, respectively, in $N^+(t_{(1)})$ or $N^-(t_{(1)})$.

CASE 1.1. If $f_{(k+1)/2} = 0$, we have $t_{(k)} \in N^-(t_{(1)})$, and $\overline{f}_{(k+1)/2} = 1$ implies $t_{(2r+k)} \in N^+(t_{(1)})$. It follows that $Y' = \{t_{(4ri+k)} \mid 0 \leq i \leq (n/r-1)/2\} \subseteq N^-(t_{(1)})$ and $Y'' = \{t_{(4ri+2r+k)} \mid 0 \leq i \leq (n/r-3)/2\} \subseteq N^+(t_{(1)})$. Similarly as above we can prove that the subtournaments $T\langle Y'\rangle$ and $T\langle Y''\rangle$ are transitive.

CASE 1.2. If $f_{(k+1)/2} = 1$, then $t_{(k)} \in N^+(t_{(1)})$ and $\overline{f}_{(k+1)/2} = 0$ which implies that $t_{(2r+k)} \in N^-(t_{(1)})$. Since the sequence e has the form $f\overline{f}\ldots\overline{f}f$ it follows that $Y' \subseteq N^+(t_{(1)})$ and $Y'' \subseteq N^-(t_{(1)})$. As in the previous case we can show that the subtournaments $T\langle Y'\rangle$ and $T\langle Y''\rangle$ are transitive. A change in the value of $f_{(k+1)/2}$ results in the reversal of arcs associated with $t_{(1)}$. However, the arcs between vertices of Y' depend on f_k only. The same reasoning applies to Y''.

CASE 2. Let k be even. This implies that $e_{k/2} = f_{k/2}$ and $e_{r+k/2} = \overline{f}_{k/2}$

determine the membership in $N^+(t_{(1)})$ or $N^-(t_{(1)})$ of $t_{(k)}$ and $t_{(2r+k)}$, respectively.

CASE 2.1. If $f_{k/2} = 0$, then $t_{(k)} \in N^+(t_{(1)})$, and $\overline{f}_{k/2} = 1$ which implies that $t_{(2r+k)} \in N^-(t_{(1)})$. Similarly as in the previous case it follows that $Y' \subseteq N^+(t_{(1)})$, $Y'' \subseteq N^-(t_{(1)})$, and the subtournaments $T\langle Y'\rangle$ and $T\langle Y''\rangle$ are transitive.

CASE 2.2. Using arguments similar to above, we can prove that if $f_{k/2} = 1$ then the sets Y' and Y'' determine transitive subtournaments $T\langle Y'\rangle$ and $T\langle Y''\rangle$, respectively. This completes the proof for $f_k = 0$.

Assume now that $f_k = 1$. This implies

$$O_{\zeta^{f_k}(k)} = O_{r+k} = \{t_{(2ri+r+k)} \mid 0 \leq i \leq n/r - 1\}. \tag{4.6}$$

In a way similar to the previous cases we define $Y' = \{t_{(4ri+r+k)} \mid 0 \leq i \leq (n/r-1)/2\}$ and $Y'' = \{t_{(4ri+3r+k)} \mid 0 \leq i \leq (n/r-3)/2\}$, and prove that $T\langle Y'\rangle$ and $T\langle Y''\rangle$ are transitive tournaments. The proof is similar to the case $f_k = 0$, however, the direction of all arcs considered is reversed since $e_k = f_k = 1$. This completes the exploration of the arc structure for orbits $O_{\zeta^{f_k}(k)} \subseteq N^+(t_{(0)})$. The result for orbits $O_{\zeta^{\overline{f}_k}(k)} \subseteq N^-(t_{(0)})$ follows since $T \cong \overline{T}$.

Last we determine the arc structure in T between vertices of $O_\ell \cap N^+(t_{(1)})$ and $O_\ell \cap N^-(t_{(1)})$. We first consider orbits $O_{\zeta^{f_k}(k)} \subseteq N^+(t_{(0)})$. Let $1 \leq k \leq r$. If $f_k = 0$, then let the layers Y' and Y'' partition vertices of O_k, where $Y' = \{t_{(4ri+k)} \mid 0 \leq i \leq (n/r-1)/2\}$ and $Y'' = \{t_{(4rj+2r+k)} \mid 0 \leq j \leq (n/r-3)/2\}$. Now, $f_{r+k} = \overline{f}_k = 1$ implies $t_{(4ri+k)} \longrightarrow t_{(4rj+2r+k)} \in \overrightarrow{H}_{2r(i+j)+r+k}$ whenever $i \leq j$. Also $t_{(4ri+k)} \longleftarrow t_{(4rj+2r+k)} \in \overrightarrow{H}_{2r(i+j)+r+k}$ whenever $i > j$. Therefore, the arcs between the two layers have a multiple fan structure rooted at Y'.

If $f_k = 1$, then let the layers Y' and Y'' partition vertices of O_{r+k}, where $Y' = \{t_{(4ri+r+k)} \mid 0 \leq i \leq (n/r-1)/2\}$ and $Y'' = \{t_{(4rj+3r+k)} \mid 0 \leq j \leq (n/r-3)/2\}$. Because $f_k = 1$ all the arcs are reversed in the previous proof. Therefore, arcs between Y' and Y'' have the multiple arc structure rooted at Y''.

The result for orbits $O_{\zeta^{\overline{f}_k}(k)} \subseteq N^-(t_{(0)})$ follows since $T \cong \overline{T}$. This completes the proof. $\square$

Next we study the structure of arcs between the orbits in $N^+(t_{(0)})$.

Theorem 4.8 *Let $n \geq 1$ and let $O_{\zeta^{f_k}(k)}, O_{\zeta^{f_\ell}(\ell)} \subseteq N^+(t_{(0)})$ be two orbits of τ^m in the Walecki tournament $W(e)$ where e has the form $f\overline{f}\ldots\overline{f}f \in E_n$, $f \in E_r$, and $1 \leq k < l \leq 2r = m$. The arcs between any two of their four layers Y'_k, Y''_k, Y'_ℓ, and Y''_ℓ have a multiple fan structure. Moreover, the root of a multiple fan structure is determined by $e_{(k+\ell)/2}$ if $\ell - k$ is even and by $e_{(k+\ell-1)/2}$ otherwise.*

Proof. The arc structure of subtournaments induced on the orbits O_ℓ for τ^m, for $1 \leq \ell \leq m$, was discussed in Theorem 4.7. Hence, we only need to consider the arc structure between layers of distinct orbits.

Case 1. We will first prove that arcs between the layers $Y'_k = \{t_{(4ri'+k)} \mid 0 \leq i' \leq (n/r-1)/2\}$ and $Y'_\ell = \{t_{(4rj'+\ell)} \mid 0 \leq j' \leq (n/r-1)/2\}$ have a multiple fan structure. There are two general cases to be considered depending on the parity of $\ell - k$.

Case 1.1. Assume that $\ell - k$ is even. Let $t = k + (\ell - k)/2 = (k + \ell)/2$. Note that $1 \leq t \leq n$.

Case 1.1.1. Let $e_t = 1$. Lemma 4.20 from [2] implies that $t_{(k)} \longrightarrow t_{(\ell)} \in \overrightarrow{H}_t$.

Let $s = 2r(i'+j')+(k+\ell)/2$. Since $s = 2r(i'+j')+t$ and $e_t = 1$ we have $e_s = 1$. Assume first that $4ri'+k-1 < 4rj'+\ell-1$. Because $k-\ell < 2r$ we have $i' \leq j'$. Equalities $\tau^{4ri'+k-1}(1) = \tau^{s-1}(\tau^{2r(i'-j')+(k-\ell)/2}(1)) = \tau^{s-1}(4r(j'-i')+\ell-k+1)$ and $\tau^{4rj'+\ell-1}(1) = \tau^{s-1}(\tau^{2r(j'-i')+(\ell-k)/2}(1)) = \tau^{s-1}(4r(j'-i')+\ell-k)$ imply $t_{(4ri'+k)} \longrightarrow t_{(4rj'+\ell)} \in \overrightarrow{H}_s$. In the last implication we used the fact that $-n < 2r(i'-j')+(k-\ell)/2 < 0$. Notice that $2r(i'-j')+(k-\ell)/2$ and $2r(j'-i')+(\ell-k)/2$ can be none of 0, n, or $-n$. If so, vertices determined by $\tau^{s-1}(\tau^{2r(i'-j')+(k-\ell)/2}(1))$ and $\tau^{s-1}(\tau^{2r(j'-i')+(\ell-k)/2}(1))$ would have been equal, which would be a contradiction with $4ri'+k-1 \neq 4rj'+\ell-1$.

Similarly we prove that $4ri'+k-1 > 4rj'+\ell-1$ implies $t_{(4ri'+k)} \longleftarrow t_{(4rj'+\ell)} \in \overrightarrow{H}_s$. Therefore, arcs between Y'_k and Y'_ℓ have a multiple fan structure rooted at Y'_k.

Case 1.1.2. Let $e_t = 0$. This implies that all arcs are reversed in the Case 1.1.1, which proves that the multiple fan arc structure is rooted at Y'_ℓ.

Case 1.2. Assume that $\ell - k$ is odd. One can prove this case similarly to the Case 1.1.

Case 2. Next we prove that arcs between the layers $Y''_k = \{t_{(4ri''+2r+k)} \mid 0 \leq i'' \leq (n/r-1)/3\}$, and $Y''_\ell = \{t_{(4rj''+2r+\ell)} \mid 0 \leq j'' \leq (n/r-1)/3\}$, have a multiple fan structure. Since $e = f\overline{f}\ldots\overline{f}f \in E_n$ and $f \in E_r$, τ^{-2r} is an automorphism which maps vertices of Y''_k to vertices of Y'_k and vertices of Y''_ℓ to vertices of Y'_ℓ. Now, $e_{(k+\ell)/2+2r} = e_{(k+\ell)/2}$ implies that arcs between Y''_k and Y''_ℓ have the same direction as the corresponding arcs between Y'_k and Y'_ℓ. Case 1 then implies that arcs between Y''_k and Y''_ℓ have a multiple fan structure. Notice that we can map Y''_k and Y''_ℓ to Y'_k and Y'_ℓ, respectively, since the latter two sets have one more element than the sets Y_k'' and Y_ℓ''.

CASE 3. Next we prove that arcs between the layers $Y_k' = \{t_{(4ri'+k)} \mid 0 \leq i' \leq (n/r-1)/2\}$ and $Y_\ell'' = \{t_{(4rj''+2r+\ell)} \mid 0 \leq j'' \leq (n/r-3)/2\}$ have a multiple fan structure. Since $e = f\overline{f}\ldots\overline{f}f \in E_n$, $f \in E_r$, and

$$e_{(k+\ell)/2+r} = \overline{e}_{(k+\ell)/2} \tag{4.7}$$

we can use substitution $\ell \to \ell - 2r$ to map vertices of Y_ℓ'' to vertices of Y_ℓ'. Equation (4.7) implies that arcs between Y_k' and Y_ℓ'' are reversed arcs of the corresponding arcs between Y_k' and Y_ℓ'. Case 1 then implies that arcs between Y_k' and Y_ℓ'' have a multiple fan structure.

CASE 4. Similarly substitution as in the previous case proves that arcs between the layers $Y_k'' = \{t_{(4ri''+2r+k)} \mid 0 \leq i'' \leq (n/r-3)/2\}$ and $Y_\ell' = \{t_{(4rj'+\ell)} \mid 0 \leq j' \leq (n/r-1)/2\}$ have a multiple fan structure. $\square$

The next result characterizes the structure of arcs between the orbits in $N^-(t_{(0)})$. It is an immediate consequence of Theorem 4.8 since $W(e) \cong \overline{W(e)}$.

Corollary 4.9 *Let $n \geq 5$ and let $O_{\zeta\overline{f}_k(k)}, O_{\zeta\overline{f}_\ell(\ell)} \subseteq N^-(t_{(0)})$ be two orbits of τ^m in the Walecki tournament $W(e)$ where e has the form $f\overline{f}\ldots\overline{f}f$, $e \in E_n$, and $1 \leq k < l \leq m$. The arcs between any two of their four layers Y_k', Y_k'', Y_ℓ', and Y_ℓ'' have a multiple fan structure. Moreover, the root of a multiple fan structure is determined by $e_{(k+\ell)/2}$ if $\ell - k$ is even, and by $e_{(k+\ell-1)/2}$ if $\ell - k$ is odd.*

Theorem 4.7, Theorem 4.8, and Corollary 4.9 imply that the arcs between distinct orbits in $N^+(t_{(0)})$ (or $N^-(t_{(0)})$) are determined by the arcs in the subtournament $W(e)\langle\{t_{(0)}\} \cup M_1\rangle$.

In the following results we consider the subtournaments induced by outsets and insets of vertices in a Walecki tournament with odd pattern. If the vertex is distinct from $t_{(0)}$ then its outset induces a subtournament that is not regular for n odd. However, the outset of $t_{(0)}$ induces a regular tournament. Similarly, for n even the outset of $t_{(0)}$ induces an almost regular tournament but the outset of any other vertex is not almost regular. This implies that $t_{(0)}$ must be fixed for any automorphism of a Walecki tournament with odd pattern.

Theorem 4.10 *Let $T = W(e)$ for $e = f\overline{f}\ldots\overline{f}f \in E_n$, $n \geq 5$, $f = (0, 0, \ldots, 0) \in E_r$, and n/r odd. For $v \in V(W(e)) - \{t_{(0)}\}$, the tournaments $T\langle N^+(v)\rangle$ are not regular and not almost regular subtournaments of T for n odd and n even, respectively.*

PROOF. Since $W(e) \cong \overline{W(e)}$, it suffices to prove the theorem for vertices in $N^+(t_{(0)})$. Furthermore, since $\tau^m \in Aut(T)$, it is sufficient to prove the theorem for the vertices in $N^+(t_{(0)}) \cap M_1$. Let $M' = M_1 \cup M_3 \cup \cdots \cup M_{n/r}$ and $M'' = M_2 \cup M_4 \cup \cdots \cup M_{n/r-1}$.

We consider $t_{(1)} \in N^+(t_{(0)}) \cap M_1$. We will count the vertices in $N^+(t_{(1)}) \cap N^+(t_{(2n)})$. First we determine the vertices in $N^+(t_{(1)})$. Since $f = (0, 0, \ldots, 0)$ Theorem 3.2 implies

$$N^+(t_{(1)}) \cap M_1 = \{t_{(2i+2)} \mid 0 \leq i \leq r-1\}. \tag{4.8}$$

Using Lemma 4.20 from [2] we have

$$N^+(t_{(1)}) \cap M_2 = \{t_{(2r+2i+1)} \mid 0 \leq i \leq r-1\}. \tag{4.9}$$

Let $X' = N^+(t_{(1)}) \cap M'$ and $X'' = N^+(t_{(1)}) \cap M''$. The odd pattern of the sequence e and equalities (4.8) and (4.9) imply

$$X' = \{t_{(4rj+2i+2)} \mid 0 \leq i \leq r-1, \mid 0 \leq j \leq (n/r-1)/2\} \tag{4.10}$$

and

$$X'' = \{t_{(4rj+2r+2i+1)} \mid 0 \leq i \leq r-1, \mid 0 \leq j \leq (n/r-3)/2\}. \tag{4.11}$$

Clearly, $N^+(t_{(1)}) = X' \cup X''$. Next we determine the vertices in $N^+(t_{(2n)})$. We have

$$N^+(t_{(2n)}) \cap M_1 = \{t_{(2i+1)} \mid 1 \leq i \leq r-1\} \tag{4.12}$$

and

$$N^+(t_{(2n)}) \cap M_2 = \{t_{(2r+1)}\} \cup \{t_{(2r+2i+2)} \mid 0 \leq i \leq r-1\}. \tag{4.13}$$

Let Y' denote $N^+(t_{(2n)}) \cap M'$ and let Y'' denote $N^+(t_{(2n)}) \cap M''$. The odd pattern of the sequence e and equalities (4.12) and (4.13) imply

$$Y' = \{t_{(4rj+2i+1)} \mid 1 \leq i \leq r-1, 0 \leq j \leq (n/r-3)/2\}. \tag{4.14}$$

Let $Y'' = \tilde{Y}'' \cup \tilde{\tilde{Y}}''$ where

$$\tilde{Y}'' = \{t_{(4rj+2r+1)} \mid 0 \leq j \leq (n/r-3)/2\} \tag{4.15}$$

and

$$\tilde{\tilde{Y}}'' = \{t_{(4rj+2r+2i+2)} \mid 0 \leq i \leq r-1, 0 \leq j \leq (n/r-3)/2\}. \tag{4.16}$$

Clearly, $N^+(t_{(2n)}) = \{t_{(0)}\} \cup Y' \cup Y''$.

Comparing the powers of τ in equalities (4.10), (4.11), (4.14), (4.15), and (4.16) for vertices in $N^+(t_{(1)})$ and $N^+(t_{(2n)})$, we deduce $(X' \cup X'') \cap (Y' \cup Y'') =$

$\tilde{Y}'$ which implies $N^+(t_{(1)}) \cap N^+(t_{(2n)}) = \tilde{Y}''$. Hence, the score of vertex $t_{(2n)}$ in $T\langle N^+(t_{(1)})\rangle$ equals $|\tilde{Y}''| = (n/r-1)/2$. If $r > 1$, then $(n/r-1)/2 < n/2-1$ which implies that $T\langle N^+(t_{(1)})\rangle$ is not regular or almost regular when n is odd or even, respectively. The proofs for the remaining vertices of $N^+(t_{(0)}) \cap M_1$ are similar and we omit them.

The arc structure of $T\langle N^+(t_{(1)})\rangle$ is different in the case when $r = 1$, that is, when $e = (0,1,0,1,\ldots,0,1,0) \in E_n$. Notice that n/r odd implies that n has to be odd. In order to verify non-regularity of $T\langle N^+(t_{(1)})\rangle$, we consider $N^+(t_{(1)}) \cap N^+(t_{(3)})$. The pattern of e implies

$$N^+(t_{(1)}) = \{t_{(2n)}\} \cup \{t_{(4i+2)}, t_{(4i+3)} \mid 0 \le i \le (n-3)/2\}. \tag{4.17}$$

Since $\tau^2 \in Aut(W(e))$ we have

$$N^+(t_{(3)}) = \{t_{(2)}\} \cup \{t_{(4i+4)}, t_{(4i+5)} \mid 0 \le i \le (n-3)/2\}. \tag{4.18}$$

Comparing the powers of τ in equalities (4.10), (4.11), (4.17), (4.18), for vertices in $N^+(t_{(1)})$ and $N^+(t_{(3)})$ we deduce that $N^+(t_{(1)}) \cap N^+(t_{(3)}) = \{t_{(2)}\}$. Hence, the score of vertex $t_{(3)}$ in $T\langle N^+(t_{(1)})\rangle$ equals 1 implying that $T\langle N^+(t_{(1)})\rangle$ is not regular. This completes the proof. □

For example, in the tournament $T = W(01010)$ the score sequence of $T\langle N^+(t_{(1)})\rangle$ is $(s(t_{(3)}), s(t_{(2)}), s(t_{(10)}), s(t_{(6)}), s(t_{(7)})) = (1,2,2,2,3)$, and in the tournament $T = W(000111000)$ the score sequence of $T\langle N^+(t_{(1)})\rangle$ is

$$(s(t_{(18)}), s(t_{(2)}), s(t_{(7)}), s(t_{(11)}), s(t_{(4)}), s(t_{(16)}), s(t_{(6)}), s(t_{(9)}), s(t_{(14)})) =$$

$$= (1,2,3,4,5,5,5,5,6).$$

Next we determine the structure of arcs in the subtournaments induced by $N^+(t_{(0)})$ and $N^-(t_{(0)})$. There are two cases to be considered for the odd pattern depending on the parity of n.

5 Regular and almost regular subtournaments

Let $e \in E_n$. We consider odd patterns for n odd. Let T denote $W(e)$. We will consider the subtournaments $T\langle N^+(t_{(0)})\rangle$ and $T\langle N^-(t_{(0)})\rangle$. We know that the out-neighbours and in-neighbours of $t_{(0)}$ are determined by f and $\overline{f}$. Therefore, r vertices of each m-set belong to $N^+(t_{(0)})$ and the other r of them to $N^-(t_{(0)})$. On the other hand, the construction of Walecki tournaments implies that out of two consecutive vertices $t_{(j)}$ and $t_{(j+1)}$ exactly one is an out-neighbour of the vertex $t_{(i)}$, whenever $j-i$ is even. Therefore one would hope that the score of each vertex in $T\langle N^+(t_{(0)})\rangle$ is no more than $2n/4 = n/2$. This would imply the regularity of the subtournaments $T\langle N^+(t_{(0)})\rangle$ and $T\langle N^-(t_{(0)})\rangle$.

Theorem 5.11 *Let n be odd and let T denote the Walecki tournament $W(e)$ for $e \in E_n$. If e has period $m < 2n$, then the subtournaments $T\langle N^+(t_{(0)})\rangle$ and $T\langle N^-(t_{(0)})\rangle$ are regular.*

PROOF. The result is clearly true for $n \le 3$. Let T be a tournament as stated in the conditions of the lemma and $n \ge 5$. We prove that $T\langle N^+(t_{(0)})\rangle$ is regular. In order to do so we first determine the score of the vertex $t_{(1)}$ in $T\langle N^+(t_{(0)})\rangle$. Let Y denote the set $N^+(t_{(0)}) \cap N^+(t_{(1)})$. We are interested in the cardinality of Y. Since $m < 2n$, Lemma 3.9 from [2] implies that $m = 2r$ and n/r is odd. Now, n odd implies that r is also odd. We proceed by proving that $|Y \cap (M_i \cup M_{i+1})| = r$, for $1 \le i \le n/r - 1$.

Let us consider the cardinality of $Y \cap (M_1 \cap M_2)$. We first determine the vertices in $N^+(t_{(1)}) \cap M_1$. If i is an integer such that $1 \le i \le r-1$, then $e_{i+1} = f_{i+1}$. There are two cases. If $f_{i+1} = 0$, then $t_{(2i+1)} \in N^-(t_{(1)})$ and $t_{(2i+2)} \in N^+(t_{(1)})$. If $f_{i+1} = 1$, then $t_{(2i+1)} \in N^+(t_{(1)})$ and $t_{(2i+2)} \in N^-(t_{(1)})$. Therefore,

$$t_{(2i+1+f_{i+1})} \in N^-(t_{(1)}) \text{ and } t_{(2i+1+\overline{f}_{i+1})} \in N^+(t_{(1)}). \qquad (5.19)$$

Since $T \cong \overline{T}$ we may assume that $e_1 = f_1 = 0$. Thus, $t_{(2)} \in N^+(t_{(1)})$. Next we consider $N^+(t_{(0)}) \cap M_1$. Similarly as above we get

$$t_{(j+1+rf_{j+1})} \in N^+(t_{(0)}) \text{ and } t_{(j+1+r\overline{f}_{j+1})} \in N^-(t_{(0)}). \qquad (5.20)$$

The neighbours of the vertex $t_{(1)}$ in the set M_2 are given by

$$t_{(2(r+i)+1+\overline{f}_{i+1})} \in N^-(t_{(1)}) \text{ and } t_{(2(r+i)+1+f_{i+1})} \in N^+(t_{(1)}), \qquad (5.21)$$

and the neighbours of the vertex $t_{(0)}$ in the set M_2 are given by

$$t_{(2r+k+1+rf_{k+1})} \in N^+(t_{(0)}) \text{ and } t_{(2r+k+1+r\overline{f}_{k+1})} \in N^-(t_{(0)}). \qquad (5.22)$$

We use (5.19), (5.20), (5.21), and (5.22) in the following case study. Let i be an integer such that $1 \le i \le (r-3)/2$. Now, $e_{i+1} = f_{i+1}$ implies that if $f_{i+1} = 0$, then $t_{(2i+2)}, t_{(2r+2i+1)} \in N^+(t_{(1)})$ and if $f_{i+1} = 1$, then $t_{(2i+1)}, t_{(2r+2i+2)} \in N^+(t_{(1)})$. Similarly, $e_{i+(r+1)/2} = f_{i+(r+1)/2}$ implies that if $f_{i+(r+1)/2} = 0$, then $t_{(r+2i+1)} \in N^+(t_{(1)})$ and if $f_{i+(r+1)/2} = 1$, then $t_{(3r+2i+1)} \in N^+(t_{(1)})$. Also, $e_{i+(r+3)/2} = f_{i+(r+3)/2}$ implies that if $f_{i+(r+3)/2} = 0$, then $t_{(3r+2i+2)} \in N^+(t_{(1)})$ and if $f_{i+(r+3)/2} = 1$, then $t_{(r+2i+2)} \in N^+(t_{(1)})$. Because $e_{2i+1} = f_{2i+1}$ it follows that if $f_{2i+1} = 0$, then $t_{(2i+1)}, t_{(2r+2i+1)} \in N^+(t_{(0)})$ and if $f_{2i+1} = 1$, then

$t_{(r+2i+1)}, t_{(3r+2i+1)} \in N^+(t_{(0)})$. Also $e_{2i+2} = f_{2i+2}$ implies that if $f_{2i+2} = 0$, then $t_{(2i+2)}, t_{(2r+2i+2)} \in N^+(t_{(0)})$ and if $f_{2i+2} = 1$, then $t_{(r+2i+2)}$, $t_{(3r+2i+2)} \in N^+(t_{(0)})$. In the following tables $\bigcirc$ represents a vertex in $N^+(t_{(0)})$, $\times$ represents a vertex in $N^+(t_{(1)})$, and $\otimes$ represents a vertex in $N^+(t_{(0)}) \cap N^+(t_{(1)})$. Table 1 displays all possible combinations of values of f_{i+1}, f_{2i+1}, f_{2i+2}, $f_{i+(r+1)/2}$, and $f_{i+(r+3)/2}$ and the membership of vertices in $N^+(t_{(0)}) \cap N^+(t_{(1)})$. It follows that exactly two of the vertices $t_{(2i+1)}$, $t_{(2i+2)}$, $t_{(r+2i+1)}$, $t_{(r+2i+2)}$, $t_{(2r+2i+1)}$, $t_{(2r+2i+2)}$, $t_{(3r+2i+1)}$, and $t_{(3r+2i+2)}$ belong to $Y \cap (M_1 \cup M_2)$ for $1 \leq i \leq (r-3)/2$.

We have yet to consider vertices $t_{(1)}$, $t_{(r)}$, $t_{(r+1)}$, $t_{(2r)}$, $t_{(2r+1)}$, $t_{(3r)}$, $t_{(3r+1)}$, and $t_{(4r)}$. Their membership in $N^+(t_{(1)}) \cap N^+(t_{(0)})$ is determined by the values of f_1, f_r and $f_{(r+1)/2}$. Notice that $e_r = f_r$ which implies that if $f_r = 0$, then $t_{(r)}, t_{(3r)} \in N^+(t_{(0)})$, and $t_{(2r)}, t_{(4r-1)} \in N^+(t_{(1)})$. If $f_r = 1$, then $t_{(2r)}, t_{(4r)} \in N^+(t_{(0)})$, and $t_{(4r)} \in N^+(t_{(1)})$. Similarly, $e_{(r+1)/2} = f_{(r+1)/2}$ implies that if $f_{(r+1)/2} = 0$, then $t_{(r+1)}, t_{(3r)} \in N^+(t_{(1)})$. If $f_{(r+1)/2} = 1$, then $t_{(r)}, t_{(3r+1)} \in N^+(t_{(1)})$. Since $f_1 = 0$ we have $e_{r+1} = \overline{f}_1 = 1$ which implies that $t_{(2r+1)} \in N^+(t_{(1)})$ and $t_{(1)}, t_{(2r+1)} \in N^+(t_{(0)})$. Clearly, $t_{(1)} \notin N^+(t_{(1)})$. Consider all four combinations of values of f_r and $f_{(r+1)/2}$ (see Table 2). It follows that exactly two of the vertices $t_{(r)}$, $t_{(r+1)}$, $t_{(2r)}$, $t_{(2r+1)}$, $t_{(3r)}$, $t_{(3r+1)}$, and $t_{(4r)}$ belong to $Y \cap (M_1 \cup M_2)$.

We have considered all vertices in $M_1 \cup M_2$ except $t_{(2)}$, $t_{(r+2)}$, $t_{(2r+2)}$, and $t_{(3r+2)}$. Their membership in $N^+(t_{(1)}) \cap N^+(t_{(0)})$ is determined by the values of f_1, f_2 and $f_{(r+3)/2}$. Now, $e_1 = f_1 = 0$ implies $t_{(2)} \in N^+(t_{(1)})$ and $e_{r+1} = \overline{f}_1 = 1$ implies $t_{(2r+2)} \in N^-(t_{(1)})$. Notice that $e_2 = f_2$ implies that if $f_2 = 0$, then $t_{(2)}, t_{(2r+2)} \in N^+(t_{(0)})$. If $f_2 = 1$, then $t_{(r+2)}, t_{(3r+2)} \in N^+(t_{(0)})$. Also,

$e_{(r+3)/2} = f_{(r+3)/2}$ implies that if $f_{(r+3)/2} = 0$, then $t_{(3r+2)} \in N^+(t_{(1)})$ and if $f_{(r+3)/2} = 1$, then $t_{(r+2)} \in N^+(t_{(1)})$. Consider all four combinations of values of f_r and $f_{(r+3)/2}$ (see Table 3). It follows that exactly one of the vertices $t_{(2)}$, $t_{(r+2)}$, $t_{(2r+2)}$, and $t_{(3r+2)}$ belong to $Y \cap (M_1 \cup M_2)$. It follows by the above observations that

$$|Y \cap (M_1 \cup M_2)| = 2(r-3)/2 + 2 + 1 = r. \tag{5.23}$$

Similar to the previous case we can show that $|Y \cap (M_i \cup M_{i+1})| = 2(r-1)/2+1 = r$, for $2 \leq i \leq n/r - 1$, and

$$|Y \cap (M_1 \cup M_{n/r})| \leq 2(r-3)/2 + 1 + 1 = r - 1. \tag{5.24}$$

Let α denote $|Y \cap M_1|$. Since $|Y \cap (M_1 \cup M_2)| = r$ it follows that $|Y \cap M_2| = r - \alpha$. Since n/r is odd we have $|Y \cap M_{n/r-1}| = r - \alpha$ and $|Y \cap M_{n/r}| = \alpha$ which implies $|Y \cap (M_1 \cup M_{n/r})| = 2\alpha \leq r-1$. Now, r is odd implies $|Y \cap (M_1 \cup M_{n/r})| \leq r - 1$. Therefore, $2|Y| \leq (r-1) + (n-r) = n - 1$ which implies

$$s(t_{(1)}) = |N^+(t_{(0)}) \cap N^+(t_{(1)})| \leq \frac{n-1}{2} < \frac{n}{2}. \tag{5.25}$$

Similar to the proof of (5.25), we can prove that

$$s(t_{(i+rf_i)}) = |N^+(t_{(0)}) \cap N^+(t_{(i+rf_i)})| \leq \frac{n-1}{2},$$

for $2 \leq i \leq r$. That is, every vertex in $N^+(t_{(0)}) \cap M_1$ has score at most $(n-1)/2$. Since τ^m is an automorphism of T by Proposition 3.6, the score of every vertex in the tournament $T\langle N^+(t_{(0)})\rangle$ is at most $(n-1)/2$. Now,

$$\binom{n}{2} = \sum_{v \in N^+(t(0))} s(v) \leq \frac{n(n-1)}{2}$$

implies $s(v) = (n-1)/2$ for every vertex $v \in N^+(t_{(0)})$. Therefore, the subtournament $T\langle N^+(t_{(0)})\rangle$ is regular. Regularity of the tournament $T\langle N^-(t_{(0)})\rangle$ follows since $T \cong \overline{T}$. $\Box$

5.1 Almost regular subtournaments for n even

When n is even, $T\langle N^+(t_{(0)})\rangle$ can not be regular. However, one can prove that it is almost regular. We follow an analog of the proof of Theorem 5.11. Oddly enough, the fact that n is even simplifies the proof.

Theorem 5.12 *Let n be even and let T denote the Walecki tournament $W(e)$ for $e \in E_n$. If e has period $m < 2n$, then the subtournaments $T\langle N^+(t_{(0)})\rangle$ and $T\langle N^-(t_{(0)})\rangle$ are almost regular.*

PROOF. The result is clearly true for $n \leq 4$. Let T be a tournament as stated in the conditions of the lemma. We first prove that $T\langle N^+(t_{(0)})\rangle$ is almost regular. In order to do so we determine the score of the vertex $t_{(1)}$ in $T\langle N^+(t_{(0)})\rangle$. Let Y denote the set $N^+(t_{(0)}) \cup N^+(t_{(1)})$. We are interested in the cardinality of Y. Since $m < 2n$, we have $m = 2r$ and n/r is odd. Now, n even implies that r is also even. We proceed by considering sets $Y \cap (M_i \cup M_{i+1})$, for $1 \leq i \leq n/r - 1$. Similar to the proof of Theorem 5.11 we can prove $|Y \cap (M_i \cup M_{i+1})| = r$, for $1 \leq i \leq n/r - 1$, and $|Y \cap (M_{n/r} \cup M_1)| \leq r$. Let α denote $|Y \cap M_1|$. Since $|Y \cap (M_1 \cup M_2)| = r$ it follows that $|Y \cap M_2| = r - \alpha$. Since n/r is odd we have $|Y \cap M_{n/r-1}| = r - \alpha$ and $|Y \cap M_{n/r}| = \alpha$ which implies $|Y \cap (M_1 \cup M_{n/r})| = 2\alpha$. Hence, $2\alpha \leq r$. Since r is even, we have $\alpha \leq r/2$ and $|Y \cap (M_1 \cup M_{n/r})| \leq r$. Therefore, $2|Y| \leq n$ which implies

$s(t_{(1)}) = |N^+(t_{(0)}) \cap N^+(t_{(1)})| \leq n/2$. Similar to the proof of last equation, we can prove that $s(t_{(i+rf_i)}) = |N^+(t_{(0)}) \cap N^+(t_{(i+rf_i)})| \leq n/2$, for $2 \leq i \leq r$. That is, every vertex in $N^+(t_{(0)}) \cap M_1$ has score at most $n/2$. Since $\tau^m \in Aut(T)$ by Proposition 3.6, the score of every vertex in the subtournament $T\langle N^+(t_{(0)})\rangle$ is at most $n/2$.

Since $T \cong \overline{T}$ with anti-automorphism η, we have $\overline{T\langle N^-(t_{(0)})\rangle} \cong T\langle N^+(t_{(0)})\rangle$. Now, $\eta(t_{(1)}) = t_{(3)}$ implying that $|N^-(t_{(0)}) \cap N^+(t_{(3)})| = s(t_{(3)}) \leq n/2$ in $\overline{T\langle N^-(t_{(0)})\rangle}$. Therefore, $|N^+(t_{(0)}) \cap N^-(t_{(1)})| \leq n/2$ in $T\langle N^+(t_{(0)})\rangle$. Since $N^+(t_{(0)}) = (N^+(t_{(0)}) \cap N^+(t_{(1)})) \cup (N^+(t_{(0)}) \cap N^-(t_{(1)})) \cup \{t_{(1)}\}$, we have $s(t_{(1)}) = |N^+(t_{(0)}) \cap N^+(t_{(1)})| \geq n/2 - 1$ in $T\langle N^+(t_{(0)})\rangle$. Using a similar argument we can prove that the score of every vertex in $T\langle N^+(t_{(0)})\rangle$ is either $n/2$ or $n/2 - 1$.

If k denotes the number of vertices with score $n/2$ in $T\langle N^+(t_{(0)})\rangle$ then the number of vertices of degree $n/2 - 1$ equals $n - k$. The equation

$$\binom{n}{2} = \sum_{v \in N^+(t(1))} s(v) = k\frac{n}{2} + (n-k)(n2-1)$$

implies that $k = n/2$. In other words, the subtournament $T\langle N^+(t_{(0)})\rangle$ is almost regular. Now, $T\langle N^-(t_{(0)})\rangle$ is also almost regular since $T \cong \overline{T}$. $\square$

In this paper we studied odd pattern Walecki tournaments. In the subsequent paper we determine the arc structure of subtournaments of even pattern Walecki tournaments.

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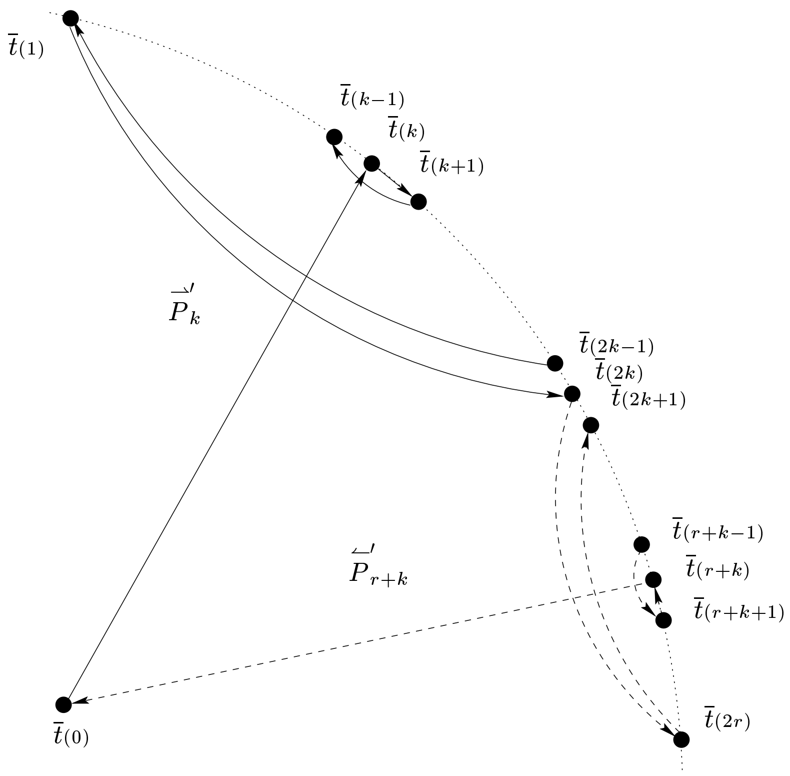

Figure 1: The diagram shows the image of the Hamilton directed cycle $\overrightarrow{H}'_k$ from the proof of Theorem 3.2.

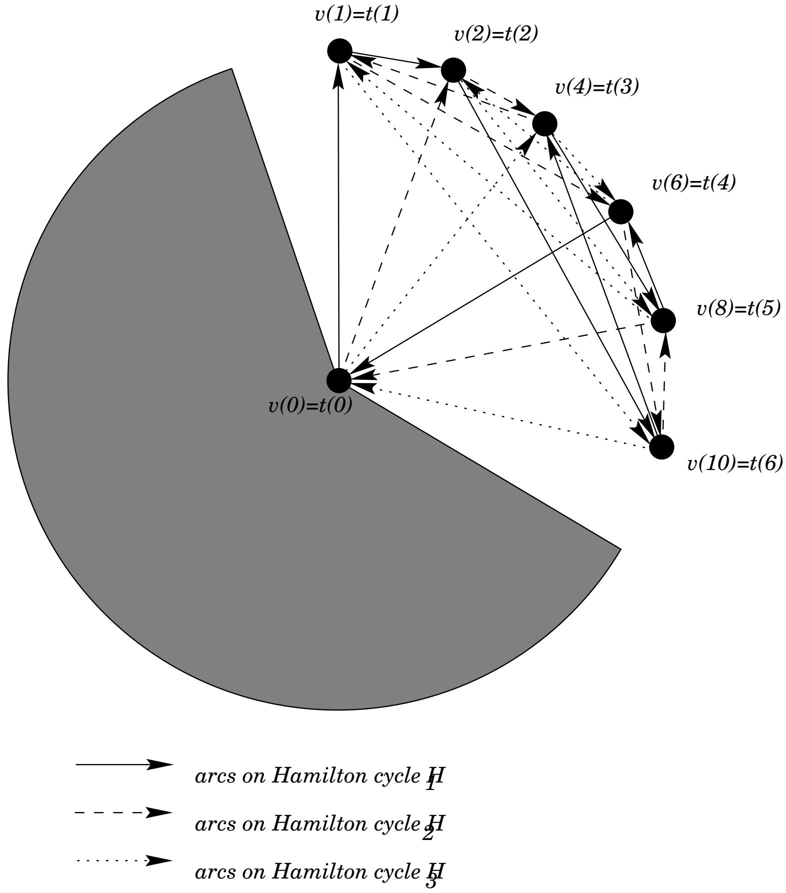

Figure 2: Walecki tournament $W(000111000)$. Only arcs of a subtournament isomorphic to $W(000)$ are drawn.

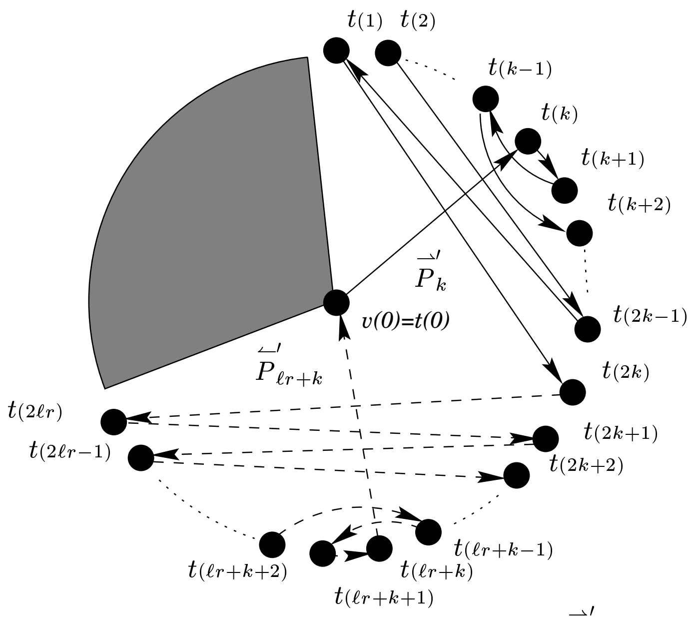

Figure 3: The diagram shows Hamilton directed cycle $\overrightarrow{H}'_k$ constructed from directed paths $\overrightarrow{P}'_k$ and $\overleftarrow{P}'_{\ell r+k}$ from the proof of Theorem 3.5. The shaded region represents vertices in $V(W(e)) - V(W(e'))$.

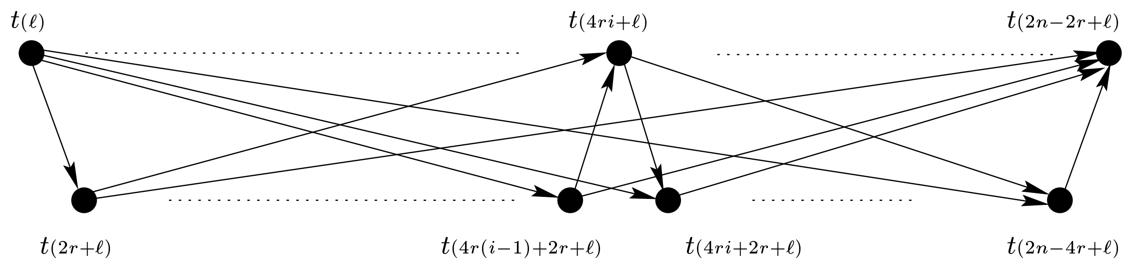

Figure 4: The diagram shows the multiple fan structure of the arcs between Y'_ℓ and Y''_ℓ rooted at Y'_ℓ.

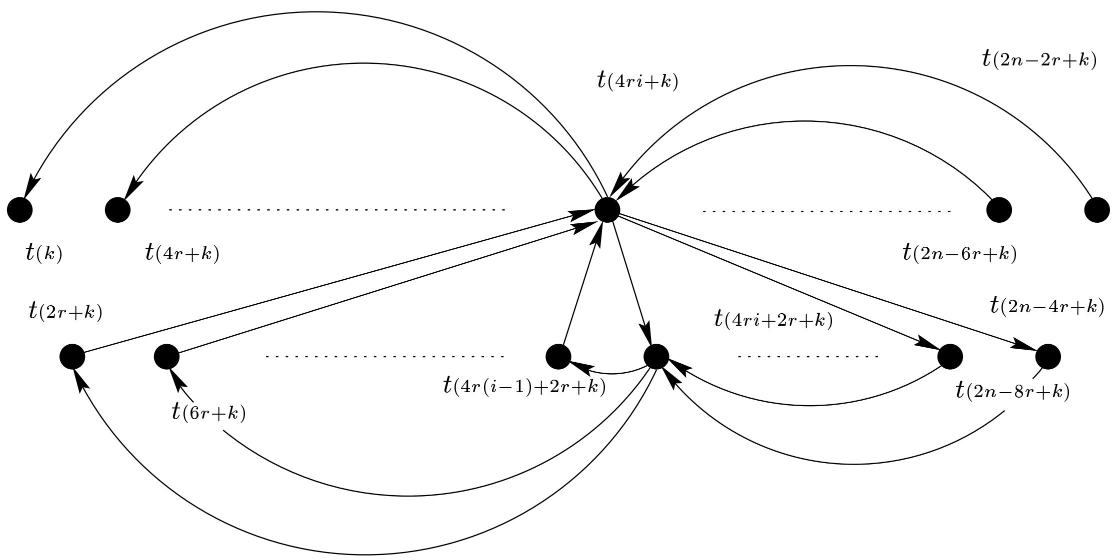

Figure 5: The diagram shows the multiple fan structure of the arcs of $T\langle O_k \rangle$ for the case when $f_k = 0$ from the proof of Theorem 4.7.

$t_{(2i+1)}$	$t_{(2i+2)}$	$t_{(r+2i+1)}$	$t_{(r+2i+2)}$	$t_{(2r+2i+1)}$	$t_{(2r+2i+2)}$	$t_{(3r+2i+1)}$	$t_{(3r+2i+2)}$	f_{i+1}	f_{2i+1}	f_{2i+2}	$f_{i+(r+1)/2}$	$f_{i+(r+3)/2}$
●	●	●	●	●	●	●	●					
○	⊗	×		⊗	○		×	0	0	0	0	0
⊗	○	×		○	⊗		×	1	0	0	0	0
	⊗	⊗		×	○	○	×	0	1	0	0	0
×	○	⊗			⊗	○	×	1	1	0	0	0
○	×	×	○	⊗			⊗	0	0	1	0	0
⊗		×	○	○	×		⊗	1	0	1	0	0
	×	⊗	○	×		○	⊗	0	1	1	0	0
×		⊗	○		×	○	⊗	1	1	1	0	0
○	⊗			⊗	○	×	×	0	0	0	1	0
⊗	○			○	⊗	×	×	1	0	0	1	0
	⊗	○		×	○	⊗	×	0	1	0	1	0
×	○	○			⊗	⊗	×	1	1	0	1	0
○	×		○	⊗		×	⊗	0	0	1	1	0
⊗			○	○	×	×	⊗	1	0	1	1	0
	×	○	○	×		⊗	⊗	0	1	1	1	0
×		○	○		×	⊗	⊗	1	1	1	1	0
○	⊗	×	×	⊗	○			0	0	0	0	1
⊗	○	×	×	○	⊗			1	0	0	0	1
	⊗	⊗	×	×	○	○		0	1	0	0	1
×	○	⊗	×		⊗	○		1	1	0	0	1
○	×	×	⊗	⊗			○	0	0	1	0	1
⊗		×	⊗	○	×		○	1	0	1	0	1
	×	⊗	⊗	×		○	○	0	1	1	0	1
×		⊗	⊗		×	○	○	1	1	1	0	1
○	⊗		×	⊗	○	×		0	0	0	1	1
⊗	○		×	○	⊗	×		1	0	0	1	1
	⊗	○	×	×	○	⊗		0	1	0	1	1
×	○	○	×		⊗	⊗		1	1	0	1	1
○	×		⊗	⊗		×	○	0	0	1	1	1
⊗			⊗	○	×	×	○	1	0	1	1	1
	×	○	⊗	×		⊗	○	0	1	1	1	1
×		○	⊗		×	⊗	○	1	1	1	1	1

Table 1: The diagram displays vertices in $N^+(t_{(0)}) \cap N^+(t_{(1)})$ for $2^5 = 32$ possible combinations of values of f_{i+1}, f_{2i+1}, f_{2i+2}, $f_{i+(r+1)/2}$, and $f_{i+(r+3)/2}$ from the proof of Theorem 5.11.

$t_{(r)}$	$t_{(r+1)}$		$t_{(2r)}$	$t_{(2r+1)}$		$t_{(3r)}$	$t_{(3r+1)}$		$t_{(4r-1)}$	$t_{(4r)}$		f_r	$f_{(r+1)/2}$
●	●		●	●		●	●		●	●			
○	×		×	⊗		⊗			×			*0*	*0*
	×		○	⊗		×				⊗		*1*	*0*
⊗			×	⊗		○	×		×			*0*	*1*
×			○	⊗			×			⊗		*1*	*1*

Table 2: The diagram displays vertices in $N^+(t_{(0)}) \cap N^+(t_{(1)})$ for $2^2 = 4$ possible combinations of values of f_r and $f_{(r+1)/2}$, and $f_1 = 0$, from the proof of Theorem 5.11.

$t_{(2)}$		$t_{(r+2)}$		$t_{(2r+2)}$		$t_{(3r+2)}$		f_2	$f_{(r+3)/2}$
●		●		●		●			
⊗				○		×		*0*	*0*
×		○				⊗		*1*	*0*
⊗		×		○				*0*	*1*
×		⊗				○		*1*	*1*

Table 3: The diagram displays vertices in $N^+(t_{(0)}) \cap N^+(t_{(1)})$ for $2^2 = 4$ possible combinations of values of f_2 and $f_{(r+3)/2}$, and $f_1 = 0$, from the proof of Theorem 5.11.