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WALECKI TOURNAMENTS: PART IV

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Abstract.

Walecki tournaments were defined by Alspach in 1966. They are regular tournaments which admit Hamilton directed cycle decomposition. Hence, they speak in favor of Kelly's conjecture from 1964. The enumeration of Walecki tournaments was presented as an open problem in a paper by Alspach in 1989. These two problems led us to study the arc structure of zero and odd pattern Walecki tournaments in the preceding papers. In this paper we determine the arc structure of subtournaments of even pattern Walecki tournaments. Some of them are almost regular, or the scores of their vertices differ for at most two. A specific permutation is proven to be an automorphism of even pattern Walecki tournaments.

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1 Introduction

In the preceding papers we studied the arc structure of Walecki tournaments with zero and odd pattern. In this paper we determine the arc structure of subtournaments of even pattern Walecki tournaments. Proving techniques differ from the ones used in the odd pattern case. An appropriate power of permutation τ is proven to be an automorphism of even pattern Walecki tournaments. Also, subtournaments induced by the outset or inset of the vertex $t(0)$ are either almost regular or have of their vertices differ by at most 2.

2 Even pattern

In this paper we consider Walecki tournaments $W(e)$ for n even, $e = f\overline{f}\dots f\overline{f} \in E_n$, $f \in E_r$, and $n/2r$ odd. Special form of e implies various symmetries in the corresponding Walecki tournament.

Proposition 2.1 *Let $m = 2r$ and let n be even. If $e = f\overline{f}\dots f\overline{f} \in E_n$, $f \in E_r$, and $n/2r$ is odd, then $e_i = e_{i+m}$, $1 \le i \le n-m$, and $e_i = \overline{e}_{i+n/2}$, $1 \le i \le n/2$,*

PROOF. Since $n/2r$ is odd we have

$$e = \underbrace{f\overline{f}\dots f\overline{f}f}_{n/2r}\underbrace{\overline{f}f\overline{f}\dots f\overline{f}}_{n/2r}.$$

The construction of Walecki tournaments and $f \in E_r$ imply the result. $\square$

In order to determine subtournaments of $W(e)$ that are isomorphic to some Walecki tournament with even pattern, we proceed in a manner similar to the odd pattern case (see Aleš [4]).

Theorem 2.2 *Let n be even, $e = f\overline{f}\dots f\overline{f} \in E_n$, and $f \in E_r$. Let T denote the Walecki tournament $W(e)$ and let $M_1, M_2, \dots, M_{n/m}$ be the m-set partition of $V(W(e)) - \{t_{(0)}\}$. If ℓ is an even integer such that $2 \le \ell \le n/r - 2$, then*

$$T\langle \{t_{(0)}\} \cup M_{n/2r-\ell/2+1} \cup M_{n/2r-\ell/2+2} \cup \dots \cup M_{n/2r+\ell/2} \rangle \cong W(\underbrace{f\overline{f}\dots f\overline{f}}_{\ell}).$$

Proof. Let ℓ be an even integer such that $2 \le \ell \le n/r-1$. Let $e' = f\overline{f}\dots f\overline{f} \in E_{\ell r}$. In the following τ is first considered as a permutation in $\mathbb{S}_{2n+1}$ used in the construction of $W(e)$. Secondly, it is considered as a permutation in $\mathbb{S}_{2\ell r+1}$ and is denoted by $\overline{\tau}$. Similarly, $t_{(i)}$ denotes a vertex of $W(e)$ and $\overline{t}_{(i)}$ denotes a vertex of $W(e')$. Note that the vertices of the subtournament $W(e')$ are chosen in $W(e)$ consecutively on the circumference starting from vertex $t_{(n-\ell r+1)}$.

We define a function

$$\psi : \{t_{(0)}\} \cup M_{n/2r-\ell/2+1} \cup M_{n/2r-\ell/2+2} \cup \dots \cup M_{n/2r+\ell/2} \longrightarrow V(W(e'))$$

by $\psi(t_{(0)}) = \overline{t}_{(0)}$ and $\psi(t_{(n+i)}) = \overline{t}_{(\ell r+i)}$, for $-\ell r + 1 \le i \le \ell r$. Clearly, ψ is a bijection. We will show that the Hamilton directed cycle $\overrightarrow{H}'_k$ in $W(e')$ is a union of ψ-images of directed paths belonging to Hamilton directed cycles $\overrightarrow{H}_{n-\ell r+k}$ and $\overrightarrow{H}_k$ in $W(e)$. The proof is similar to the proof of Theorem 3.2 from [4]. We omit the details.

Let $\overrightarrow{P}_{n-\ell r+k}$ denote the directed path $[t_{(0)}, t_{(n-\ell r+k)}, \dots, t_{(n-\ell r+2k)}]$ on $\overrightarrow{H}_{n-\ell r+k}$ and let $\overrightarrow{P}_k$ denote the directed path $[t_{(n-\ell r+2k)}, \dots, t_{(n+k)}, t_{(0)}]$ on $\overrightarrow{H}_k$. We have $\psi(\overrightarrow{P}_{n-\ell r+k}) = \overrightarrow{P}'_{n-\ell r+k}$ and $\psi(\overrightarrow{P}_k) = \overrightarrow{P}'_k$. Now, n and ℓ are even, and e has even pattern. This implies that if $e_k = 0$, then $e_{n-\ell r+k} = 0$ and $\overrightarrow{H}'_k = \overrightarrow{P}'_{n-\ell r+k} \cup \overrightarrow{P}'_k$ (see Figure 1). If $e_k = 1$, then $e_{n-\ell r+k} = 1$ and $\overrightarrow{H}'_k = \overleftarrow{P}'_k \cup \overleftarrow{P}'_{n-\ell r+k}$. This completes the proof. □

Even pattern Walecki tournaments not only contain even pattern subtournaments, but also odd pattern ones, as the following result demonstrates.

Theorem 2.3 *Let $n \geq 6$ and let $e = f\overline{f} \ldots f\overline{f} \in E_n$, and $f \in E_r$. Let T denote the Walecki tournament $W(e)$ and let $M_1, M_2, \ldots, M_{n/m}$ be the m-set partition of $V(W(e)) - \{t(0)\}$. If ℓ is an odd integer such that $1 \leq \ell \leq n/2r$, then*

$$T\langle \{t(0)\} \cup M_1 \cup M_2 \cup \cdots \cup M_\ell \rangle \cong W(\underbrace{f\overline{f} \ldots \overline{f}f}_{\ell}).$$

Proof. The proof is similar to that of the odd pattern (see Theorem 4.5 from [4]) the only difference being that instead of $1 \leq \ell \leq n/r$ we now have $1 \leq \ell \leq n/2r$. □

Even pattern Walecki tournaments also contain proper zero pattern Walecki subtournaments.

Theorem 2.4 *For $n \in \mathbb{Z}^+$, $n \geq 6$, $e = f\overline{f} \ldots f\overline{f} \in E_n$, $f \in E_r$, $n/2r$ even, and $n/2r > 1$, $h = (0, 0, \ldots, 0) \in E_{n/2r}$, and $R_\ell = \{t(2rk+\ell) \mid 0 \leq k \leq n/r - 1\}$,*

$$W(e)\langle \{t(0)\} \cup R_\ell \rangle \cong W(h),$$

where ℓ is an integer such that $1 \leq \ell \leq 2r$.

Proof. Let T denote $W(e)\langle \{t(0)\} \cup R_\ell \rangle$ with the corresponding defining binary sequence $e' = (e_\ell, e_{2r+\ell}, \ldots, e_{n-2r+\ell}) = (f_\ell, f_\ell, \ldots, f_\ell)$ for some ℓ such that $1 \leq \ell \leq 2r$. Since $W(e) \cong \overline{W(e)}$ we may assume that $f_\ell = 0$. All arcs in T are either defined with binary sequence elements $e_{2ri+\ell}$ for some i, $1 \leq i \leq n/2r - 1$, or $e_{2rj+r+\ell}$ for some j, $0 \leq j \leq n/2r - 2$. These elements equal either f_ℓ or

$f_{r+\ell} = \overline{f}_\ell$. We omit the details. $\square$

3 Automorphism σ^m

We remind the reader of the definition of the permutation σ and $\tau \in \mathbb{S}_{2n+1}$ (see Aleš [2]): $\sigma = (1\ 2\ 4\ \cdots 2n-4\ 2n-2)(3\ 5\ 7 \cdots\ 2n-3\ 2n-1\ 2n)(0)$ and $\tau = (1\ 2\ \cdots\ 2n)(0)$.

Proposition 3.5 *If m divides n, then*

$$\sigma^m(t_{(i)}) = \begin{cases} \tau^m(t_{(i)} & \text{if } 1 \leq i \leq n-m, \\ \tau^{m-n}(t_{(i)}) & \text{if } n-m+1 \leq i \leq n, \\ \tau^{n-m}(t_{(i)}) & \text{if } n+1 \leq i \leq n+m, \\ \tau^{-m}(t_{(i)}) & \text{if } n+m+1 \leq i \leq 2n. \end{cases} \tag{3.1}$$

PROOF. The result follows directly from the definitions of permutations σ and τ. $\square$

In the following three results we prove that σ^m is an element of the automorphism group of certain Walecki tournaments with even pattern. The result is a generalization of the proof of Theorem 5.23 from [2].

Lemma 3.6 *Let n be even, $e = f\overline{f}\ldots f\overline{f} \in E_n$, $f \in E_r$, and $m = 2r$. If $1 \leq i \leq n-m-1$, then τ^m is dominance-preserving on $\overrightarrow{H}_i$ and τ^{-m} is dominance-preserving on $\overrightarrow{H}_{i+m}$.*

PROOF. Let $m = 2r$. Since $e = f\overline{f}\ldots f\overline{f} \in E_n$ and $f \in E_r$, we have $e_i = e_{i+m}$ for $1 \leq i \leq n-m$. The result now follows from the definition of the Hamilton directed cycles $\overrightarrow{H}_1, \overrightarrow{H}_2, \ldots, \overrightarrow{H}_n$ comprising $W(e)$. $\square$

Lemma 3.7 *Let n be even, $e = f\overline{f}\dots f\overline{f} \in E_n$, $f \in E_r$, and $m = 2r$. Consider the Hamilton directed cycle $\vec{H}_i$, $1 \le i \le n$, in the Walecki tournament $W(e)$. Let u and w be end vertices of any arc on $\vec{H}_i$ of the form $u = v_{(\tau^{i-1}(2j+1))}$ and $w = v_{(\tau^{i-1}(2j+2))}$, $0 \le j \le n-m-1$, or $u = v_{(\tau^{i-1}(2j))}$ and $w = v_{(\tau^{i-1}(2j+1))}$, $1 \le j \le n-m-1$. Define ϱ by letting $\varrho = \tau^{-m}$ on $v_{(\tau^{i-1}(2j+1))}$, $1 \le j \le n-m-1$, and $\varrho = \tau^{m}$ on $v_{(\tau^{i-1}(2j))}$, $1 \le j \le n-m-1$. Then ϱ is dominance-preserving on the arc joining u and w.*

PROOF. The arc joining $\varrho(u)$ and $\varrho(w)$ also lies on $\vec{H}_i$. Moreover, $\varrho(u) \longrightarrow \varrho(w)$ if and only if $u \longrightarrow w$ which follows from the way $\vec{H}_i$ is constructed. More rigorously, the fact that $2j+2m+2 \le 2n$, and the chain of equalities

$$\varrho(u) = \varrho\big(v_{(\tau^{i-1}(2j+1))}\big) = \tau^{-m}\big(v_{(\tau^{i-1}(2j+1))}\big)$$

$$= v_{(\tau^{-m}(\tau^{i-1}(2j+1)))} = v_{(\tau^{i-1}(\tau^{-m}(2j+1)))} = v_{(\tau^{i-1}(2j+2m+1))}$$

and

$$\varrho(w) = \varrho\big(v_{(\tau^{i-1}(2j+2))}\big) = \tau^{m}\big(v_{(\tau^{i-1}(2j+2))}\big)$$

$$= v_{(\tau^{m}(\tau^{i-1}(2j+2)))} = v_{(\tau^{i-1}(\tau^{m}(2j+2)))} = v_{(\tau^{i-1}(2j+2m+2))}$$

imply that the arc joining $\varrho\big(v_{(\tau^{i-1}(2j+1))}\big)$ and $\varrho\big(v_{(\tau^{i-1}(2j+2))}\big)$ lies on the Hamilton directed cycle $\vec{H}_i$, $1 \le j \le n-m-1$. Arcs on the Hamilton directed cycle $\vec{H}_i$ are either of the form $v_{(\tau^{i-1}(k))} \longrightarrow v_{(\tau^{i-1}(k+1))}$ or $v_{(\tau^{i-1}(k+1))} \longrightarrow v_{(\tau^{i-1}(k))}$, depending on e_i. Therefore, $v_{(\tau^{i-1}(2j+1))} \longrightarrow v_{(\tau^{i-1}(2j+2))}$ if and only if $\varrho\big(v_{(\tau^{i-1}(2j+1))}\big) \longrightarrow \varrho\big(v_{(\tau^{i-1}(2j+2))}\big)$.

Similarly, we prove the remaining case. We have $\varrho\big(v_{(\tau^{i-1}(2j))}\big) = v_{(\tau^{i-1}(2j+2m))}$ and $\varrho\big(v_{(\tau^{i-1}(2j+1))}\big) = v_{(\tau^{i-1}(2j+2m+1))}$, it follows that the arc joining $\varrho\big(v_{(\tau^{i-1}(2j))}\big)$

and $\varrho(v_{(\tau^{i-1}(2j+1))})$ lies on the Hamilton directed cycle $\overrightarrow{H}_i$, $1 \leq j \leq n-m-1$. Furthermore, $v_{(\tau^{i-1}(2j))} \longrightarrow v_{(\tau^{i-1}(2j+1))} \in \overrightarrow{H}_i$ if and only if $\varrho(v_{(\tau^{i-1}(2j))}) \longrightarrow \varrho(v_{(\tau^{i-1}(2j+1))}) \in \overrightarrow{H}_i$, for $1 \leq j \leq n-m-1$. This proves that ϱ is dominance-preserving on the arc joining u and w. $\square$

Lemma 3.8 *Let n be even, $e = f\overline{f}\ldots f\overline{f} \in E_n$, $f \in E_r$, and $m = 2r$. Consider the Hamilton directed cycle $\overrightarrow{H}_i$, $1 \leq i \leq n$, in the Walecki tournament $W(e)$. Let u and w be end vertices of any arc on $\overrightarrow{H}_i$ of the form $u = v_{(\tau^{i-1}(2j+1))}$ and $w = v_{(\tau^{i-1}(2j+2))}$, $n-m+1 \leq j \leq n-2$, or $u = v_{(\tau^{i-1}(2j))}$ and $w = v_{(\tau^{i-1}(2j+1))}$, $n-m+1 \leq j \leq n-1$. Define ϱ by letting $\varrho = \tau^{n-m}$ on $v_{(\tau^{i-1}(2j+1))}$, $n-m+1 \leq j \leq n-1$, and on $v_{(\tau^{i-1}(2n))}$. Furthermore, define $\varrho = \tau^{m-n}$ on $v_{(\tau^{i-1}(2j))}$, $n-m+1 \leq j \leq n-1$. Then ϱ is dominance-preserving on the arc joining u and w.*

Proof. Since $2j-2(n-m) \geq 2$ we have $\varrho(v_{(\tau^{i-1}(2j+1))}) = v_{(\tau^{i-1}(2j-2(n-m)+1))}$ and $\varrho(v_{(\tau^{i-1}(2j+2))}) = v_{(\tau^{i-1}(2j-2(n-m)+2))}$, which implies that the arc joining $\varrho(v_{(\tau^{i-1}(2j+1))})$ and $\varrho(v_{(\tau^{i-1}(2j+2))})$ lies on the Hamilton directed cycle $\overrightarrow{H}_i$, for $n-m+1 \leq j \leq n-1$. Moreover, $v_{(\tau^{i-1}(2j+1))} \longrightarrow v_{(\tau^{i-1}(2j+2))} \in \overrightarrow{H}_i$ if and only if $\varrho(v_{(\tau^{i-1}(2j+1))}) \longrightarrow \varrho(v_{(\tau^{i-1}(2j+2))}) \in \overrightarrow{H}_i$.

We also need to consider arcs joining vertices $v_{(\tau^{i-1}(2n-1))}$ and $v_{(\tau^{i-1}(2n))}$. Since $\varrho(v_{(\tau^{i-1}(2n-1))}) = v_{(\tau^{n-m+i-1}(2n-1))}$ and $\varrho(v_{(\tau^{i-1}(2n))}) = v_{(\tau^{n-m+i-1}(2n))}$, the arc joining $\varrho(v_{(\tau^{i-1}(2n-1))})$ and $\varrho(v_{(\tau^{i-1}(2n))})$ lies on the Hamilton directed cycle $\overrightarrow{H}_{n-m+i}$ whenever $1 \leq i \leq m$. Moreover, since $e_{n-m+i} = e_{m(n/m-1)+i} = e_i$, the Hamilton directed cycles $\overrightarrow{H}_i$ and $\overrightarrow{H}_{n-m+i}$ have the same orientation for $1 \le i \le m$. It follows that $v(\tau^{i-1}(2n-1)) \longrightarrow v(\tau^{i-1}(2n)) \in \overrightarrow{H}_i$ if and only if $\varrho(v(\tau^{i-1}(2n-1))) \longrightarrow \varrho(v(\tau^{i-1}(2n))) \in \overrightarrow{H}_{n-m+i}$, for $1 \le i \le m$.

If $m+1 \le i \le n$ then the considered arc lies on the Hamilton directed cycle $\overrightarrow{H}_{i-m}$. Since $e_{i-m} = e_i$, the Hamilton directed cycles $\overrightarrow{H}_{i-m}$ and $\overrightarrow{H}_i$ have the same orientation. It follows that $v(\tau^{i-1}(2n-1)) \longrightarrow v(\tau^{i-1}(2n)) \in \overrightarrow{H}_i$ if and only if $\varrho(v(\tau^{i-1}(2n-1))) \longrightarrow \varrho(v(\tau^{i-1}(2n))) \in \overrightarrow{H}_{i-m}$, for $m+1 \le i \le n$.

Similarly we prove the remaining case. Since $\varrho(v(\tau^{i-1}(2j))) = v(\tau^{i-1}(2j-2(n-m)))$ and $\varrho(v(\tau^{i-1}(2j+1))) = v(\tau^{i-1}(2j-2(n-m)+1))$, the arc joining $\varrho(v(\tau^{i-1}(2j)))$ and $\varrho(v(\tau^{i-1}(2j+1)))$ lies on the Hamilton directed cycle $\overrightarrow{H}_i$, $n-m+1 \le j \le n-1$. Moreover, $v(\tau^{i-1}(2j)) \longrightarrow v(\tau^{i-1}(2j+1)) \in \overrightarrow{H}_i$ if and only if $\varrho(v(\tau^{i-1}(2j))) \longrightarrow \varrho(v(\tau^{i-1}(2j+1))) \in \overrightarrow{H}_i$. This proves that ϱ is dominance-preserving on the arc joining u and w. $\square$

Theorem 3.9 *Let n be even and let $\sigma \in \mathbb{S}_{2n+1}$, where*

$$\sigma = (1\ 2\ 4\ \cdots 2n-4\ 2n-2)(3\ 5\ 7 \cdots\ 2n-3\ 2n-1\ 2n)(0).$$

If $e = f\overline{f}\ldots\overline{f}f \in E_n$, $f \in E_r$, $n/2r$ is odd, and $m = 2r$, then σ^m is an element of the automorphism group $Aut(W(e))$.

Proof. We want to show that σ^m is dominance-preserving on all of $W(e)$. Let us partition the vertices of $V(W(e)) - \{t(0)\}$ into four sets: $U' = \{t(1), t(2), \ldots, t(n-m)\}$, $U'' = \{t(n-m+1), t(n-m+2), \ldots, t(n)\}$, $W' = \{t(n+m), t(n+m+1), \ldots, t(2n)\}$, and $W'' = \{t(n+1), t(n+2), \ldots, t(n+m-1)\}$. Let $U = U' \cup U''$ and $W = W' \cup W''$.

By definition, σ^m fixes $t(0)$. It cyclically permutes the vertices of U. That is, for a fixed integer i, $0 \le i \le m-1$, it cyclically permutes vertices $t(i+jm)$, $1 \le j \le$

n/m. The structure of e then implies that σ^m is dominance-preserving on the arcs joining $t(0)$ and vertices in U. Similarly, σ^m cyclically permutes the vertices of W, that is, it cyclically permutes vertices $t(i+mj)$, $n/m \leq j \leq 2n/m-1$, where $1 \leq i \leq m$. The structure of e implies that σ^m is dominance-preserving on the arcs joining $t(0)$ and vertices in W. Therefore, σ^m is dominance-preserving on the arcs incident with $t(0)$.

Note that σ^m restricted to U' has the same action as τ^m. It then follows from Lemma 3.6 that σ^m is dominance-preserving on any arc both of whose end vertices lie in U'. Similarly, σ^m is dominance-preserving on any arc both of whose end vertices lie in W'.

By Lemma 3.7, σ^m is dominance-preserving on any arc with one end vertex in U' and the other end vertex in W' because σ^m acts like τ^m on U' and τ^{-m} on W'. Similarly, σ^m acts like τ^{m-n} on U'' and τ^{n-m} on W''. Hence, Lemma 3.8 implies that σ^m is dominance-preserving on any arc between U'' and W''.

It remains to show that σ^m is dominance-preserving on the arcs with exactly one end vertex in either U'' or W''. There are many cases to consider. We prove one and leave the rest to the reader. Let $0 \leq i \leq m-1$.

Case 1. Let us first consider arcs joining vertices $v(\tau^{n+i}(1)) \in W''$ and $v(\tau^j(1)) \in W' \cup U'$. We consider two subcases depending on the parity of $j-i$.

Case 1.1. Let $j-i$ be even. We divide the proof into two more subcases depending on the range of index j.

Case 1.1.1. Let $n+m \leq j \leq 2n-1$. Since $n+i < j$, we can use determine the Hamilton directed cycle that contains the arc joining $v(\tau^{n+i}(1)) = v(2(n-i)+1)$

and $v(\tau^j(1)) = v(2(2n-j)+1)$. Let

$$k = \frac{i+j+n}{2} + 1. \tag{3.2}$$

Since $n+1 \le n+m/2+1 \le k \le (3n+m)/2 \le 2n$, the Hamilton directed cycle $\vec{H}_{k-n}$ contains the considered arc. We have to determine the orientation of this arc. Since $(-3n+1)/2 \le (i-j-n)/2 \le -n-1/2$ and since τ has period $2n$, we have $1 \le (n+1)/2 \le (i-j+3n)/2 \le n-1/2 < n$. Now, $\tau^j(1) = \tau^{k-n-1}(i-j+3n+1)$ and $\tau^{n+i}(1) = \tau^{k-n-1}(i-j+3n)$. Thus,

$$e_{k-n} = 0 \text{ if and only if } v(\tau^{n+i}(1)) \longrightarrow v(\tau^j(1)) \in \vec{H}_{k-n}\,. \tag{3.3}$$

Now let us consider the σ^m images of $v(\tau^{n+i}(1))$ and $v(\tau^j(1))$. Clearly,

$$\sigma^m\big(v(\tau^{n+i}(1))\big) = v(\tau^{2n-m+i}(1)) \tag{3.4}$$

and

$$\sigma^m\big(v(\tau^j(1))\big) = v(\tau^{j-m}(1)). \tag{3.5}$$

Using $j-m < 2n-m+i$ we determine the index of the Hamilton directed cycle containing $v(\tau^{2n-m+i}(1))$ and $v(\tau^{j-m}(1))$. If

$$\ell = \frac{i+j}{2} + n - m + 1 \tag{3.6}$$

then the considered arc lies on the Hamilton directed cycle $\vec{H}_{\ell-n}$ since $n+1 \le (3n-m)/2+1 \le \ell \le 2n-m/2 \le 2n$. Moreover, since $j-i$ is even, $-n-1/2 \le (i-j)/2 \le -(n+1)/2 < 0$ implies $v(\tau^{2n-m+i}(1)) = v(\tau^{\ell-n-1}(j-i+1))$ and $v(\tau^{j-m}(1)) = v(\tau^{\ell-n-1}(j-i))$. Therefore,

$$e_{\ell-n} = 1 \text{ if and only if } \sigma^m\big(v(\tau^{n+i}(1))\big) \longrightarrow \sigma^m\big(v(\tau^j(1))\big) \in \vec{H}_{\ell-n}\,. \tag{3.7}$$

To determine the orientation of the considered arcs one needs to compare e_{k-n} and $e_{\ell-n}$. Using equations (3.2) and (3.6) we consider two cases depending on the value of k. The bounds for k are $n+m/2+1 \leq k \leq (3n+m)/2$. If $k \leq 3n/2$, then Proposition 2.1 and inequalities $n/2 - m/2 + 1 \leq k - n/2 - m \leq n - m$ and $n/2 + m/2 + 1 \leq k - n/2 \leq n$ imply $e_{\ell-n} = \overline{e}_{k-n}$. Similarly, $3n/2 + 1 \leq k$ implies $e_{\ell-n} = \overline{e}_{k-n}$. Therefore, σ^m is dominance-preserving on the considered arcs (see Figure 2).

CASE 1.1.2. $0 \leq j \leq n - m - 1$.

CASE 1.2. Let $j - i$ be odd. We divide the proof into two subcases depending on the range of index j: $n + m \leq j \leq 2n - 1$ and $0 \leq j \leq n - m - 1$.

CASE 2. Next we consider arcs joining vertices $v(\tau^{n-m+i}(1)) \in U''$ and $v(\tau^j(1)) \in W' \cup U'$. There are two subcases depending on the parity of $j - i$.

CASE 2.1. Let $j - i$ be even. We divide the proof into two more subcases depending on the range of index j: $n + m \leq j \leq 2n - 1$ and $0 \leq j \leq n - m - 1$.

CASE 2.2. Let $j - i$ be odd. We divide the proof into two subcases depending on the range of index j: $n + m \leq j \leq 2n - 1$ and $0 \leq j \leq n - m - 1$.

Now, σ^m is dominance-preserving on the considered arcs. Therefore, σ^m is dominance-preserving on all of $W(e)$. This completes the proof. □

The significance of the permutation τ^m for the automorphism groups of Walecki tournaments with odd pattern can be recognized immediately from the odd pattern sequences. By contrast, the fact that σ^m is a permutation in the automorphism group of some Walecki tournaments with even pattern for n odd was previously unknown. However, once even pattern sequences were determined as a potential source of Walecki tournaments with non-trivial automorphism groups, permutation σ^m became a natural candidate for their generator.

4 Almost regular subtournaments

In order to prove that vertex $t_{(0)}$ must be fixed for any automorphism of a Walecki tournament T with even pattern we prove that the subtournament induced by the outsets of vertices on the circumference do not have the same structure as $T\langle N^+(v_{(0)})\rangle$.

Theorem 4.10 *Let $T = W(e)$ for $e = f\overline{f}\dots f\overline{f} \in E_n$, n even, and $f = (0, 0, \dots, 0) \in E_r$. Let $v \in V(T) - \{t_{(0)}\}$. If $r \geq 2$, then there exists a vertex in the subtournament $T\langle N^+(v)\rangle$ whose score equals $n/2r$. If $r = 1$, then there exists a vertex in the subtournament $T\langle N^+(v)\rangle$ whose score equals* 1.

PROOF. Let T denote $W(e)$. Since $T \cong \overline{T}$ it suffices to prove the theorem for vertices in $N^+(t_{(0)})$. Furthermore, since $\sigma^{2r} \in Aut(T)$, it suffices to prove the theorem for the vertices in $N^+(t_{(0)}) \cap M_1$ and $N^+(t_{(0)}) \cap M_{n/r}$. Let $M' = M_1 \cup M_3 \cup \dots \cup M_{n/r}$ and $M'' = M_2 \cup M_4 \cup \dots \cup M_{n/r-1}$.

We first consider $t_{(1)} \in N^+(t_{(0)}) \cap M_1$. We will count the vertices in $N^+(t_{(1)}) \cap N^+(t_{(2)})$. First we determine the vertices in $N^+(t_{(1)})$. Since $f = (0, 0, \dots, 0)$ Theorem 3.2 from [4] implies

$$N^+(t_{(1)}) \cap M_1 = \{t_{(2i+2)} \mid 0 \leq i \leq r-1\}, \tag{4.8}$$

and

$$N^+(t_{(1)}) \cap M_2 = \{t_{(2r+2i+1)} \mid 0 \leq i \leq r-1\}. \tag{4.9}$$

Let $X' = N^+(t_{(1)}) \cap M'$ and $X'' = N^+(t_{(1)}) \cap M''$. The even pattern of the sequence e implies

$$X' = \{t_{(4rj+2i+2)} \mid 0 \le i \le r-1, 0 \le j \le n/2r - 1\} \tag{4.10}$$

and

$$X'' = \{t_{(4rj+2i+1)} \mid 0 \le i \le r-1, 0 \le j \le n/2r - 1\}. \tag{4.11}$$

Clearly, $N^+(t_{(1)}) = X' \cup X''$ and $|N^+(t_{(1)})| = n$.

Next we determine vertices in $N^+(t_{(2)})$. Since $f = (0, 0, \ldots, 0)$ it follows that

$$N^+(t_{(2)}) \cap M_1 = \{t_{(2i+1)} \mid 1 \le i \le r-1\} \cup \{t_{(2r)}\}, \tag{4.12}$$

$$N^+(t_{(2)}) \cap M_2 = \{t_{(2r+2i+2)} \mid 0 \le i \le r-2\}, \tag{4.13}$$

$$N^+(t_{(2)}) \cap M_3 = \{t_{(4r+2i+1)} \mid 0 \le i \le r-1\} \cup \{t_{(6r)}\}, \tag{4.14}$$

and

$$N^+(t_{(2)}) \cap M_{n/r} = \{t_{(2n-2r+2i+2)} \mid 0 \le i \le r-1\}. \tag{4.15}$$

Let Y' denote $N^+(t_{(2)}) \cap M'$ and let Y'' denote $N^+(t_{(2)}) \cap M''$. We use equations (4.12) and (4.14) to prove the following two statements. Let $Y' = \tilde{Y}' \cup \tilde{\tilde{Y}}'$ where

$$\tilde{Y}' = N^+(t_{(2)}) \cap M_1 = \{t_{(2i+1)} \mid 1 \le i \le r-1\} \cup \{t_{(2r)}\} \tag{4.16}$$

and

$$\begin{aligned} \tilde{\tilde{Y}}' \;=\; & N^+(t_{(2)}) \cap (M' - M_1) = \\ & \{t_{(4rj+2i+1)} \mid 0 \le i \le r-1, 1 \le j \le n/2r - 1\} \cup \\ & \cup \{t_{(4rj+2r)} \mid 1 \le j \le n/2r - 1\}. \end{aligned} \tag{4.17}$$

The even pattern of the sequence e and equations (4.13) and (4.15) imply

$$Y'' = \{t_{(4rj+2r+2i+2)} \mid 0 \le i \le r-2, 0 \le j \le n/2r - 2\} \cup \tag{4.18}$$

$$\cup\{t_{(2n-2r+2i+2)} \mid 0 \le i \le r-2\} \cup \{t_{(2n)}\}.$$

Clearly, $N^+(t_{(2)}) = Y' \cup Y''$ and $|N^+(t_{(2)})| = n$.

By comparing the parity of powers of τ in equations (4.10), (4.11), (4.16), (4.17), and (4.18) for vertices in $N^+(t_{(1)})$ and $N^+(t_{(2)})$, we deduce $X'' \cap Y'' = \emptyset$. Moreover, setting $j = 0$ and $i = r-1$ in equation (4.10) implies $X' \cap \tilde{Y}' = \{t_{(2r)}\}$ and

$$X' \cap \tilde{\tilde{Y}}' = \{t_{(4rj+2r)} \mid 1 \le j \le n/2r - 1\}.$$

Furthermore, set $i = r-1$ in equation (4.10) and let $1 \le j \le n/2r - 1$ which implies

$$N^+(t_{(1)}) \cap N^+(t_{(2)}) = \{t_{(2r)}\} \cup \{t_{(4rj+2r)} \mid 1 \le j \le n/2r - 1\}.$$

Hence, the score of vertex $t_{(2)}$ in $T\langle N^+(t_{(1)})\rangle$ equals $n/2r$. The proofs for the remaining vertices of $N^+(t_{(0)}) \cap M_1$ and for the vertices in $N^+(t_{(0)}) \cap M_{n/r}$ are similar and we omit them. Since $\sigma^{2r} \in Aut(T)$, $s(\sigma^{2r}(t_{(2)})) = n/2r$ in $N^+(\sigma^{2r}(t_{(1)}))$. Applying a similar argument $n/2r$ times for each orbit for σ^{2r} proves the result.

The proof above suffices to show that $T\langle N^+(t_{(1)})\rangle$ is not almost regular if $r \ge 2$. The arc structure of $T\langle N^+(t_{(1)})\rangle$ is different in the case when $r = 1$, that is, when $e = (0,1,0,1,\ldots,0,1,0) \in E_n$. We consider $N^+(t_{(1)}) \cap N^+(t_{(2n-1)})$. The pattern of e implies

$$N^+(t_{(1)}) = \{t_{(4i+2)}, t_{(4i+3)} \mid 0 \le i \le n/2 - 1\} \tag{4.19}$$

and

$$N^+(t_{(2n-1)}) = \{t_{(0)}\} \cup \{t_{(2)}\} \cup \{t_{(4(i+1))}, t_{(4(i+1)+1)} \mid 0 \le i \le n/2 - 2\}. \tag{4.20}$$

By comparing the parity of powers of τ in equations (4.19) and (4.20) for vertices in $N^+(t_{(1)})$ and $N^+(t_{(2n-1)})$ we deduce that $N^+(t_{(1)}) \cap N^+(t_{(2n-1)}) = \{t_{(2)}\}$. Hence, the score of vertex $t_{(2n-1)}$ in $T\langle N^+(v_{(1)})\rangle$ equals 1. Since $\sigma^2 \in Aut(T)$, $s(\sigma^2(t_{(2n-1)})) = 1$ in $N^+(\sigma^2(t_{(1)}))$. Using a similar argument $n/2$ times for each orbit for σ^2 proves the result. $\square$

Examples of tournaments for Theorem 4.10 are $T_6 = W(010101)$ for $r = 1$, with the score sequence of $T_6\langle N^+(t_{(1)})\rangle$ being $(s(t_{(11)}), s(t_{(2)}), s(t_{(12)}), s(t_{(6)}), s(t_{(10)}), s(t_{(7)})) = (1, 2, 2, 2, 3, 5)$, and $T_9 = W(000111000)$ for $r = 3$ with the score sequence of $T_9\langle N^+(t_{(1)})\rangle$ being $(s(t_{(2)}), s(t_{(4)}), s(t_{(15)}), s(t_{(7)}), s(t_{(17)}), s(t_{(5)}), s(t_{(14)}), s(t_{(12)})) = (2, 3, 3, 3, 4, 4, 4, 5)$.

Next we consider the subtournaments of a Walecki tournament $W(e)$ with an even pattern ($e = f\overline{f}\ldots f\overline{f} \in E_n$ and $f \in E_r$) induced by $N^+(t_{(0)})$ and $N^-(t_{(0)})$. Since n is even, $W(e)\langle N^+(t_{(0)})\rangle$ can not be regular. Moreover, it is not necessarily almost regular in general. For example, in the tournament $T_8 = W(00011110)$ with $n = 8$ and $r = 4$ the score sequence of $T_8\langle N^+(t_{(0)})\rangle$ is $(s(t_{(1)}), s(t_{(3)}), s(t_{(15)}), s(t_{(2)}), s(t_{(14)}), s(t_{(13)}), s(t_{(12)}), s(t_{(8)})) = (2, 3, 3, 4, 4, 4, 4, 4)$. $W(e)\langle N^+(t_{(0)})\rangle$ is not necessarily almost regular even in the case when f has zero pattern . An example of such a Walecki tournament is $T'_8 = W(01010101)$ with $n = 8$ and $r = 1$ whose subtournament $T'_8\langle N^+(t_{(0)})\rangle$ has the score sequence $(s(t_{(3)}), s(t_{(7)}), s(t_{(1)}), s(t_{(16)}), s(t_{(14)}), s(t_{(5)}), s(t_{(12)}), s(t_{(10)})) = (2, 2, 4, 4, 4, 4,$

$4, 4)$. Notice, that the scores in $T'_8\langle N^+(t_{(0)})\rangle$ differ by at most 2. This turns out to be true in general if f has a zero pattern. We prove a slightly stronger statement for the special case when $n/2r$ and r are odd in the following result.

Theorem 4.11 *Let $n \geq 6$, n even, and let $W(e)$ be a Walecki tournament where $e = f\overline{f} \dots f\overline{f} \in E_n$. The bounds on a score of vertex $v \in N^+(t_{(0)})$ in the subtournament $W(e)\langle N^+(t_{(0)})\rangle$ are*

$$n/2 - 2 \leq s(v) \leq n/2.$$

Moreover, if $n/2r$ and r are odd, then the subtournaments $W(e)\langle N^+(t_{(0)})\rangle$ are almost regular.

PROOF. Let $W(e)$ be a tournament as stated in the conditions of the theorem and let $U = M_1 \cup M_2 \cup \cdots \cup M_{n/2r}$ and $W = M_{n/2r+1} \cup M_{n/2r+2} \cup \cdots \cup M_{n/r}$. Assume first that $n/2r$ and r are odd.

We first consider the case when $r = 1$, that is, $e = (0, 1, 0, 1, \dots, , 0, 1) \in E_n$. The pattern of e implies that the out-neighbours of $v_{(0)}$ are

$$N^+(t_{(0)}) \cap U = \{t_{(2i+1)} \mid 0 \leq i \leq n/2 - 1\} \tag{4.21}$$

and

$$N^+(t_{(0)}) \cap W = \{t_{(n+2i+2)} \mid 0 \leq i \leq n/2 - 1\}. \tag{4.22}$$

Since e has even pattern, Theorem 2.3 implies

$$W(e)\langle\{t_{(0)}\} \cup U\rangle \cong W(\underbrace{f\overline{f} \dots \overline{f}f}_{n/2}).$$

Moreover, since $n/2$, is odd Theorem 5.11 from [4] implies that $W(e)\langle\{t_{(0)}\} \cup U\rangle$ is a regular tournament of degree $n/2$ and $W(e)\langle N^+(t_{(0)}) \cap U\rangle$ is a regular tournament of degree $(n/2-1)/2$. Similar to the previous proof we can prove $|N^+(t_{(0)}) \cap N^+(t_{(1)})| = n/2 - 1$. Since $\sigma^2 \in Aut(W(e))$, we have $|N^+(t_{(0)}) \cap$

$N^+(v)| = n/2-1$ for all vertices $v \in N^+(t_{(0)}) \cap U$. One can also prove $|N^+(t_{(0)}) \cap N^+(t_{(2n)})| = n/2$. Since $\sigma^2 \in Aut(W(e))$, we have $|N^+(t_{(0)}) \cap N^+(v)| = n/2$ for all vertices $v \in N^+(t_{(0)}) \cap W$. This proves that $W(e)\langle N^+(t_{(0)})\rangle$ is almost regular in the case when $r = 1$.

Assume next $r \geq 3$ and r odd. Since e has even pattern, Theorem 2.3 implies

$$W(e)\langle \{t_{(0)}\} \cup U\rangle \cong W(\underbrace{f\overline{f}\ldots\overline{f}f}_{n/2}).$$

Moreover, since $n/2$ is odd, Theorem 5.11 from [4] implies that $W(e)\langle N^+(t_{(0)}) \cap U\rangle$ is a regular tournament of degree $(n/2-1)/2$. Therefore,

$$|N^+(t_{(0)}) \cap N^+(t_{(1)}) \cap U| = (n/2-1)/2. \tag{4.23}$$

We leave it to the reader to prove

$$|N^+(t_{(0)}) \cap N^+(t_{(1)}) \cap W| = (n/2-1)/2. \tag{4.24}$$

Equations (4.23) and (4.24) imply $|N^+(t_{(0)}) \cap N^+(t_{(1)})| = n/2-1$. Since $\sigma^{2r} \in Aut(W(e))$, we have $|N^+(t_{(0)}) \cap N^+(v)| = n/2-1$ for all vertices $v \in O_{t(1)}$, where $O_{t(1)}$ is the orbit of $t_{(1)}$ for the permutation σ^{2r}. Similarly we obtain $|N^+(t_{(0)}) \cap N^+(t_{(2n)})| = \frac{n}{2}$. Hence, the score of $t_{(2n)}$ in the subtournament $W(e)\langle N^+(t_{(0)})\rangle$ equals $n/2$. We have determined the scores of vertices $t_{(1)} \in M_1$ and $t_{(2n)} \in M_{n/r}$. Similarly we can determine scores for all vertices in M_1 and $M_{n/r}$. The scores of the vertices in each of the two sets alternate between $n/2-1$ and $n/2$. Then $\sigma^{2r} \in Aut(W(e))$ implies that the number of vertices with score $n/2$ is $n/2$, which proves that $W(e)\langle N^+(t_{(0)})\rangle$ is almost regular.

In the case when $n/2r$ or r is even, the subtournament on the outset of vertex $t_{(0)}$ is not necessarily almost regular. However, we will prove that the scores differ by at most 2. Let $Y = N^+(t(0)) \cap N^+(t(1))$. In a way similar to the proof of Theorem 5.12 from [4], one can prove $|Y \cap (M_i \cup M_{i+1})| = r$, for $1 \leq i \leq n/2r - 1$. Now $|M_i \cup M_{i+1}| = 4r$, for $1 \leq i \leq n/2r - 1$, which implies

$$|Y \cap U| \leq n/4. \tag{4.25}$$

Also, $|Y \cap (M_i \cup M_{i+1})| = r$, for $n/2r + 1 \leq i \leq n/r - 1$ implying

$$|Y \cap W| \leq n/4. \tag{4.26}$$

We deduce that

$$n/4 - 1 \leq |Y \cap U| \tag{4.27}$$

and

$$n/4 - 1 \leq |Y \cap W|. \tag{4.28}$$

Equations (4.25), (4.26), (4.27), and (4.28) imply $n/2 - 2 \leq s(t(1)) = |N^+(t(0)) \cap N^+(t(1))| \leq n/2$. A similar argument can be applied to any vertex $v \in N^+(t(0)) \cap M_1$ $v \in N^+(t(0)) \cap M_{n/r}$. Since $\sigma^{2r} \in Aut(W(e))$, bounds for the scores of any vertex $v \in N^+(t(0))$ are $n/2 - 2 \leq s(t(1)) \leq n/2$. This completes the proof. □

5 Transitive subtournaments and multiple fan structure

Let $U = M_1 \cup M_2 \cup \cdots \cup M_{n/2r}$ and $W = M_{n/2r+1} \cup M_{n/2r+2} \cup \cdots \cup M_{n/r}$. In a way similar to the odd pattern case we partition the vertices of $V(W(e)) - \{t(0)\} = U \cup W$ into sets $Q_1, Q_2, \ldots, Q_m \subseteq U$ and $R_1, R_2, \ldots, R_m \subseteq W$ of cardinality n/m, where $m = 2r$. These sets are orbits for the permutation σ^m. However,

σ^m is an automorphism of $W(e)$ only in the case (see Aleš [4]) when n/m is odd. For this reason we call them *pre-orbits*.

We remind the reader of the definition of the permutation $\zeta \in \mathbb{S}_{2r}$ acting on the set $\{1, 2, \ldots, 2r\}$: $\zeta = (1\ \ r+1)(2\ \ r+2)\cdots(r\ \ 2r)$. Pre-orbits for σ^m can then be denoted by $Q_{\zeta^{f_k}(k)}$, $Q_{\zeta^{\overline{f_k}}(k)}$, $R_{\zeta^{f_k}(k)}$, $R_{\zeta^{\overline{f_k}}(k)}$, for $1 \le k \le r$. For a clearer understanding of the structure of pre-orbits, we write a vertex representative belonging to either m-set M_1 or $M_{n/2r+1}$ for each pre-orbit: $t_{(rf_k+k)} \in Q_{\zeta^{f_k}(k)} \cap M_1, t_{(r\overline{f_k}+k)} \in Q_{\zeta^{\overline{f_k}}(k)} \cap M_1, t_{(n+r\overline{f_k}+k)} \in R_{\zeta^{f_k}(k)} \cap M_{n/2r+1}, t_{(n+rf_k+k)} \in R_{\zeta^{\overline{f_k}}(k)} \cap M_{n/2r+1}$, where $1 \le k \le r$. All vertices for a particular pre-orbit can then be obtained by applying σ^m $n/2r - 1$ times on a vertex representative. It follows that $Q_{\zeta^{f_k}(k)}$, $R_{\zeta^{f_k}(k)} \subseteq N^+(v_{(0)})$, and $Q_{\zeta^{\overline{f_k}}(k)}$, $R_{\zeta^{\overline{f_k}}(k)} \subseteq N^-(v_{(0)})$. Therefore,

$$N^+(v_{(0)}) = \bigcup_{k=1}^{r} \left(Q_{\zeta^{f_k}(k)} \cup R_{\zeta^{f_k}(k)} \right) \tag{5.29}$$

and

$$N^-(v_{(0)}) = \bigcup_{k=1}^{r} \left(Q_{\zeta^{\overline{f}_k}(k)} \cup R_{\zeta^{\overline{f}_k}(k)} \right). \tag{5.30}$$

The multiple fan arc structure that is present in Walecki tournaments with odd pattern also occurs in the even pattern case. We omit proofs since they are similar to the proofs of Theorem 4.7 from [4].

Theorem 5.12 *Let $n \ge 6$, n even, and let T denote the Walecki tournament $W(e)$ for $e \in E_n$. If $e = f\overline{f}\ldots f\overline{f} \in E_n$, then the pre-orbits $Q_1, Q_2, \ldots, Q_m$ and $R_1, R_2, \ldots, R_m$ for the permutation σ^m induce regular subtournaments $T\langle Q_1\rangle$, $T\langle Q_2\rangle, \ldots, T\langle Q_m\rangle, T\langle R_1\rangle, T\langle R_2\rangle, \ldots, T\langle R_m\rangle$. If ℓ is an integer such that $1 \le \ell \le m$, the subtournaments $T\langle Q_\ell \cap N^+(t_{(1)})\rangle$, $T\langle Q_\ell \cap N^-(t_{(1)})\rangle$, $T\langle R_\ell \cap N^+(t_{(1)})\rangle$,* *and $T\langle R_\ell \cap N^-(t_{(1)})\rangle$ are transitive. Furthermore, arcs between $Q_\ell \cap N^+(t_{(1)})$ and $Q_\ell \cap N^-(t_{(1)})$, and arcs between $R_\ell \cap N^+(t_{(1)})$ and $R_\ell \cap N^-(t_{(1)})$ have a multiple fan structure.*

Theorem 5.13 *Let $n \geq 6$, $e = f\overline{f}\ldots f\overline{f} \in E_n$, $f \in E_r$, and $1 \leq k < l \leq 2r = m$. Let Y_k and Y_ℓ be two pre-orbits from the set $\{Q_{\zeta^{f_i}(i)}, Q_{\zeta^{f_j}(j)}, R_{\zeta^{f_i}(i)}, R_{\zeta^{f_j}(j)}\}$ $\subseteq N^+(t_{(0)})$, where $1 \leq i, j \leq m$. The arcs between any two of their four layers Y_k', Y_k'', Y_ℓ', and Y_ℓ'' have a multiple fan structure.*

Corollary 5.14 *Let $n \geq 6$, $e = f\overline{f}\ldots f\overline{f} \in E_n$, $f \in E_r$, and $1 \leq k < l \leq 2r = m$. Let Y_k and Y_ℓ be two pre-orbits from the set $\{Q_{\zeta^{\overline{f_i}}(i)}, Q_{\zeta^{\overline{f_j}}(j)}, R_{\zeta^{\overline{f_i}}(i)}, R_{\zeta^{\overline{f_j}}(j)}\}$ $\subseteq N^-(t_{(0)})$, where $1 \leq i, j \leq m$. The arcs between any two of their four layers Y_k', Y_k'', Y_ℓ', and Y_ℓ'' have a multiple fan structure.*

6 Research problems

We have characterized the arc structure of subtournaments of Walecki tournaments with zero, odd, and even pattern (see Aleš [2, 4]). That is, for all Walecki tournaments with periodic patterns. However, the arc structure of subtournaments of aperiodic Walecki tournaments still remains unknown. Automorphism groups of Walecki tournaments for initial cases and zero pattern were also determined (see Aleš [3]).

We suspect that automorphism groups of Walecki tournaments with odd or even pattern are cyclic groups generated by τ^{2r} or σ^{2r}, where $e = f\overline{f}\ldots f$ or $e = f\overline{f}\ldots f\overline{f}$, respectively, and $f \in E_r$. Moreover, we have a strong belief that the automorphism groups of Walecki tournaments with aperiodic pattern are trivial. Computational results (see Aleš [3]) support our predictions.

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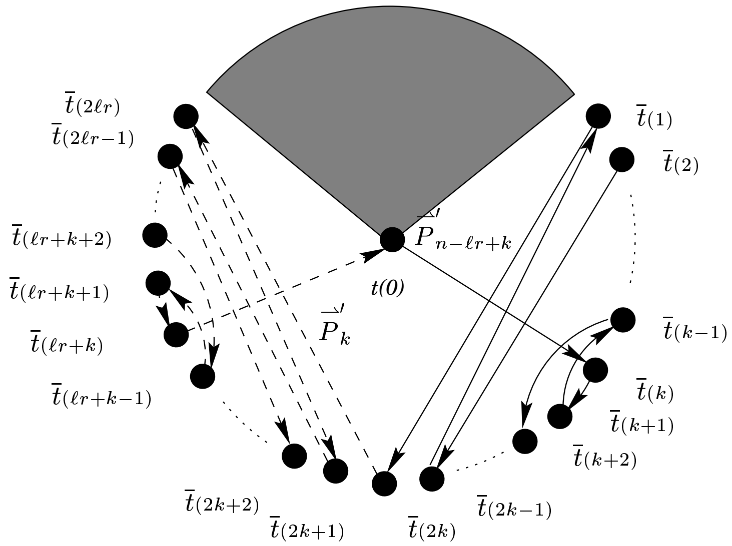

Figure 1: The diagram shows Hamilton directed cycle $\overrightarrow{H}'_k$ constructed from directed paths $\overrightarrow{P}'_{n-\ell r+k}$ and $\overleftarrow{P}'_k$ from the proof of Theorem 2.2.

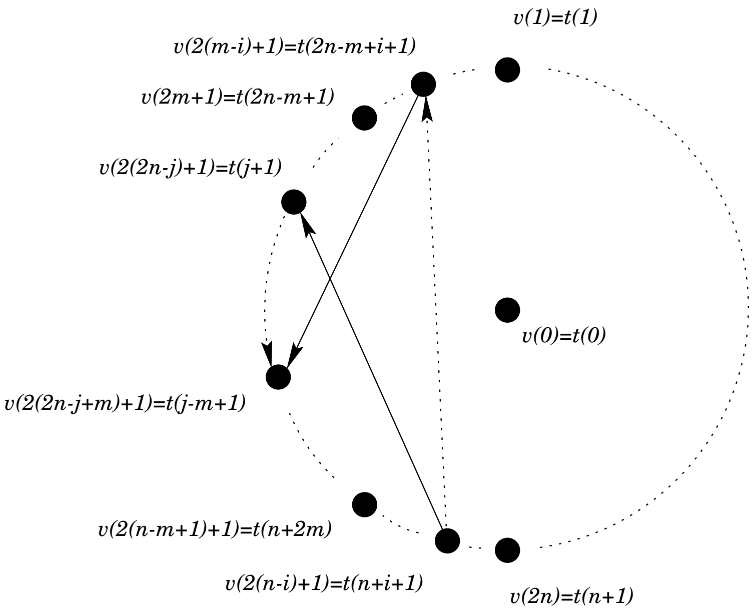

Figure 2: The diagram shows the action of permutation $\sigma^m \in \mathbb{S}_{2n+1}$ from Case 1.1.1 with $e_{k-n} = 0$ in the proof of Theorem 3.9.