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THE OKA PRINCIPLE, LIFTING  
OF HOLOMORPHIC MAPS AND  
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# THE OKA PRINCIPLE, LIFTING OF HOLOMORPHIC MAPS AND REMOVAL OF INTERSECTIONS

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**Abstract.** In section 1 we survey results on the Oka principle for sections of holomorphic submersions over Stein manifolds. In section 2 we apply these results to the lifting problem for holomorphic mappings, and in section 3 we apply them to removability of intersections of holomorphic maps from Stein manifolds with closed complex subvarieties of the target space.

## &1. The Oka principle for sections of holomorphic submersions.

Our main references for this section are the papers by Grauert [Gra], Cartan [Car], Gromov [Gro], and Prezelj and the author [FP1, FP2, FP3]. Let  $h: Z \rightarrow X$  be a holomorphic map of a complex manifold  $Z$  onto a complex manifold  $X$ . A *section* of  $h$  is a holomorphic map  $g: X \rightarrow Z$  such that  $hg = id_X$ . We shall say  $h$  satisfies the basic **Oka principle** if the following holds:

**The basic Oka principle.** *Given any continuous section  $f_0: X \rightarrow Z$  of  $h: Z \rightarrow X$ , there exists a homotopy of continuous sections  $f_t: X \rightarrow Z$  ( $t \in [0, 1]$ ) such that  $f_1$  is holomorphic on  $X$ .*

Stronger versions concern parametrized families of sections with parameter in a compact Hausdorff space (see [Gro] and [FP2]). The parametric form is essentially equivalent to the validity of

**The strong Oka principle.** *The inclusion  $\iota: \Gamma_h(X, Z) \hookrightarrow \Gamma_c(X, Z)$  of the space of holomorphic sections of  $h: Z \rightarrow X$  to the space of continuous sections is a weak homotopy equivalence, that is,  $\iota$  induces an isomorphism of all fundamental groups of the two spaces (when endowed with the compact-open topology).*

In 1957 H. Grauert established the strong Oka principle for certain types of holomorphic fiber bundles over Stein manifolds (and reduced Stein spaces) whose fiber is a complex Lie group or a complex homogeneous space [Gra, Car]. Grauert's result implies that the classification of holomorphic vector bundles over Stein spaces agrees with their topological classification (see the survey [Lei]). In 1986 a different proof of Grauert's theorem was given by Henkin and Leiterer [HL1, HL2]. In 1989 M. Gromov [Gro] gave an important extension of Grauert's theorem which we now explain.

**Definition 1.** [Gro, 1.1.B]) A **spray** associated to a holomorphic submersion  $h: Z \rightarrow X$  is a triple  $(E, p, s)$ , where  $p: E \rightarrow Z$  is a holomorphic vector bundle and  $s: E \rightarrow Z$  is a holomorphic map such that for each  $z \in Z$  we have (i)  $s(E_z) \subset Z_{h(z)}$  (equivalently,  $hs = hp$ ), (ii)  $s(0_z) = z$ , and (iii) the derivative  $ds: T_{0_z}E \rightarrow T_zZ$  maps the subspace  $E_z \subset T_{0_z}E$  surjectively onto  $VT_z(Z) := \ker dh_z$ . A spray on a complex manifold  $Z$  is by definition a spray associated to the trivial submersion  $Z \rightarrow \text{point}$ .

*Example.* Suppose that  $V_1, V_2, \dots, V_N$  are  $\mathbb{C}$ -complete holomorphic vector fields on  $Z$  which are vertical with respect to a submersion  $h: Z \rightarrow X$  (i.e., they are tangent to the fibers of  $h$ ). Let  $\phi_j^t$  denote the flow of  $V_j$ .  $\mathbb{C}$ -completeness of  $V_j$  means that  $\phi_j^t$  is defined for all complex values of the time parameter  $t$  and hence  $\{\phi_j^t: t \in \mathbb{C}\}$  is a complex one-parameter group of holomorphic automorphisms of  $Z$  preserving the fibers of  $h$ . The map

$$s: Z \times \mathbb{C}^N \rightarrow Z, \quad s(z; t_1, \dots, t_N) = \phi_1^{t_1} \phi_2^{t_2} \cdots \phi_N^{t_N}(z),$$

satisfies properties (i) and (ii) of sprays, and it also satisfies (iii) when the vector fields  $V_1, \dots, V_N$  span the vertical tangent space  $VT_zZ$  at each point  $z \in Z$ . ♠

The following is the Main Theorem in [Gro] and Theorem 1.2 in [FP2].

**1.1 Theorem.** *Let  $h: Z \rightarrow X$  be a holomorphic submersion of a complex manifold  $Z$  onto a Stein manifold  $X$ . If each point  $x \in X$  has an open neighborhood  $U \subset X$  such that the submersion  $h: Z_U = h^{-1}(U) \rightarrow U$  admits a spray then the strong Oka principle holds for sections of  $h$ . This holds in particular if  $h: Z \rightarrow X$  is a holomorphic fiber bundle whose fiber admits a spray.*

A complete proof and some further extensions can be found in the papers [FP1, FP2, FP3]; see also the recent preprint [Lar] by F. Lárusson. Perhaps the most important application (besides Grauert's classification of holomorphic vector bundles) has been the embedding theorem for Stein manifolds into euclidean spaces of minimal dimension, due to Eliashberg and Gromov [EGr] and Schürmann [Sch] (see also [Pre] for an embedding–interpolation theorem). We give another application in section 3.

**Additions to Theorem 1.1.** A1. If  $K \subset\subset X$  is a compact holomorphically convex set and the initial section  $f_0$  (or family of sections) is holomorphic in an open set containing  $K$  then the homotopy can be chosen to approximate  $f_0$  uniformly on  $K$  (Theorem 1.5 in [FP2]).

A2. If  $X_0$  is a closed complex subvariety in  $X$  and the initial section  $f_0: X \rightarrow Z$  is holomorphic on  $X_0$ , we can choose the homotopy  $f_t$  ( $t \in [0, 1]$ ) as above such that  $f_t|_{X_0} = f_0|_{X_0}$  for all  $t$  (Theorem 1.4 in [FP3]). In this case it suffices to assume that  $h: Z \rightarrow X$  admits a spray over a small neighborhood of any point  $x \in X \setminus X_0$  (no spray is needed over points in  $X_0$ ).

A3. Theorem 1.1 also holds if  $X$  is a reduced Stein space which is stratified by a finite descending chain of closed complex subspaces  $X = X_0 \supset X_1 \supset \cdots \supset X_m = \emptyset$  such that each stratum  $S_k = X_k \setminus X_{k+1}$  is non-singular and the restriction of the submersion  $h: Z \rightarrow X$  to  $S_k$  admits a spray over an open neighborhood (in  $S_k$ ) of any point  $x \in S_k$ ; see the Appendix to [FP3]. ♠

We now give a small extension of (A2), and therefore of Theorem 1.4 in [FP3]; we shall need this in section 3 below.

**Definition 2.** Let  $h: Z \rightarrow X$  be a holomorphic submersion,  $X_0 \subset X$  a closed complex subvariety of  $X$ , and  $\mathcal{S} \subset \mathcal{O}_X$  a coherent sheaf of analytic ideals with support  $X_0$  (i.e.,  $\mathcal{S}_x = \mathcal{O}_{X,x}$  precisely when  $x \in X \setminus X_0$ ). We say that local holomorphic sections  $f_0$  and  $f_1$  of  $h: Z \rightarrow X$  in a neighborhood of a point  $x \in X_0$  are  $\mathcal{S}$ -tangent at  $x$  if there is a neighborhood  $V \subset Z$  of the point  $z = f_0(x) = f_1(x) \in Z$  and a biholomorphic map  $\phi: V \rightarrow V' \subset \mathbb{C}^N$  ( $N = \dim_{\mathbf{t}} Z$ ) such that the germ at  $x$  of (any component of) the map  $\phi f_0 - \phi f_1: U \rightarrow \mathbb{C}^N$  belongs to  $\mathcal{S}_x$ . If  $f_0$  and  $f_1$  are holomorphic in an open set containing  $X_0$  and  $\mathcal{S}$ -tangent at each  $x \in X_0$ , we say that  $f_0$  and  $f_1$  are  $\mathcal{S}$ -tangent and write  $\delta(f_0, f_1) \in \mathcal{S}$ .

It is easily seen that the property of being  $\mathcal{S}$ -tangent is independent of the choice of local coordinates on  $Z$ .

**1.2 Theorem.** Let  $h: Z \rightarrow X$  be a holomorphic submersion onto a Stein manifold  $X$ , let  $X_0$  be a closed complex subvariety of  $X$ , and assume that  $h$  admits a spray over a small neighborhood of any point  $x \in X \setminus X_0$ . Let  $\mathcal{S} \subset \mathcal{O}_X$  be a coherent sheaf of analytic ideals with support  $X_0$ . Given a continuous section  $f_0: X \rightarrow Z$  which is holomorphic in a neighborhood of  $X_0$ , there is a homotopy of continuous sections  $f_t: X \rightarrow Z$  ( $t \in [0, 1]$ ) such that for each  $t \in [0, 1]$ ,  $f_t$  is holomorphic in a neighborhood of  $X_0$  and satisfies  $\delta(f_0, f_t) \in \mathcal{S}$ , and the section  $f_1$  is holomorphic on  $X$ .

The analogous result holds for parametrized families of sections and with uniform approximation on compact holomorphically convex subsets of  $X$ . Furthermore, the result holds if  $X$  is a reduced Stein space. Theorem 1.2 shows that the validity of the Oka principle for sections of a holomorphic submersion  $h: Z \rightarrow X$  extends from complex subspace  $X_0 \subset X$  to all of  $X$  provided that  $h$  admits a spray over a neighborhood of any point in  $X \setminus X_0$ .

*Proof.* Theorem 1.2 was proved in [FP3] in the case when  $\mathcal{S}$  is an integral power of the ideal sheaf of  $X_0$ . To prove theorem 1.2 in general the following modification must be made. On line 4 in the proof of Theorem 5.2 in [FP3] we chose finitely many holomorphic functions  $h_1, \dots, h_m$  on  $X$  such that

$$X_0 = \{x \in X: h_j(x) = 0, 1 \leq j \leq m\} \quad (1.1)$$

and each  $h_j$  vanishes to a given order  $r \in \mathbb{N}$  on  $X_0$ . We now replace this by the requirement that *the functions  $h_j$  be sections of the sheaf  $\mathcal{S}$* . (A finite

collection of sections of  $\mathcal{S}$  satisfying (1.1) can be constructed by using Cartan's Theorem A; one inductively lowers the dimension of the superfluous irreducible components of their common zero set.) Once this replacement is made, the proof of Theorem 1.4 in [FP3, section 6] gives theorem 1.2 above, and no other changes are needed. ♠

We give an important special case of theorem 1.2. Let  $h: E \rightarrow X$  be a holomorphic vector bundle of rank  $q$  over a reduced Stein space  $X$ . For each  $x \in X$  we denote by  $\widehat{E}_x \approx \mathbf{CP}^q$  the compactification of the fiber  $E_x \cong \mathbf{C}^q$  obtained by adding to  $E_x$  the hyperplane at infinity  $\Lambda_x \approx \mathbf{CP}^{q-1}$ . The resulting fiber bundle  $\widehat{h}: \widehat{E} \rightarrow X$  with fibers  $\widehat{E}_x \approx \mathbf{CP}^q$  is again holomorphic. (The essential observation is that the transition maps, which are  $\mathbf{C}$ -linear automorphisms of fibers  $E_x$ , extend to projective linear automorphisms of  $\widehat{E}_x$ ).

**1.3 Corollary. (Avoiding subvarieties by holomorphic sections.)** *Let  $h: E \rightarrow X$  be a holomorphic vector bundle of rank  $q$  over a reduced Stein space  $X$  and let  $\widehat{h}: \widehat{E} \rightarrow X$  be the associated bundle with fiber  $\mathbf{CP}^q$ . Let  $\widehat{\Sigma} \subset \widehat{E}$  be a closed complex subvariety and set  $\Sigma = \widehat{\Sigma} \cap E$ . If for each  $x \in X$  the fiber  $\Sigma_x = \Sigma \cap E_x$  is of complex codimension at least two in  $E_x$  then the strong Oka principle holds for sections of  $h: E \setminus \Sigma \rightarrow X$ .*

Notice that the fibers  $\Sigma_x$  are algebraic by Chow's theorem. Under stronger hypothesis on  $\Sigma$  this is Corollary 1.8 in [FP2].

*Proof of corollary 1.3.* In [FP2, Lemma 8.3] the following was shown: *If  $\Lambda_x \not\subset \widehat{\Sigma}_x$  for some  $x \in X$  then there is an open neighborhood  $U \subset X$  of  $x$  such that the submersion  $h: h^{-1}(U) \setminus \Sigma \rightarrow U$  admits a spray.* Granted this lemma we prove the corollary by induction on  $n = \dim_{\mathbf{C}} X$  as follows. For  $n = 0$  the result is trivial. Assume now that  $n \geq 1$  and that the result holds over Stein spaces of dimension  $< n$ . Let  $\Lambda \subset \widehat{E}$  be the complex hypersurface with fibers  $\Lambda_x = \widehat{E}_x \setminus E_x$ . If an irreducible component of  $\Lambda$  is contained in  $\widehat{\Sigma}$ , we can remove this component from  $\widehat{\Sigma}$  without changing the assumption on  $\Sigma = \widehat{\Sigma} \cap E$ . Thus we may assume that  $\widehat{\Sigma}$  and  $\Lambda$  have no common irreducible components. The set

$$X_1 = \{x \in X: \Lambda_x \subset \widehat{\Sigma}_x\} \cup X_{\text{sing}}$$

is then a closed complex subspace of  $X$  with  $\dim X_1 \leq n - 1$ . The restriction of the submersion  $h: E \setminus \Sigma \rightarrow X$  to  $X_1$  satisfies the hypothesis of corollary 1.3 and hence the conclusion holds over  $X_1$ . Any holomorphic section over  $X_1$  (or any compact family of sections) extends holomorphically to an open neighborhood  $V \subset X$  of  $X_1$  such that they still avoid  $\Sigma$  over  $V$ . If  $x \in X \setminus X_1$  then  $\Lambda_x \not\subset \widehat{\Sigma}_x$  and hence the submersion  $h: E \setminus \Sigma \rightarrow X$  admits a spray over a neighborhood of  $x$  (see the lemma stated at the beginning of the proof). By theorem 1.2 the Oka principle extends from  $X_1$  to  $X$ . ♠

## &2. Lifting of holomorphic mappings.

In this section we show that the results discussed in section 1 give the Oka

principle for liftings of holomorphic maps. I wish to thank Takeo Ohsawa who asked this question during my talk at the Hayama conference on December 16, 2000.

Let  $h: Z \rightarrow X$  be a holomorphic map. Given a complex manifold  $Y$  and a holomorphic map  $f: Y \rightarrow X$ , the problem is to find a holomorphic map  $g: Y \rightarrow Z$  such that  $hg = f$ ; any such  $g$  will be called a **holomorphic lifting** of  $f$  (with respect to  $h$ ).

$$\begin{array}{ccc} & & Z \\ & \nearrow g & \downarrow h \\ Y & \xrightarrow{f} & X \end{array}$$

An obvious necessary condition is the existence of a **continuous lifting**, and the main question is when is this condition also sufficient.

**2.1 Theorem. (The Oka principle for liftings.)** *Let  $h: Z \rightarrow X$  be a holomorphic map. Suppose that  $Y$  is a Stein manifold,  $f: Y \rightarrow X$  is a holomorphic map and  $g_0: Y \rightarrow Z$  is a continuous map such that  $hg_0 = f$ . Assume that for each  $y \in Y$  the point  $f(y) \in X$  has an open neighborhood  $U \subset X$  such that  $h: Z_U = h^{-1}(U) \rightarrow U$  is a holomorphic submersion onto  $U$  which admits a spray (definition 1). Then there exists a homotopy of continuous maps  $g_t: Y \rightarrow Z$  such that  $hg_t = f$  for each  $t \in [0, 1]$  and the map  $g_1$  is holomorphic on  $Y$ . If  $g_0$  is holomorphic on a closed complex subvariety  $Y_0 \subset Y$  then we can choose the homotopy  $g_t$  to be fixed on  $Y_0$ . If  $g_0$  is holomorphic in a neighborhood of a compact holomorphically convex subset  $K \subset\subset Y$ , the homotopy  $g_t$  can be chosen to approximate  $g_0$  uniformly on  $K$ .*

*Proof.* The main idea is to apply theorem 1.2 above to a pull-back submersion  $\tilde{h}: \tilde{Z} = f^*Z \rightarrow Y$  which is constructed such that liftings of  $f$  correspond to section of  $\tilde{h}$  and sprays for  $h$  induce sprays for  $\tilde{h}$ . Set

$$\begin{aligned} \tilde{Z} &= \{(y, z): y \in Y, z \in Z, f(y) = h(z)\}, \\ \tilde{h}(y, z) &= y \in Y, \quad \sigma(y, z) = z \in Z. \end{aligned} \tag{2.1}$$

It is easily seen that  $\tilde{Z}$  is a closed complex submanifold of  $Y \times Z$ , the maps  $\tilde{h}: \tilde{Z} \rightarrow Y$  and  $\sigma: \tilde{Z} \rightarrow Z$  are holomorphic and satisfy  $f\tilde{h} = h\sigma$ . Furthermore, since  $h$  is a submersion in a neighborhood of any point  $f(y) \in X$  for  $y \in Y$ , it follows that  $\tilde{h}$  is a holomorphic submersion of  $\tilde{Z}$  onto  $Y$ . For any section  $\tilde{g}: Y \rightarrow \tilde{Z}$  of  $\tilde{h}: \tilde{Z} \rightarrow Y$  the map  $g = \sigma\tilde{g}: Y \rightarrow Z$  is an  $h$ -lifting of  $f$  since

$$hg = h(\sigma\tilde{g}) = (h\sigma)\tilde{g} = (f\tilde{h})\tilde{g} = f(\tilde{h}\tilde{g}) = f.$$

Moreover, we claim that any lifting  $g$  of  $f$  is of this form. To see this, observe that  $h(g(y)) = f(y)$  for  $y \in Y$  implies that the point  $\tilde{g}(y) := (y, g(y)) \in Y \times Z$  belongs to the subset  $\tilde{Z} \subset Y \times Z$  defined by (2.1) and hence  $\tilde{g}: Y \rightarrow \tilde{Z}$  is a section of  $\tilde{h}: \tilde{Z} \rightarrow Y$ . Furthermore we have  $\sigma(\tilde{g}(y)) = \sigma(y, g(y)) = g(y)$  and hence the lifting  $g$  of  $f$  is indeed obtained from the section  $\tilde{g}$ . Therefore theorem 2.1 will follow from theorem 1.2 in section 1 once we prove the following.

**2.2 Lemma. (Pulling back sprays.)** *Let  $f: Y \rightarrow X$  and  $h: Z \rightarrow X$  be holomorphic maps. Assume that  $U \subset X$  is an open set such that map  $h: Z_U = h^{-1}(U) \rightarrow U$  is a submersion onto  $U$  which admits a spray. Then the map  $\tilde{h}: \tilde{Z} = f^*Z \rightarrow Y$  defined by (2.1) is a submersion with spray over the open set  $V = f^{-1}(U) \subset Y$ .*

*Proof.* Let  $(E, p, s)$  be a spray associated to the submersion  $h: Z_U \rightarrow U$ . Set  $V = f^{-1}(U) \subset Y$  and observe that  $\sigma$  maps the set  $\tilde{Z}_V = \tilde{h}^{-1}(V)$  to  $Z_U$ . Let  $\tilde{p}: \tilde{E} \rightarrow \tilde{Z}_V$  denote the pull-back of the holomorphic vector bundle  $p: E \rightarrow Z_U$  by the map  $\sigma: \tilde{Z}_V \rightarrow Z_U$ . Explicitly we have

$$\begin{aligned} \tilde{E} &= \{(\tilde{z}, e): \tilde{z} \in \tilde{Z}_V, e \in E; \sigma(\tilde{z}) = p(e)\} \\ &= \{(y, z, e): y \in V, z \in Z, e \in E; f(y) = h(z), p(e) = z\} \end{aligned}$$

and  $\tilde{p}(\tilde{z}, e) = \tilde{z}$ . Consider the map  $\tilde{s}: \tilde{E} \rightarrow \tilde{Z}_V$  defined by  $\tilde{s}(y, z, e) = (y, s(e))$ . We claim that  $(\tilde{E}, \tilde{p}, \tilde{s})$  is a spray associated to the submersion  $\tilde{h}: \tilde{Z}_V \rightarrow V$ . We first check that the map  $\tilde{s}$  is well defined. If  $(y, z, e) \in \tilde{E}$  then  $p(e) = z$  and  $h(z) = f(y)$ . Since  $s$  is a spray for  $h$ , we have  $h(s(e)) = h(z) = f(y)$  which shows that the point  $\tilde{s}(y, z, e) = (y, s(e)) \in Y \times Z$  belongs to the fiber  $\tilde{Z}_y$ . This verifies property (i) in definition 1. Clearly  $\tilde{s}(y, z, 0_{(y,z)}) = (y, s(0_z)) = (y, z)$  which verifies property (ii) in definition 1. It is also immediate that  $\tilde{s}$  satisfies property (iii) provided that  $s$  does since the vertical derivatives of the two maps coincide under the identifications  $\tilde{Z}_y \approx Z_{f(y)}$  and  $\tilde{E}_{(y,z)} \approx E_z$ . ♠

*Proof of theorem 2.1 for submersions with stratified sprays.* Assume that  $X$  is stratified by a finite descending chain of closed complex subvarieties  $X = X_0 \supset X_1 \supset \dots \supset X_m = \emptyset$  such that for each  $k = 0, \dots, m-1$  the stratum  $S_k = X_k \setminus X_{k+1}$  is non-singular and the submersion  $h: Z_k = h^{-1}(S_k) \rightarrow S_k$  admits a spray over an open neighborhood of any point  $x \in S_k$ . Let  $f: Y \rightarrow X$  be a holomorphic map. Set  $Y'_k = f^{-1}(X_k) \subset Y$ . Then  $Y = Y'_0 \supset Y'_1 \supset \dots \supset Y'_m = \emptyset$  is a stratification of  $Y$ . There are two problems: (1) the strata  $\Sigma'_k = Y'_k \setminus Y'_{k+1}$  may have singularities, and (2) the subvariety  $Y' \subset Y$  (over which the initial map  $g_0: Y \rightarrow Z$  is holomorphic) need not be included in the sets  $Y'_k$ . To rectify this we take  $\tilde{Y}_k = Y'_k \cup Y'$  to get a stratification  $Y = \tilde{Y}_0 \supset \tilde{Y}_1 \supset \dots \supset \tilde{Y}_m = Y'$ . We delete any possible repetitions and pass to a refinement  $Y = Y_0 \supset Y_1 \supset \dots \supset Y_l = Y'$  in which the strata  $\Sigma_k = Y_k \setminus Y_{k+1}$  are nonempty and regular.

By the construction  $f$  maps each stratum  $\Sigma_k = Y_k \setminus Y_{k+1}$  to a stratum  $S_j = X_j \setminus X_{j+1}$  for some  $j = j(k)$ . Lemma 2.2 now provides local stratified sprays for the pull-back submersion  $\tilde{h}: \tilde{Z} \rightarrow Y$  defined by (2.1). More precisely, for each  $y \in \Sigma_k$  we get an open neighborhood  $V$  of  $y$  in  $\Sigma_k$  and a spray on the submersion  $\tilde{h}: \tilde{h}^{-1}(V) \rightarrow V$  by pulling back a spray from a neighborhood  $U \subset S_j$  of the point  $f(y) \in S_j$  as in lemma 2.2. Hence theorem 2.1 follows from the version of theorem 1.1 for submersion with stratified sprays. ♠

### &3. Removability of intersections.

The main reference for this section is [Fo] and [FP3]. Let  $Z$  be a complex manifold and  $\Sigma \subset Z$  a closed complex subvariety of  $Z$ . Given a complex manifold  $X$  and a holomorphic map  $f: X \rightarrow Z$ , we write

$$f^{-1}(\Sigma) = \{x \in X: f(x) \in \Sigma\} = Y \cup \tilde{Y} \quad (3.1)$$

where each of the sets  $Y$  and  $\tilde{Y}$  is a union of connected components of the preimage  $f^{-1}(\Sigma)$  and  $Y \cap \tilde{Y} = \emptyset$ . We say that the set  $\tilde{Y}$  is **holomorphically removable** from  $f^{-1}(\Sigma)$  if the following holds:

**(H-rem)** *There is a holomorphic homotopy  $f_t: X \rightarrow Z$  ( $t \in [0, 1]$ ), with  $f = f_0$ , such that for each  $t \in [0, 1]$  the set  $Y$  is a union of connected components of  $f_t^{-1}(\Sigma)$  and we have  $f_1^{-1}(\Sigma) = Y$ .*

The following is clearly a necessary condition for the validity of (H-rem):

**(C-rem)** *There are an open set  $U \subset X$  containing  $Y$  and a homotopy of continuous maps  $\tilde{f}_t: X \rightarrow Z$  ( $t \in [0, 1]$ ) with  $\tilde{f}_0 = f$  such that for each  $t \in [0, 1]$  we have  $\tilde{f}_t|_U = f_t|_U$ , and  $\tilde{f}_1^{-1}(\Sigma) = Y$ .*

The validity of the Oka principle means that (C-rem) $\Rightarrow$ (H-rem).

#### 3.1 Theorem. (The Oka principle for removability of intersections)

*Let  $X$  be a Stein manifold and  $f: X \rightarrow Z$  a holomorphic map to a complex manifold  $Z$ . Let  $\Sigma$  be a closed complex subvariety of  $Z$  and write  $f^{-1}(\Sigma) = Y \cup \tilde{Y}$  as in (3.1). Suppose that (C-rem) holds for  $\tilde{Y}$ . Then:*

- (a) *If the manifold  $Z \setminus \Sigma$  admits a spray, there is a continuous homotopy  $f'_t: X \rightarrow Z$  ( $t \in [0, 1]$ ) satisfying (C-rem) such that the map  $f'_1$  is holomorphic on  $X$ .*
- (b) *If both  $Z$  and  $Z \setminus \Sigma$  admit a spray then (C-rem) $\Rightarrow$ (H-rem).*

We postpone the proof for a moment and give a couple of examples. As our first example we let  $Z = \mathbb{C}^d$  for some  $d \geq 1$ . The map  $s: \mathbb{C}^d \times \mathbb{C}^d \rightarrow \mathbb{C}^d$ ,  $s(z, w) = z + w$ , is clearly a spray on  $\mathbb{C}^d$ . A closed complex subvariety  $\Sigma \subset \mathbb{C}^d$  is said to be **tame** if there is a holomorphic automorphism  $\Phi$  of  $\mathbb{C}^d$  such that  $\Phi(\Sigma) \subset \{(z', z_d) \in \mathbb{C}^d: |z_d| \leq 1 + |z'|\}$ . In particular, any algebraic subvariety of  $\mathbb{C}^d$  is tame. For discrete subsets of  $\mathbb{C}^d$  this notion of tameness coincides with the one introduced by Rosay and Rudin [RR]. The following is a corollary of theorem 3.1.

**3.2 Corollary.** (Theorem 3.1 in [Fo]) *Let  $\Sigma$  be a closed complex subvariety of  $\mathbb{C}^d$  satisfying one of the following conditions:*

- (a)  *$\Sigma$  is tame and  $\dim \Sigma \leq d - 2$ ;*
- (b) *a complex Lie group acts holomorphically and transitively on  $\mathbb{C}^d \setminus \Sigma$ .*

*Then (C-rem) $\Rightarrow$ (H-rem) holds for any data  $(X, f, \tilde{Y})$  as in theorem 3.1.*

*Proof.* In each of the two cases the manifold  $\mathbb{C}^d \setminus \Sigma$  admits a spray. ♠



Corollary 3.2 applies in particular when  $\Sigma = \{0\} \subset \mathbb{C}^d$  and gives the result of Forster and Ramspott [FR] from 1966 to the effect that the Oka principle holds in the problem of *complete intersections* on Stein manifolds.

We mention that for each  $d \geq 1$  there exist discrete sets  $\Sigma \subset \mathbb{C}^d$  such that corollary 3.2 fails. In fact, there exists a discrete set  $B \subset \mathbb{C}^d$  which is *unavoidable*, in the sense that every entire map  $G: \mathbb{C}^d \rightarrow \mathbb{C}^d \setminus B$  whose image avoids  $B$  has rank  $< d$  at each point (Rosay and Rudin [RR]). Choose a point  $p \in \mathbb{C}^d \setminus B$  and set  $\Sigma = B \cup \{p\}$ . Take  $X = \mathbb{C}^d$  and let  $f: \mathbb{C}^d \rightarrow \mathbb{C}^d$  be the identity map  $f(z) = z$ . Also take  $Y = \{p\}$  and  $\tilde{Y} = B$ . Then clearly (C-rem) holds but (H-rem) fails for the pair  $(f, \tilde{Y})$  (since  $\text{rank} G < d$  for a holomorphic map  $G: \mathbb{C}^d \rightarrow \mathbb{C}^d \setminus B$  implies that  $G^{-1}(p)$  contains no isolated points, and hence  $p$  cannot be a connected component of  $G^{-1}(p)$ ). ♠

As our second example we let  $\Sigma$  be an affine complex subspace of  $Z = \mathbb{C}\mathbb{P}^d$ . Since  $\mathbb{C}\mathbb{P}^d$  and  $\mathbb{C}\mathbb{P}^d \setminus \Sigma$  are complex homogeneous spaces (the group of all affine complex automorphisms acts transitively on  $\mathbb{C}\mathbb{P}^d$ , and the group of those affine automorphisms which fix  $\Sigma$  acts transitively on  $\mathbb{C}\mathbb{P}^d \setminus \Sigma$ ), we get

**3.3 Corollary.** *For any Stein manifold  $X$  the implication (C-rem)  $\Rightarrow$  (H-rem) holds for intersections of holomorphic maps  $X \rightarrow \mathbb{C}\mathbb{P}^d$  with affine complex subspaces of  $\mathbb{C}\mathbb{P}^d$ .*

*Problem.* Does corollary 3.3 hold if  $\Sigma \subset \mathbb{C}\mathbb{P}^d$  is an algebraic subvariety of codimension at least two? Does  $\mathbb{C}\mathbb{P}^d \setminus \Sigma$  admit a spray for every such  $\Sigma$ ? In particular, does  $\mathbb{C}\mathbb{P}^d \setminus \{p, q\}$  admit a spray?

*Proof of theorem 3.1 (a).* By assumption there is a continuous homotopy  $\tilde{f}_t: X \rightarrow Z$  ( $t \in [0, 1]$ ) satisfying (C-rem) for the given initial map  $f = \tilde{f}_0$  and the subset  $\tilde{Y} \subset f^{-1}(\Sigma)$ . Thus  $\tilde{f}_1: X \rightarrow Z$  is a continuous map which is holomorphic near the set  $Y = \tilde{f}_1^{-1}(\Sigma)$ . Assuming that  $Z \setminus \Sigma$  admits a spray, we must show that one can deform  $\tilde{f}_1$  to a holomorphic map  $\tilde{f}_2: X \rightarrow Z$  by a continuous homotopy  $\tilde{f}_t: X \rightarrow Z$  ( $1 \leq t \leq 2$ ) which is holomorphic in a neighborhood of  $Y$  and satisfies  $\tilde{f}_t^{-1}(\Sigma) = Y$  for all  $t \in [1, 2]$ . The homotopy  $\tilde{f}'_t = \tilde{f}_{2t}$  ( $t \in [0, 1]$ ) will then satisfy part (a) in theorem 3.1.

A complete proof was given in [Fo] for  $Z = \mathbb{C}^d$ , but unfortunately it uses the linear structure on  $\mathbb{C}^d$ . A small modification is needed in the general case. To simplify the notation we replace  $f$  by  $f_1$ ; hence  $f$  is holomorphic near  $Y = f^{-1}(\Sigma)$ . We define a coherent sheaf of ideals  $\mathcal{R}$  on  $X$  which measures the order of contact of  $f$  with  $\Sigma$  along  $Y$ . Fix a point  $x \in Y$  and let  $z = f(x) \in \Sigma$ . Let  $g_1, \dots, g_k$  be holomorphic functions which generate the sheaf of ideals of  $\Sigma$  in some neighborhood of  $z$  in  $Z$ . We take the functions  $g_j \circ f$ ,  $1 \leq j \leq k$ , as the local generators of the sheaf  $\mathcal{R}$  near  $x$ . Furthermore we take  $\mathcal{S} = \mathcal{R} \cdot \mathcal{J}_Y^r$ , where  $\mathcal{J}_Y$  is the sheaf of ideals of the subvariety  $Y$  and  $r$  is a fixed positive integer. The purpose of introducing the sheaf  $\mathcal{S}$  is explained by the following lemma.

**3.4 Lemma.** (Notation as above) *If  $U \subset X$  is an open set containing  $Y$  and  $g: U \rightarrow Z$  is a holomorphic map which satisfies  $\delta(f, g) \in \mathcal{S}$  then there is an open set  $V$ , with  $Y \subset V \subset U$ , such that  $\{x \in V: g(x) \in \Sigma\} = Y$ . The analogous conclusion holds for continuous families of sections with parameter in a compact space.*

Lemma 3.4 is proved in [Fo, Section 3]. We continue with the proof of theorem 3.1. We identify maps  $X \rightarrow Z$  with sections of the trivial submersion  $h: \tilde{Z} = X \times Z \rightarrow X$  without changing the notation. Let  $\tilde{Z}' = \tilde{Z} \setminus \tilde{\Sigma} = X \times (Z \setminus \Sigma)$  where  $\tilde{\Sigma} = (X \times \Sigma)$ . This is a trivial submersion with a spray (since there is a spray on the fiber  $Z \setminus \Sigma$ ). Then  $f$  is a section of  $\tilde{Z}$  whose image outside the subvariety  $Y$  belongs to  $\tilde{Z}'$ .

We now apply the proof of Theorem 1.4 in [FP3]; more precisely, we apply the second version in which the sections are holomorphic near  $Y$  and the patching of sections takes place over small sets in  $X \setminus Y$ . There is only one apparent difficulty: The sections have values in  $\tilde{Z}$  over points in  $Y$  and they must have values in the smaller set  $\tilde{Z}'$  over points in  $X \setminus Y$ . Fortunately this change of codomain is only a virtual difficulty which can be avoided as follows.

At a typical step of the procedure in [FP3] we have a compact, holomorphically convex set  $K \subset X$  and a pair of sections  $(a, b)$  satisfying:

- (i)  $a: X \rightarrow \tilde{Z}$  is a continuous section which is holomorphic in an open set  $\tilde{A} \subset X$  with  $Y \cup K \subset \tilde{A}$ ,
- (ii)  $a(x) \in \tilde{\Sigma}$  precisely when  $x \in Y$ ,
- (iii)  $b: \tilde{B} \rightarrow \tilde{Z}'$  is a holomorphic section with values in  $\tilde{Z}'$ , defined in an open neighborhood  $\tilde{B}$  of a (small) compact set  $B \subset X \setminus Y$ ,
- (iv) there is a homotopy of holomorphic sections of  $\tilde{Z}'$  over the intersection  $\tilde{C} = \tilde{A} \cap \tilde{B}$  which connects  $a$  to  $b$ .

The goal is to patch the two sections  $(a, b)$  into a single section  $\tilde{a}: X \rightarrow \tilde{Z}$  which satisfies conditions (i) and (ii) above over an open neighborhood of  $Y \cup K \cup B$  and which approximates  $a$  uniformly on  $K$ . We need to assume that  $(K, B)$  is a Cartan pair (see [FP2] and [FP3]). The patching of  $a$  and  $b$  is achieved by performing the following steps. We shall follow the notation in [FP3] as much as possible, with  $X_0 = Y$ .

In the proof of Theorem 5.2 in [FP3] we constructed a holomorphic map  $s_1: V \rightarrow \tilde{Z}$  from an open set  $V \subset \tilde{A} \times \mathbb{C}^N$  containing  $\tilde{A} \times \{0\}$  (for some large  $N \in \mathbb{N}$ ) such that  $s_1(x, 0) = a(x)$  for all  $x \in \tilde{A}$  and the map  $\xi \rightarrow s_1(x, \xi) \in \tilde{Z}_x$  has derivative of maximal rank at  $\xi = 0$  for  $x \in \tilde{A} \setminus Y$ . However, for  $x \in Y$  we have  $s_1(x, \xi) = a(x)$  for all  $\xi \in \mathbb{C}^N$  near  $\xi = 0$ .

In the construction of  $s_1$  in [FP3] we used certain holomorphic functions  $h_1, \dots, h_m$  on  $X$  which vanish to a given order  $r$  on  $Y = X_0$  and whose common zero set is precisely  $Y$ . The only change in our current situation is to choose these functions  $h_j$  to be sections of the sheaf  $\mathcal{S}$  constructed above. Furthermore, we let  $s_2: \tilde{B} \times \mathbb{C}^N \rightarrow \tilde{Z}'$  be the holomorphic map as in the proof of Theorem

5.2 in [FP3] (which is obtained from the spray on the submersion  $\tilde{Z}' \rightarrow X$  over the open set  $\tilde{B}$ ).

The patching of  $a$  and  $b$  now proceeds in two steps. In the first step we use the holomorphic homotopy between the two sections over  $\tilde{C}$  in order to replace  $b$  by a section of  $\tilde{Z}'$  (over a smaller neighborhood of  $B$ ) which approximates  $a$  sufficiently well over a neighborhood of  $K \cap B$ . This is done exactly as in [FP3]. In the second step the two sections (which are now close over a neighborhood of  $K \cap B$ ) are patched by forming a transition map  $\psi$  satisfying  $s_1(x, \xi) = s_2(x, \psi(x, \xi))$  (see (5.2) in [FP3]) and then solving the equation  $\psi(x, \alpha(x)) = \beta(x)$  for  $x$  near  $K \cap B$  (see (4.1) in [FP3]). The solutions  $\alpha$  (which is holomorphic over a neighborhood of  $Y \cup K$ ) and  $\beta$  (which is holomorphic over a neighborhood of  $B$ ) then give a single section  $\tilde{a}$  of  $\tilde{Z}$  over an open neighborhood of  $Y \cup K \cup B$ , defined by  $\tilde{a}(x) = s_1(x, \alpha(x)) = s_2(x, \beta(x))$ . By construction the two expressions agree for  $x \in K \cap B$ .

We claim that this resulting section  $\tilde{a}$  satisfies all requirements, provided that it approximates  $a$  sufficiently well in a neighborhood of  $K$  (as we may assume to be the case). By construction of the map  $s_1$  we have  $\delta(\tilde{a}, a) \in \mathcal{S}$ , that is, the two sections are  $\mathcal{S}$ -tangent along  $Y$ . Lemma 3.4 implies that  $\tilde{a}(x) \in \tilde{Z}$  if  $x$  is sufficiently close to  $Y$  but not in  $Y$ . If  $\tilde{a}$  approximates  $a$  sufficiently well on  $K$  it follows that  $\tilde{a}(x) \in \tilde{Z}'$  for all  $x \in K \setminus Y$ . For  $x \in B$  the same is true since  $s_2$  has range in  $\tilde{Z}'$  and we have  $\tilde{a}(x) = s_2(x, \beta(x))$ . Finally we extend  $\tilde{a}$  to a continuous section over  $X$  by patching it with  $a$  outside a suitable neighborhood of  $Y \cup K \cup B$ .

This completes the induction step. The final section  $f'_1$  which is holomorphic on  $X$  is obtained as a locally uniform limit of sections obtained by this procedure. The same construction can be done for parametrized families of sections and we get a required homotopy  $f'_t$  ( $t \in [0, 1]$ ) satisfying theorem 3.1 (a).

*Proof of theorem 1.3 (b).* Suppose now that  $Z$  also admits a spray. Let  $\{f'_t\}$  ( $t \in [0, 1]$ ) be a homotopy satisfying theorem 3.1 (a). Applying theorem 1.2 in section 1 to  $\{f'_t\}$ , with the sheaf  $\mathcal{S}$  defined at the beginning of the proof, we obtain a two-parameter homotopy of maps  $h_{t,s}: X \rightarrow Z$  ( $t, s \in [0, 1]$ ) which are holomorphic in a neighborhood of  $Y$  and satisfy the following properties:

- (i)  $h_{t,0} = f'_t$  for all  $t \in [0, 1]$ ,
- (ii)  $h_{0,s} = f'_0$  and  $h_{1,s} = f'_1$  for all  $s \in [0, 1]$
- (iii)  $\delta(h_{t,0}, h_{t,s}) \in \mathcal{S}$  for all  $s, t \in [0, 1]$ , and
- (iv) the map  $f_t := h_{t,1}$  is holomorphic on  $X$  for each  $t \in [0, 1]$ .

It follows from (iii) and lemma 3.4 above that there is a neighborhood  $V \subset X$  of  $Y$  such that  $h_{t,s}^{-1}(\Sigma) \cap V = Y$  for all  $s, t \in [0, 1]$ . The homotopy  $\{f_t: t \in [0, 1]\}$ , defined by (iv) above, then satisfies (H-rem) for the initial map  $f = f_0$  and the set  $\tilde{Y} \subset f^{-1}(\Sigma)$ .  $\spadesuit$

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