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Preprint series, Vol. 39 (2001), 745

CUBES POLYNOMIAL AND ITS DERIVATIVES

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ISSN 1318-4865

February 21, 2001

Ljubljana, February 21, 2001

Cubes polynomial and its derivatives

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February 21, 2001

Abstract

Let $\alpha_i(G)$ be the number of induced i-cubes of a graph G. Then the cubes polynomial $c(G,x)$ of G is introduced as $\sum_{i\geq 0}\alpha_i(G)x^i$. It is shown that any function f with two related, natural properties, is up to the factor $f(K_1,x)$ the cubes polynomial. The derivation ∂G of a median graph G is also introduced and it is proved that the cubes polynomial is the only function f with the property $f'(G,x) = f(\partial G,x)$ provided that $f(G,0) = |V(G)|$. Several relations that generalize many previous results for median graphs are also given. For instance, for any $s \geq 0$ we have $c^{(s)}(G,x+1) = \sum_{i\geq s}\frac{c^{(i)}(G,x)}{(i-s)!}$.

*Supported by the Ministry of Science and Technology of Slovenia under the grant 101–504.

†Supported by the Ministry of Science and Technology of Slovenia under the grant J1-0502-0101-00.

1 Introduction

Several graph polynomials have been introduced in the literature, see the book [3]. In this paper we introduce the *cubes polynomial* of a graph G as $c(G,x) = \sum_{i\geq 0} \alpha_i(G)x^i$, where $\alpha_i(G)$ denotes the number of induced i-cubes of G.

We first show that the cubes polynomial has three nice properties: amalgamation, product, and expansion property. In fact, any function f with the amalgamation and the expansion property is up to the factor $f(K_1,x)$ the cubes polynomial. Weaker conditions suffice to reach similar conclusions if we restrict to the class of median graphs. Then we introduce the derivation graph ∂G of a median graph G and prove that the cubes polynomial is the only function f with the property $f'(G,x) = f(\partial G, x)$ provided that its value in $x = 0$ equals the number of vertices. In the last section we prove several relations for median graphs involving the cubes polynomial, for instance, for any $s \geq 0$ we have

$$c^{(s)}(G,x) = \sum_{i\geq s} \frac{(-1)^{i-s}}{(i-s)!} c^{(i)}(G, x+1)\,.$$

These relations (widely) generalize some previous results [7, 12, 13].

The *Cartesian product* $G \Box H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ in which the vertex (a,x) is adjacent to the vertex (b,y) whenever $ab \in E(G)$ and $x = y$, or $a = b$ and $xy \in E(H)$. The Cartesian product of k copies of K_2 is a *hypercube* or *k-cube* Q_k. Isometric subgraphs of hypercubes are called *partial cubes.* A graph G is a *median graph* if there exists a unique vertex x to every triple of vertices u, v, and w such that x lies simultaneously on a shortest u,v-path, a shortest u,w-path, and a shortest w,v-path. Median graphs are partial cubes, cf. [10, 4]. For more information on median graphs see [4, 8, 9, 10].

Two edges $e = xy$ and $f = uv$ of G are in the Djoković-Winkler [2, 15] relation Θ if $d(x,u) + d(y,v) \neq d(x,v) + d(y,u)$. Winkler [15] proved that a bipartite graph is a partial cube if and only if Θ is transitive.

Let $\mathcal{G}$ denote the set of all finite graphs, and $\mathcal{M}$ the class of all median graphs. Finally, Let $A_1, \ldots, A_n$ be sets and let $\mathcal{I} = \{1,\ldots,n\}$. Then the *inclusion-exclusion* property for sets $A_1,\ldots,A_n$ says:

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{\mathcal{J}\subseteq\mathcal{I}} (-1)^{|\mathcal{J}|-1} |\bigcap_{i\in\mathcal{J}} A_i|. \tag{1}$$

2 The cubes polynomial

For a graph G, let $\alpha_i(G)$ denote the number of induced i-cubes of G. Note that $\alpha_0(G) = |V(G)|$ and $\alpha_1(G) = |E(G)|$. Let $c(G,x)$ be the generating function of the sequence $(\alpha_i(G))_{i=0}^{\infty}$, that is,

$$c(G,x) = \sum_{i \geq 0} \alpha_i(G) x^i \, .$$

For finite graphs G, $c(G,x)$ is a polynomial and we call it the *cubes polynomial of* G. For instance, let T be a tree on n vertices, then $c(T,x) = (n-1)x + n$. Note also that $c(Q_n, x) = (x+2)^n$ and that

$$\alpha_k(G) = \frac{c^{(k)}(G,0)}{k!}. \tag{2}$$

A *cover* $\mathcal{C}$ of a graph G is a set of induced subgraphs $\mathcal{C} = \{G_1, \ldots, G_n\}$ of G such that $G = G_1 \cup G_2 \cup \cdots \cup G_n$. We say that a cover $\mathcal{C}$ is *proper*, if every induced hypercube of G is contained in at least one of the graphs of $\mathcal{C}$. Let $\mathcal{I}_n = \{1, \ldots, n\}$. For any subset $\mathcal{A} \subseteq \mathcal{I}_n$, let $G_{\mathcal{A}}$ be the intersection (possibly empty and possibly disconnected) of the graphs G_i $(i \in \mathcal{I})$.

Proposition 1 *Let $\mathcal{C} = \{G_1, \ldots, G_n\}$ be a proper cover of a graph G. Then*

$$c(G,x) = \sum_{\mathcal{A} \subseteq \mathcal{I}_n} (-1)^{|\mathcal{A}|-1} c(G_{\mathcal{A}}, x). \tag{3}$$

Proof. For $i \geq 0$ and $j \in \{1, 2, \ldots, n\}$, let A_i^j be the set of induced i-cubes of the graph G_j. Then $\alpha_i(G) = |A_i^1 \cup A_i^2 \cup \cdots \cup A_i^n|$. Hence, by (1), we infer

$$\alpha_i(G) = \sum_{\mathcal{A} \subseteq \mathcal{I}_n} (-1)^{|\mathcal{A}|-1} |\bigcap_{j \in \mathcal{A}} A_i^j|.$$

In addition, $c(G_{\mathcal{A}}, x) = \sum_{i \geq 0} |\cap_{j \in \mathcal{A}} A_i^j| \, x^i$. Therefore,

$$\begin{aligned} c(G,x) &= \sum_{i \geq 0} \alpha_i(G) \, x^i \\ &= \sum_{i \geq 0} \Big(\sum_{\mathcal{A} \subseteq \mathcal{I}_n} (-1)^{|\mathcal{A}|-1} |\bigcap_{j \in \mathcal{A}} A_i^j| \Big) \, x^i \\ &= \sum_{\mathcal{A} \subseteq \mathcal{I}_n} \sum_{i \geq 0} \Big((-1)^{|\mathcal{A}|-1} |\bigcap_{j \in \mathcal{A}} A_i^j| \Big) \, x^i \\ &= \sum_{\mathcal{A} \subseteq \mathcal{I}_n} (-1)^{|\mathcal{A}|-1} \, c(G_{\mathcal{A}}, x) \, . \end{aligned}$$

$\square$

Because of Proposition 1 we say that a function $f : \mathcal{G} \times \mathbb{R} \to \mathbb{R}$ has the *amalgamation property* if

$$f(G,x) = \sum_{\mathcal{A} \subseteq \mathcal{I}_n} (-1)^{|\mathcal{A}|-1} f(G_{\mathcal{A}}, x). \tag{4}$$

whenever $\{G_1, \ldots, G_n\}$ is a proper cover of G.

As hypercubes are the simplest Cartesian product graphs, the cubes polynomial should behave nicely with respect to the Cartesian product. Indeed, observe that an induced r-cube of $G \square H$ is uniquely representable as $Q_s \square Q_{r-s}$, where Q_s is an induced s-cube of G and Q_{r-s} an induced $(r-s)$-cube of H. Hence, for every $k \geq 0$,

$$\alpha_k(G \square H) = \sum_{i=0}^{k} \alpha_i(G) \alpha_{k-i}(H) \,.$$

From here we easily conclude:

Proposition 2 *For any graphs G and H, $c(G \square H, x) = c(G,x)c(H,x)$.*

Observe that the identity $|E(G \square H)| = |E(G)|\,|V(H)| + |E(H)|\,|V(G)|$ immediately follows from Proposition 2. We say that a function $f : \mathcal{G} \times \mathbb{R} \to \mathbb{R}$ has the *product property* if for any graphs G and H,

$$f(G \square H, x) = f(G,x) f(H,x) \,.$$

Let G be a connected graph. The *expansion* G^* of G with respect to the proper cover $\{G_1, G_2\}$ of G is the graph constructed as follows. Let G_i^* be an isomorphic copy of G_i, for $i = 1, 2$, and, for any vertex u of $G_0 = G_1 \cap G_2$, let u_i be the corresponding vertex in G_i^*. Then G^* is obtained from the disjoint union of G_1^* and G_2^*, where for each u of G_0 the vertices u_1 and u_2 are joined by an edge. (Note that, since the cover is proper, a vertex of $G_1 \setminus G_2$ cannot be adjacent to a vertex of $G_2 \setminus G_1$.)

It is easy to prove the following claim. Just observe that via the expansion the subgraph isomorphic to $G_0 \square K_2$ gives rise to new/larger hypercubes.

Proposition 3 *Let G^* be a graph constructed by the expansion with respect to the proper cover $\{G_1, G_2\}$ (over G_0). Then $c(G^*, x) = c(G_1, x) + c(G_2, x) + x\, c(G_0, x)$.*

Because of this result, the following definition seems reasonable. A function $f : \mathcal{G} \times \mathbb{R} \to \mathbb{R}$ has the *expansion property* if

$$f(G^*, x) = f(G_1, x) + f(G_2, x) + x\, f(G_0, x),$$

whenever G^* is the expansion with respect to the proper cover $\{G_1, G_2\}$ (over G_0).

Theorem 4 *Let $f : \mathcal{G} \times \mathbb{R} \to \mathbb{R}$ be a function with the amalgamation and the expansion property. Then for any graph G,*

$$f(G, x) = f(Q_0, x)\, c(G, x)\,.$$

Proof. The proof is by induction on the number of vertices of a graph. The n-cube Q_n, $n \geq 1$, can be obtained as the expansion of the proper cover $\{Q_{n-1}, Q_{n-1}\}$ (over Q_{n-1}). Now, by the expansion property we have $f(Q_n, x) = (2 + x) f(Q_{n-1}, x)$. Hence,

$$f(Q_n, x) = (2 + x)^n f(Q_0, x) = c(Q_n, x) f(Q_0, x).$$

Suppose now that G is a graph different from hypercubes. Let $\mathcal{C}$ be the set comprised of graphs $G - v$ for every $v \in V(G)$. Since G is not a hypercube, $\mathcal{C}$ is a proper cover and every graph of $\mathcal{C}$ is smaller than G. Using amalgamation property for c, and the induction hypothesis for graphs G_i, we infer that

$$\begin{aligned} f(G, x) &= \sum_{\mathcal{A} \subseteq \mathcal{I}_n} (-1)^{|\mathcal{A}|-1} f(G_{\mathcal{A}}, x) \\ &= \sum_{\mathcal{A} \subseteq \mathcal{I}_n} (-1)^{|\mathcal{A}|-1} f(Q_0, x)\, c(G_{\mathcal{A}}, x) \\ &= f(Q_0, x)\, c(G, x). \end{aligned}$$

□

From the above proof we can also deduce that if f has the amalgamation property and $f(Q_n, x) = (x + 2)^n$ holds for every $n \in \mathbb{N}_0$, then $f \equiv c$.

In the case of median graphs we can further strengthen the result of Theorem 4. Recall that Mulder [9, 10] proved that a graph is a median graph if and only if it can be obtained from K_1 by a sequence of expansions in which $G_0 = G_1 \cap G_2$ is always convex, cf. also [8, 11].

Corollary 5 *Let* $f : \mathcal{M} \times \mathbb{R} \to \mathbb{R}$ *be a function with the expansion property. Then* $f(G,x) = f(Q_0,x)\, c(G,x)$. *If, in addition,* f *has the product property, then either* $f \equiv 0$ *or* $f \equiv c$.

Proof. Let f has the expansion property. The proof of the first assertion is by induction on the number of expansion steps needed to obtain a median graph. The conclusion is clear for $G = K_1$. Let now G^* be a median graph obtained by a (convex) expansion with respect to the proper cover $\{G_1, G_2\}$ (over G_0). Then, $f(G^*, x) = f(G_1, x) + f(G_2, x) + x\, f(G_0, x)$. From the induction hypothesis we obtain

$$f(G^*, x) = f(Q_0, x)c(G_1, x) + f(Q_0, x)c(G_2, x) + x\, f(Q_0, x)c(G_0, x)\,,$$

and so $f(G^*, x) = f(Q_0, x)(c(G_1, x) + c(G_2, x) + x\, c(G_0, x)) = f(Q_0, x)c(G^*)$.

If f also has the product property, then, since $Q_0 \Box Q_0 = Q_0$, we have $f(Q_0, x) = [f(Q_0, x)]^2$. Thus either $f(Q_0, x) = 0$ or $f(Q_0, x) = 1$. Combining this with the the fact that $f(G,x) = f(Q_0,x)\, c(G,x)$ completes the proof. □

3 Derivation graphs of median graphs

Let F be a Θ-class of a median graph and let $e = uv \in F$. Then it is well-known that F forms a matching. Moreover, the endvertices of edges of F that are closer to u than to v induce a median graph isomorphic to the subgraph induced by the endvertices of edges of F that are closer to v than to u. We denote this median subgraph by U_e. Let $\mathcal{F}(G)$ be the set of edges consisting of representatives of the Θ-classes of G. Then we define the *derivation* of a median graph G as the graph

$$\partial\, G = \bigcup_{e \in \mathcal{F}(G)} U_e,$$

that is, as the disjoint union of the sets U_e, $e \in \mathcal{F}(G)$. (Note that transitivity of Θ implies that the graph $\partial\, G$ is well-defined.) For instance, $\partial\,(P_n \Box P_m)$ is the disjoint union of $n-1$ copies of P_m and $m-1$ copies of P_n.

The reason for calling the graph $\partial\, G$ "the derivation" of G is the following property.

Proposition 6 (Derivation property) *Let* G *be a median graph. Then,*

$$c'(G,x) = c(\partial\, G, x). \tag{5}$$

Proof. Note that for any Q_n in G its edges lie in n Θ-classes ($n \geq 1$), and the corresponding graphs U_e induce $(n-1)$-cubes—altogether there are n Q_{n-1}'s for each n-cube of G. Hence, $n\,\alpha_{n-1}(G)$ equals the number of $(n-1)$-cubes of $\partial\, G$. In other words:

$$c'(G,x) = \sum_{e \in \mathcal{F}} c(U_e, x) = c(\partial\, G, x).$$

□

We say that a function $f : \mathcal{M} \times \mathrm{I\!R} \to \mathrm{I\!R}$ has the *derivation property* if

$$f'(G,x) = f(\partial\, G, x).$$

We next prove that the cubes polynomial is the only function on median graphs with the derivation property such that its value in $x = 0$ equals the number of vertices.

Theorem 7 *Let $f : \mathcal{M} \times \mathrm{I\!R} \to \mathrm{I\!R}$ be a function with the derivation property, such that $f(G,0) = |V(G)|$. Then $f \equiv c$.*

Proof. First, since Q_0 has no Θ-classes, $\partial\, Q_0$ is empty. Thus $f(Q_0, x)$ is a constant, and $f(Q_0, 0) = 1$ implies $f(Q_0, x) = 1$. Hence $f(Q_0, x) = c(Q_0, x)$.

The proof proceeds by induction on the number of vertices of a median graph. Suppose that for any median graph H with less than k vertices we have $f(H,x) = c(H,x)$, and let G has k vertices. Recall that for every $e \in E(G)$, the graph U_e is a median graph. Since U_e has less than k vertices, $f(U_e, x) = c(U_e, x)$ holds by the induction assumptions. By the derivation properties of c and f we infer that $f'(G,x) = c'(G,x)$. Thereby $f(G,x) = c(G,x) + C$, where C is a constant. Since $f(Q_0, 0) = c(Q_0, 0)$, we derive that $C = 0$, thereof $f \equiv c$. □

Let us denote by $\mathcal{M}^*$ the class of all graphs whose connected components are median graphs. Thus, $G \in \mathcal{M}^*$ can be written as $G = G_1 \cup G_2 \cup \cdots \cup G_s$, where every G_i is a median graph. Then we can extend the concept of the derivation graph to the graphs from $\mathcal{M}^*$ by setting $\partial\, G = \partial\, G_1 \cup \partial\, G_2 \cup \cdots \cup \partial\, G_s$ and also define the higher derivations in the following way. For $k \geq 0$, set

$$\partial^k\, G = \begin{cases} \partial\,(\partial^{k-1}\, G) & k \geq 1 \\ G & k = 0. \end{cases} \tag{6}$$

Then one can easily extend Proposition 6 to the graphs from $\mathcal{M}^*$ as well as to generalize it to the higher derivatives in the following way:

$$c^{(k)}(G,x) = c(\partial^k\, G, x). \tag{7}$$

4 Relations for median graphs

Throughout this section let G be a median graph with k Θ-classes. The following two relations are known:

$$\sum_{i\geq 0}(-1)^i\,\alpha_i(G)=1 \qquad \text{and} \qquad k=-\sum_{i\geq 0}(-1)^i\,i\,\alpha_i(G). \tag{8}$$

The first of these relations is due to Soltan and Chepoi [13, Theorem 4.2.(6)]. It was later independently obtained by Škrekovski in [12], where the second relation is also proved. Note that the first equality presents a generalization of the well-known equality "$n-m=1$" for trees, while the second one applied to trees says that "$k=m$", which is another characterizing property of trees. These relations also imply the Euler-type formulas from [7, 6].

Denote by the $\theta_s(G)$ $(s\geq 0)$ the number of components in the graph $\partial^s G$. Thus, $\theta_0(G)=1$ and $\theta_1(G)=k$. Then we can extend (8) as follows. (Recall that a graph is a median graph if and only if it can be obtained from hypercubes by a sequence of convex amalgams, a result due to Bandelt and van de Vel [1], cf. also [8, 14].)

Proposition 8 *Let G be a median graph and $s\geq 0$. Then*

$$\theta_s(G)=c^{(s)}(G,-1)\,.$$

Proof. The proof is by induction. So, let first $s=0$. If $G\cong Q_n$ $(n\geq 0)$ then $c(Q_n,-1)=1=\theta_0(Q_n)$. Now assume that G is the amalgam of G_1 and G_2 over G_0. Then, by the induction assumption,

$$c(G,-1)=c(G_1,-1)+c(G_2,-1)-c(G_0,-1)=1+1-1=1=\theta_0(G).$$

Suppose now that the claim holds for all integers smaller that s $(s\geq 1)$ and for all median graphs with less vertices than G. Since $c^{(s+1)}(G,x)=\sum_{e\in\mathcal{F}(G)}c^{(s)}(U_e,x)$ we can use the induction hypothesis for graphs U_e and derive

$$c^{(s+1)}(G,-1)=\sum_{e\in\mathcal{F}(G)}\theta_s(U_e)=\theta_{s+1}(G).$$

□

Theorem 9 *Let G be a median graph and $s \geq 0$. Then,*

$$c^{(s)}(G, x+1) = \sum_{i \geq s} \frac{c^{(i)}(G, x)}{(i-s)!}, \tag{9}$$

$$c^{(s)}(G, x) = \sum_{i \geq s} \frac{(-1)^{i-s}}{(i-s)!} c^{(i)}(G, x+1). \tag{10}$$

Proof. The proof of the first equality is by induction on the number of amalgamation steps. Suppose first that $G \cong Q_n$. Since $c(Q_n, x) = (x+2)^n$ it follows that $c^{(s)}(Q_n, x+1) = \frac{n!}{(n-s)!}(x+3)^{n-s}$. Using binomial formula we obtain:

$$\begin{aligned} c^{(s)}(Q_n, x+1) &= \frac{n!}{(n-s)!} \sum_{j=0}^{n-s} \binom{n-s}{j} (x+2)^j \\ &= \sum_{j=0}^{n-s} \frac{n!}{(n-s)!} \cdot \frac{(n-s)!}{j!(n-s-j)!} (x+2)^j \\ &= \sum_{j \geq 0} \frac{c^{(s+j)}(Q_n, x)}{j!} \\ &= \sum_{i \geq s} \frac{c^{(i)}(Q_n, x)}{(i-s)!}, \end{aligned}$$

and so the desired formula follows. If G is not a hypercube then it can be obtained by an amalgamation of G_1 and G_2 over G_0. By the induction hypothesis we deduce:

$$\begin{aligned} c^{(s)}(G, x+1) &= c^{(s)}(G_1, x+1) + c^{(s)}(G_2, x+1) - c^{(s)}(G_0, x+1) \\ &= \sum_{i \geq s} \frac{1}{(i-s)!} (c^{(i)}(G_1, x) + c^{(i)}(G_2, x) - c^{(i)}(G_0, x)) \\ &= \sum_{i \geq s} \frac{c^{(i)}(G, x)}{(i-s)!}. \end{aligned}$$

This proves the first relation.

The second equality can be proved in a similar way. Alternatively, one can write down the first equality for every $0 \leq s \leq r$, where r is the dimension of a largest hypercube of G, and invert the obtained system of equations. □

Corollary 10 *Let G be a median graph, $\alpha_i = \alpha_i(G)$ the number of induced i-cubes in G, and $\theta_i = \theta_i(G)$ the number of components in $\partial^i G$. Then for any $s \in \mathbb{N}_0$,*

$$(a) \quad \alpha_s = \frac{1}{s!} \sum_{i \geq s} \frac{\theta_i}{(i-s)!} \qquad \textit{and} \qquad \theta_s = s! \sum_{i \geq 0} (-1)^{i-s} \binom{i}{s} \alpha_i \,.$$

$$(b) \quad \sum_{i \geq 0} (-1)^i \, 2^i \, \alpha_i = \sum_{i \geq 0} (-1)^i \, \frac{\theta_i}{i!} \,.$$

Proof. In order to proof the first relation of (a), we set $x = -1$ in relation (10) to get

$$(-1)^s \, c^{(s)}(G, -1) = \sum_{i \geq s} (-1)^i \, \frac{c^{(i)}(G, 0)}{(i-s)} \,.$$

Therefore

$$(-1)^s \, \frac{c^{(s)}(G, -1)}{s!} = \sum_{i \geq s} (-1)^i \, \frac{c^{(i)}(G, 0)}{s!(i-s)!} \,.$$

Now, using Proposition 8 and formula (2), we obtain

$$(-1)^s \, \frac{\theta_s}{s!} = \sum_{i \geq s} (-1)^i \binom{i}{s} \alpha_i \,.$$

Finally note that $\binom{i}{s} = 0$ if $i < s$.

The proof of the second formula is obtained analogously as above by setting $x = -1$ in relation (9).

In order to obtain the relation (b) just sum up equalities from the first formula of (a) and use a basic property of binomial coefficients. $\square$

Note that relations from (8) can be obtained from the second formula of Corollary 10(a) by setting $s = 0$ and $s = 1$, respectively.

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