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CYCLE DECOMPOSITIONS III:  
COMPLETE GRAPHS AND  
FIXED LENGTH CYCLES

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# Cycle decompositions III: Complete graphs and fixed length cycles\*

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## Abstract

We show that the necessary conditions for the decomposition of the complete graph of odd order into cycles of a fixed even length and for the decomposition of the complete graph of even order minus a 1-factor into cycles of a fixed odd length are also sufficient.

## 1 Introduction

### 1.1 The problem

This paper examines the conditions under which a complete graph admits a decomposition into cycles of some fixed length. Since the existence of such a decomposition requires that the degrees of all vertices be even, the complete graph must have an odd number of vertices. However, this question can be extended to graphs with

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an even number of vertices in which every vertex has even degree. A natural way of creating such graphs that are very “close” to complete graphs is to remove a 1-factor from a complete graph with an even number of vertices. The question now becomes the following: when does  $K_n$  or  $K_n - I$ , whichever is appropriate, admit a decomposition into cycles of a fixed length  $m$ ?

In addition to the condition on the parity of  $n$ , there are two more obvious necessary conditions, namely, that  $3 \leq m \leq n$  and that the cycle length  $m$  divides the number of edges in either  $K_n$ , that is,  $\frac{n(n-1)}{2}$ , or  $K_n - I$ , that is,  $\frac{n(n-2)}{2}$ .

Alspach and Gavlas [1] have recently shown that for the case when  $m$  and  $n$  are either both odd or both even, the necessary conditions mentioned above are also sufficient. In this paper we extend their results to the case  $m$  and  $n$  of opposite parity, thus completely solving the problem. That is, we give a constructive proof of the following two theorems:

**Theorem 1.1.1** *Let  $n$  be an even integer and  $m$  be an odd integer with  $3 \leq m \leq n$ . The graph  $K_n - I$  can be decomposed into cycles of length  $m$  whenever  $m$  divides the number of edges in  $K_n - I$ .*

**Theorem 1.1.2** *Let  $n$  be an odd integer and  $m$  be an even integer with  $3 \leq m \leq n$ . The graph  $K_n$  can be decomposed into cycles of length  $m$  whenever  $m$  divides the number of edges in  $K_n$ .*

We mention only some of the most important partial results towards proving Theorems 1.1.1 and 1.1.2 to date (a survey can be found in [8]). The necessary conditions have been shown to be sufficient for all even  $m$  and  $n \equiv 1 \pmod{2m}$  (Kotzig [7], Rosa [10]), for all  $n$  odd and  $m$  twice an odd prime power (Alspach and Varma [2]), and for all  $n$  odd and  $m \leq 50$  (Bell [3]). In addition, Rodger [9] showed that if the necessary conditions are sufficient for all odd  $n$  in the range  $m \leq n < 3m$ , then they are sufficient for all odd  $n \geq m$ . The equivalent result for  $n$  even has been proved in part by El-Zanati [5] and, independently and completely, by the author [12].

## 1.2 Definitions and terminology

We begin with a few basic definitions. All graphs in this paper are simple. We use  $K_n$  to denote the complete graph on  $n$  vertices and, for  $n$  even,  $K_n - I$  to denote  $K_n$  with a 1-factor  $I$  removed.  $\overline{K_n}$  denotes the complement of  $K_n$ ,  $K_{m,n}$  denotes the complete bipartite graph with the bipartition sets of sizes  $m$  and  $n$ , and  $K_{g(m)}$  denotes the complete  $g$ -partite graph with  $m$  vertices in each part. An  $m$ -cycle, that is, a cycle of length  $m$ , is denoted by  $C_m$ . An  $n$ -cycle in a graph with  $n$  vertices is called a *Hamilton cycle*.

**Definition 1.2.1** A graph  $G$  is said to be *decomposed* into its subgraphs  $H_1$  and  $H_2$ , which is denoted by  $G = H_1 \oplus H_2$ , if  $G$  is the edge disjoint union of  $H_1$  and  $H_2$ . If  $G = H_1 \oplus \cdots \oplus H_k$ , where  $H_1, \dots, H_k$  are all isomorphic to  $H$ , then  $G$  is said to be  *$H$ -decomposable* and  $\{H_1, \dots, H_k\}$  is an  *$H$ -decomposition* of  $G$ . In particular,  $G$  is  *$C_m$ -decomposable* if it can be decomposed into subgraphs isomorphic to an  $m$ -cycle.

**Definition 1.2.2** The *join*  $G \bowtie H$  of graphs  $G$  and  $H$  is the graph with vertex set  $V(G) \cup V(H)$  and edge set  $E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$ .

Notice that  $K_n$  is isomorphic to  $K_{n-1} \bowtie K_1$  and, for  $n$  even,  $K_n - I$  is isomorphic to  $(K_{n-2} - I) \bowtie \overline{K_2}$ . In these representations, the vertices of  $K_1$  and  $\overline{K_2}$ , respectively, are called *central vertices*.

**Definition 1.2.3** Let  $G$  be a graph with vertex set  $V(G) = \{x_0, \dots, x_{k-1}\}$ . We define  $G(2)$  to be the graph with vertex set  $V(G(2)) = \{x_i^j : x_i \in V(G), j \in \mathbb{Z}_2\}$  and edge set  $E(G(2)) = \{x_{i_1}^{j_1} x_{i_2}^{j_2} : x_{i_1} x_{i_2} \in E(G), j_1, j_2 \in \mathbb{Z}_2\}$ .

Notice that  $K_k(2)$  is isomorphic to the join  $K_k \bowtie K_k$  with a 1-factor removed, that is, to  $K_{2k} - I$ .

For a set  $S \subseteq \mathbb{Z}$  we write  $-S = \{-s : s \in S\}$  and, for any  $x \in \mathbb{Z}$ ,  $S+x = \{s+x : s \in S\}$ .

**Definition 1.2.4** Let  $k$  be a positive integer and  $L$  a subset of  $\{1, 2, \dots, \lfloor \frac{k}{2} \rfloor\}$ . A *circulant*  $X = X(k; L)$  is a graph with vertex set  $V(X) = \{u_0, u_1, \dots, u_{k-1}\}$  and edge set  $E(X) = \{u_i u_{i+\ell} : i \in \mathbb{Z}_k, \ell \in L\}$ . The edge  $u_i u_{i+\ell}$ , where  $\ell \in L$ , is said to

be of length  $\ell$ , and  $L$  is called the *edge length set* of the circulant  $X$ . When  $k$  is even, the edge length  $\frac{k}{2}$  is called the *diameter length* and edges  $u_i u_{i+\ell}$  and  $u_{i+\frac{k}{2}} u_{i+\frac{k}{2}+\ell}$  of length  $\ell$  are called *diametrically opposed*.

Note: In the literature, a circulant is more often described by its symbol  $S$ , where  $S = L \cup (-L)$ , so that  $S \subseteq \{1, \dots, k-1\}$  and  $-S = S$ , rather than by its edge length set  $L$ . However, we find the description via the edge length set more convenient for the purposes of this paper.

Notice that  $K_n$  is isomorphic to the circulant  $X(n; \{1, \dots, \frac{n-1}{2}\})$  for  $n$  odd, and  $K_n - I$  is isomorphic to the circulant  $X(n; \{1, \dots, \frac{n}{2} - 1\})$  for  $n$  even.

For any automorphism  $\gamma$  of a graph  $G$ , the subgroup of the automorphism group  $\text{Aut}(G)$  generated by  $\gamma$  is denoted by  $\langle \gamma \rangle$ . For any vertex  $v$  in a graph  $G$  and a subgroup  $\Gamma$  of  $\text{Aut}(G)$ , the orbit of  $\Gamma$  containing  $v$  is denoted by  $\Gamma(v)$ . The action of  $\text{Aut}(G)$  can be naturally extended to the edges of  $G$  and to the subgraphs of  $G$ .

**Definition 1.2.5** Let  $X = X(k; L)$  be a circulant with vertex set  $\{u_0, \dots, u_{k-1}\}$ . By the *rotation*  $\rho$  we mean the cyclic permutation  $(u_0 \dots u_{k-1})$ .

Notice that  $\langle \rho \rangle = \{\rho^i : i = 0, \dots, k-1\}$  is a subgroup of  $\text{Aut}(X)$  whose edge orbits are the sets  $\{u_i u_{i+\ell} : i \in \mathbb{Z}_k\}$  for all  $\ell \in L$ ; that is, the sets containing all edges of the same length. For  $k$  even,  $\rho^2$  is the permutation  $(u_0 u_2 \dots u_{k-2})(u_1 u_3 \dots u_{k-1})$  so that  $\langle \rho^2 \rangle$  is a subgroup of  $\langle \rho \rangle$  with edge orbits  $\{u_i u_{i+\ell} : i \in \mathbb{Z}_k, i \text{ even}\} = \langle \rho^2 \rangle(u_0 u_\ell)$  and  $\{u_i u_{i+\ell} : i \in \mathbb{Z}_k, i \text{ odd}\} = \langle \rho^2 \rangle(u_1 u_{1+\ell})$  for each  $\ell \in L$ . Hence each set of edges of the same length splits into an “odd” and an “even” orbit of  $\langle \rho^2 \rangle$ . Both  $\langle \rho \rangle$  and  $\langle \rho^2 \rangle$  play an important role in our decompositions into  $m$ -cycles.

When representing  $K_n$  and  $K_n - I$  as the joins  $K_{n-1} \bowtie K_1$  and  $(K_{n-2} - I) \bowtie \overline{K_2}$ , respectively, we define  $\rho$  to be the permutation that acts as a rotation on the circulant part of the graph ( $K_{n-1}$  or  $K_{n-2} - I$ ) and leaves the central vertices fixed.

A  $p$ -*path* is a path of length  $p$ ; that is, a path with  $p$  edges and  $p+1$  vertices. In a path  $P = x_0 x_1 \dots x_p$ , the vertices encountered first, third,  $\dots$  (that is,  $x_0, x_2, \dots$ ) are called the *odd* vertices, and the vertices encountered second, fourth,  $\dots$  (that is,  $x_1, x_3, \dots$ ) are called the *even* vertices. The vertices  $x_0$  and  $x_p$  are called the

*endpoints* of  $P$ ,  $x_0$  being the *initial* vertex and  $x_p$  being the *terminal* vertex. The vertices  $x_1, \dots, x_{p-1}$  are called *internal*.

If  $P = x_0x_1 \dots x_p$  is a  $p$ -path and  $Q = y_0y_1 \dots y_q$  is a  $q$ -path and  $x_p = y_0$  is the only vertex the two paths have in common, then  $PQ$  will denote the  $(p + q)$ -path  $x_0x_1 \dots x_py_1 \dots y_q$ , the concatenation of  $P$  and  $Q$ . Furthermore,  $\overleftarrow{P}$  denotes the path  $x_px_{p-1} \dots x_1x_0$ , the reverse of  $P$ . Clearly,  $P$  and  $\overleftarrow{P}$  represent the same graph, however, the order in which the vertices are listed is important in concatenation.

**Definition 1.2.6** A path  $P$  in a circulant  $X(k; L)$  with the property that no two edges of  $P$  are of the same length is called a *zig-zag* path. The set of all edge lengths represented in  $P$ , denoted by  $L(P)$ , is called the *edge length set* of  $P$ . A *copy* of a zig-zag path  $P$  in  $X(k; L)$  is a zig-zag path of the form  $\rho^i(P)$  for  $i \in \{0, 1, \dots, k-1\}$ .

The name zig-zag path comes from the way such a path can be constructed from its edge length set. It is easy to see that  $\{a_1, \dots, a_p\} \subseteq L$ , where  $1 \leq a_1 < \dots < a_p \leq \lfloor \frac{k}{2} \rfloor$ , is the edge length set of each of the zig-zag paths  $u_0u_{a_1}u_{a_1-a_2} \dots u_{a_1-a_2+\dots+(-1)^{p-1}a_p}$  and  $u_0u_{-a_1}u_{-a_1+a_2} \dots u_{-a_1+a_2-\dots+(-1)^pa_p}$  in  $X(k; L)$ . A copy of a zig-zag path of one of these forms will be called *basic*. Notice that, if  $k$  is even and  $a_p < \frac{k}{2}$ , then a basic path  $P$  and its diametrically opposed path  $\rho^{\frac{k}{2}}(P)$  are vertex-disjoint. Almost all cycles in our decompositions are constructed from zig-zag paths.

### 1.3 A brief discussion of techniques used in the constructions

In this section we introduce the three basic types of  $m$ -cycles used in our constructions. First we would like to explain what we mean by the term peripheral cycle.

**Definition 1.3.1** Let  $G = X(k; L)$  be a circulant with vertex set  $\{u_0, \dots, u_{k-1}\}$ . Let  $d = \gcd(k, m)$ ,  $n' = \frac{k}{d}$ , and  $m' = \frac{m}{d}$ . We are assuming that  $d \geq 3$ .

A *peripheral cycle* is an  $m$ -cycle  $C$  of  $G$  that fits one of the following two descriptions:

1.  $C$  is the concatenation of  $m'$ -paths  $P, \rho^{n'}(P), \rho^{2n'}(P), \dots, \rho^{(d-1)n'}(P)$ ; or
2.  $C$  is made up of  $(m' - 1)$ -paths  $P, \rho^{n'}(P), \rho^{2n'}(P), \dots, \rho^{(d-1)n'}(P)$  together with  $d$  linking edges.

In both cases the path  $P$  is said to *generate* the peripheral cycle  $C$ .

In almost all of our constructions the path  $P$  that generates a peripheral cycle is a zig-zag path. The following conditions are sufficient for a path  $P$  to generate a peripheral cycle: in Case 1 of the definition, if the sets of internal vertices of the paths  $\rho^{jn'}(P)$ ,  $j = 0, \dots, d-1$ , are mutually disjoint and if the terminal vertex of  $P$  coincides with the initial vertex of  $\rho^{n'}(P)$ ; and in Case 2, if the paths  $\rho^{jn'}(P)$  are pairwise vertex-disjoint and if there exist suitable linking edges.

If  $C$  is a peripheral cycle in  $G = X(k; L)$ , then so are  $\rho(C), \rho^2(C), \dots, \rho^{n'-1}(C)$ . These cycles are said to be *generated* by the same  $(m' - 1)$ - or  $m'$ -path  $P$  as  $C$ . It is not difficult to see that if  $C$  is generated by a zig-zag  $m'$ -path  $P$  with edge length set  $L_P$ , then  $\{\rho^i(C) : i = 0, \dots, n' - 1\}$  is a  $C_m$ -decomposition of  $X(k; L_P)$ .

In some of our constructions, for every peripheral cycle  $C$  in the  $C_m$ -decomposition,  $\rho(C), \rho^2(C), \dots, \rho^{n'-1}(C)$  are used as well. In the cases with  $m \leq n < 2m$ , however,  $n'$  is even and for each peripheral cycle  $C$  there are two options: either  $C, \rho(C), \dots, \rho^{n'-1}(C)$  are all used in the  $C_m$ -decomposition — in which case  $C$  is called *coupled* — or only  $C, \rho^2(C), \dots, \rho^{n'-2}(C)$  are used as such, whereas the copies of the generating path that appear in  $\rho(C), \rho^3(C), \dots, \rho^{n'-1}(C)$  are incorporated into other cycles — in which case  $C$  is called *solitary*.

**Using an auxiliary circulant to connect  $(m' - 1)$ -paths into a peripheral  $m$ -cycle.** This technique is due to Alspach and Gavlas [1]. The assumption is that we have zig-zag  $(m' - 1)$ -paths  $P_{i,0}$ ,  $i = 1, \dots, c$ , with the same initial vertex  $u_0$  and the same terminal vertex  $u_t$  (in our constructions  $t \in \{-1, n' - 1, n' + 1\}$ ), whose edge length sets  $L_i$  are pairwise disjoint. In addition, the  $(m' - 1)$ -paths within each of the families  $\mathcal{P}_i = \{P_{i,j} = \rho^{jn'}(P_{i,0}) : j = 0, \dots, d-1\}$  are pairwise vertex-disjoint. To choose the connecting edges we use Hamilton cycles in an auxiliary circulant  $X$  with vertex set  $\{v_0, \dots, v_{d-1}\}$ .

Suppose we are given a Hamilton cycle  $C$  in  $X$ . Arbitrarily orient the cycle to obtain a Hamilton directed cycle  $\vec{C}$ . Let  $v_{j_1}v_{j_2}$  be an edge of length  $j_2 - j_1$  in  $C$ . If  $\vec{C}$  uses the arc from  $v_{j_1}$  to  $v_{j_2}$ , connect the terminal vertex of  $P_{i,j_1}$  (that is,  $u_{j_1n'+t}$ ) with the initial vertex of  $P_{i,j_2}$  (that is,  $u_{j_2n'}$ ), thus using an edge of length  $(j_2 - j_1)n' - t$  (or, exceptionally,  $(j_1 - j_2)n' + t$  if  $j_2 - j_1 = 1$  and  $t = n' + 1$ ) in  $G$ . Conversely, if  $\vec{C}$  contains the arc from  $v_{j_2}$  to  $v_{j_1}$ , connect the initial vertex of  $P_{i,j_1}$

(that is,  $u_{j_1 n'}$ ) with the terminal vertex of  $P_{i, j_2}$  (that is,  $u_{j_2 n' + t}$ ), thus using an edge of length  $(j_2 - j_1)n' + t$  in  $G$ . In either case, the family  $\mathcal{P}_i$  of  $(m' - 1)$ -paths together with the linking edges arising from a directed cycle  $\vec{C}$  in  $X$  form an  $m$ -cycle. Since the two arcs corresponding to a fixed edge of the auxiliary circulant  $X$  result in two connecting edges of distinct lengths in  $G$ , each Hamilton cycle in  $X$  can be used to link two families of  $(m' - 1)$ -paths, one orientation for each family.

We now explain how to decompose the auxiliary circulant  $X$  into Hamilton cycles. Let the edge length set of  $X$  be  $\{\ell_1, \dots, \ell_b\}$ . First we decompose  $X$  into circulants of degree 2 and 4; namely, circulants of the forms  $X(d; \{\ell_i\})$ , where  $\gcd(d, \ell_i) = 1$ , and  $X(d; \{\ell_i, \ell_j\})$ , where the edge lengths  $\ell_i$  and  $\ell_j$  are chosen in such a way that the circulant is connected. A circulant of the first form is clearly a  $d$ -cycle, and a circulant of the second form can be decomposed into two Hamilton cycles by the following theorem by Bermond, Favaron, and Maheo [4].

**Theorem 1.3.2** [4] *Any connected circulant of degree 4 can be decomposed into Hamilton cycles.*

Examples of circulants of degree 4 that are connected include circulants of the forms  $X(d; \{\ell, \ell + 1\})$ ,  $X(d; \{2\ell - 1, 2\ell + 1\})$ , and if  $d$  is odd,  $X(d; \{2\ell, 2\ell + 2\})$ .

We thus have a Hamilton decomposition of the auxiliary circulant  $X$ . A cycle  $C$  in this decomposition either uses a single edge length  $\ell_i$  or two distinct edge lengths  $\ell_i$  and  $\ell_j$ . In the first case, the connecting edges arising from one orientation of  $C$  will all have the same length; namely, either  $\ell_i n' - t$  or  $\ell_i n' + t$ . After applying  $\rho, \rho^2, \dots, \rho^{n'-1}$  to the resulting  $m$ -cycle in  $G$ , all edges of this length will be covered. We thus have a choice of either using both or just one of the two orientations of  $C$ , thus linking up one or two families of  $(m' - 1)$ -paths. If  $C$  uses two distinct edge lengths  $\ell_i$  and  $\ell_j$ , however, a single orientation of  $C$  may produce connecting edges of four different edge lengths; namely,  $\ell_i n' - t$ ,  $\ell_i n' + t$ ,  $\ell_j n' - t$ , and  $\ell_j n' + t$ . In order to account for all the edges of these lengths in  $G$ , we therefore must use both orientations of  $C$ . A circulant  $X(d; \{\ell_i, \ell_j\})$  is thus used to link four families of  $(m' - 1)$ -paths. It is now easy to see that, provided that a sufficient number of suitable edge lengths  $\ell_1, \dots, \ell_b$  can be found, any number of families of  $(m' - 1)$ -paths can be linked into  $m$ -cycles. We shall give the details of how to choose the



edge lengths in  $X$  and how to decompose  $X$  into circulants of degree 2 and 4 for each of the constructions that uses this method when needed.

We now introduce the other two types of cycles.

**Definition 1.3.3** Let  $m$  and  $k$  be even positive integers and let  $G = X(k; L)$  be a circulant with  $\frac{k}{2} \in L$ . A *diameter cycle*  $C_D$  is an  $m$ -cycle in  $G$  consisting of two zig-zag  $(\frac{m}{2} - 1)$ -paths  $P$  and  $\rho^{\frac{k}{2}}(P)$  together with two edges of diameter length  $\frac{k}{2}$ . The zig-zag  $(\frac{m}{2} - 1)$ -path  $P$  is said to *generate* the diameter cycle  $C_D$ .

**Definition 1.3.4** An  $m$ -cycle in  $K_n$  or  $K_n - I$  that is neither a peripheral cycle nor a diameter cycle is called *central*.

As we shall see, the construction of central cycles greatly varies from case to case — hence the open-ended definition.

## 2 Decomposition of $K_n - I$ into cycles of odd length

### 2.1 Proof of Theorem 1.1.1

In this short section we present the framework for the proof of Theorem 1.1.1. The details of the construction, however, will follow in Sections 2.2 and 2.3.

Hoffman, Lindner, and Rodger [6] proved that for  $m$  and  $n$  both odd, if  $K_n$  is  $C_m$ -decomposable for all  $n$  in the range  $m \leq n < 3m$  that satisfy the necessary condition  $n(n - 1) \equiv 0 \pmod{2m}$ , then  $K_n$  is  $C_m$ -decomposable for all  $n \geq m$  satisfying  $n(n - 1) \equiv 0 \pmod{2m}$ . In a similar way, the author [12] proved the equivalent statement for  $n$  even:

**Theorem 2.1.1** [12] *For  $m$  odd and  $n$  even, if  $K_n - I$  is  $C_m$ -decomposable for all  $n$  in the range  $m \leq n < 3m$  that satisfy the necessary condition  $n(n - 2) \equiv 0 \pmod{2m}$ , then  $K_n - I$  is  $C_m$ -decomposable for all  $n$  satisfying  $n(n - 2) \equiv 0 \pmod{2m}$ .*

**PROOF OF THEOREM 1.1.1.** By Theorem 2.1.1, it is sufficient to find a  $C_m$ -decomposition of  $K_n - I$  for all  $n$  in the range  $m \leq n < 3m$  that satisfy the necessary

condition  $n(n-2) \equiv 0 \pmod{2m}$ . This is done in two parts: Lemma 2.2.4 of Section 2.2 settles the case  $2m \leq n < 3m$ , and Lemma 2.3.4 of Section 2.3 settles the case  $m \leq n < 2m$ .  $\square$

We note here that the following graphs have already been shown to be  $C_m$ -decomposable:

**Lemma 2.1.2** [12] *The graphs  $K_{2m} - I$  and  $K_{2m+2} - I$  are  $C_m$ -decomposable.*

## 2.2 $C_m$ -decomposition of $K_n - I$ for $2m \leq n < 3m$

Throughout this section, let  $m$  and  $n$  be positive integers satisfying the necessary conditions for the existence of a  $C_m$ -decomposition of  $K_n - I$ , where  $m$  is odd and  $2m \leq n < 3m$ . Let  $k = \frac{n}{2}$ . As we shall see, it is convenient to view the graph  $K_n - I$  as  $K_k(2)$ .

The basic idea of this construction is very simple. First we decompose  $K_k$  into  $m$ -cycles and  $m$ -paths, and then we carry this decomposition over to  $K_k(2)$ . The next lemma describes how to construct the  $m$ -paths in  $K_k$  and how these give rise to  $m$ -cycles in  $K_k(2)$ . Lemma 2.2.2 explains how the  $m$ -cycles in  $K_k$  are carried over to form  $m$ -cycles in  $K_k(2)$ .

**Lemma 2.2.1** *Let  $k = \frac{n}{2}$  be even and  $m \geq 5$ . Furthermore, let  $X = X(k; S_p)$  be a circulant with edge length set  $S_p = \{a_1, a_2, \dots, a_{\frac{m-1}{2}}, \frac{k}{2}\}$  such that  $1 \leq a_1 < a_2 < \dots < a_{\frac{m-1}{2}} < \frac{k}{2}$  and  $\frac{k}{2} - a_1 \in S_p$ . Then  $X(2)$  is  $C_m$ -decomposable.*

PROOF. First we decompose  $X$  into  $m$ -paths. Let the vertex set of the circulant  $X$  be  $\{u_0, u_1, \dots, u_{k-1}\}$ . Find  $p$ ,  $1 \leq p \leq \frac{m-1}{2}$ , such that  $\frac{k}{2} - a_1 = a_p$ . Let  $h = \frac{m-1}{2}$  and  $A = a_1 - a_2 + \dots + (-1)^p a_{p-1} + (-1)^{p+1} a_p$ , and let  $R$  and  $Q$  be the following paths:

$$\begin{aligned} R &= u_{a_1} u_{a_1 - a_2} \dots u_{a_1 - a_2 + \dots + (-1)^p a_{p-1}} \text{ and} \\ Q &= u_A u_{A + (-1)^{p+2} a_{p+1}} \dots u_{A + (-1)^{p+2} a_{p+1} + \dots + (-1)^h a_{h-1} + (-1)^{h+1} a_h} \\ &\quad u_{A + (-1)^{p+2} a_{p+1} + \dots + (-1)^h a_{h-1} + (-1)^{h+1} a_h + \frac{k}{2}} u_{A + (-1)^{p+2} a_{p+1} + \dots + (-1)^h a_{h-1} + \frac{k}{2}} \\ &\quad \dots u_{A + (-1)^{p+2} a_{p+1} + \frac{k}{2}} u_{A + \frac{k}{2}}. \end{aligned}$$

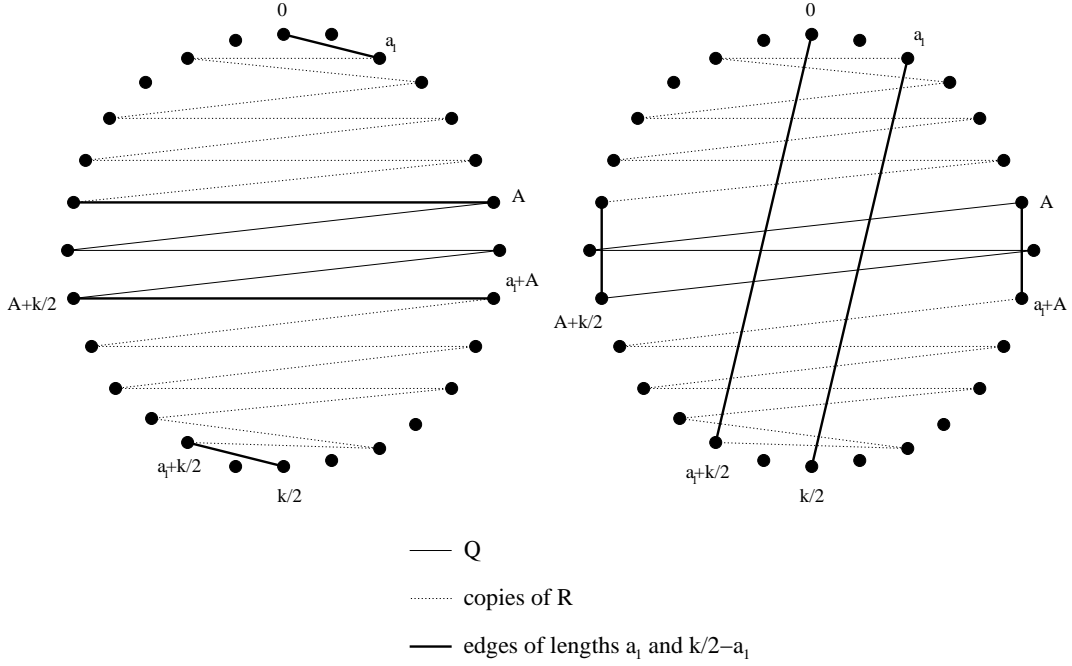


Figure 1: Lemma 2.2.1:  $m$ -paths  $P_0$  and  $\tilde{P}_0$ .

Notice that in the path  $R$ , each of the lengths  $a_2, \dots, a_{p-1}$  occurs precisely once, and in the path  $Q$ , the diameter length  $\frac{k}{2}$  occurs exactly once while each of the lengths  $a_{p+1}, \dots, a_h$  occurs precisely twice.

We now define  $m$ -paths  $P_0$  and  $\tilde{P}_0$  as follows:

$$P_0 = u_0 u_{a_1} R u_{a_1 - a_2 + \dots + (-1)^p a_{p-1}} u_A Q u_{A + \frac{k}{2}} u_{a_1 - a_2 + \dots + (-1)^p a_{p-1} + \frac{k}{2}} \rho^{\frac{k}{2}}(\overleftarrow{R}) u_{a_1 + \frac{k}{2}} u_{\frac{k}{2}}, \text{ and}$$

$$\tilde{P}_0 = u_{\frac{k}{2}} u_{a_1} R u_{a_1 - a_2 + \dots + (-1)^p a_{p-1}} u_{A + \frac{k}{2}} \overleftarrow{Q} u_A u_{a_1 - a_2 + \dots + (-1)^p a_{p-1} + \frac{k}{2}} \rho^{\frac{k}{2}}(\overleftarrow{R}) u_{a_1 + \frac{k}{2}} u_0.$$

It is easy to see that each of the  $m$ -paths  $P_0$  and  $\tilde{P}_0$  contains a pair of diametrically opposed edges of each of the lengths  $a_1, \dots, a_h$  together with one edge of the diameter length  $\frac{k}{2}$ . Each of the families  $\mathcal{P} = \{\rho^i(P_0) : i = 0, \dots, \frac{k}{2} - 1\}$  and  $\tilde{\mathcal{P}} = \{\rho^i(\tilde{P}_0) : i = 0, \dots, \frac{k}{2} - 1\}$  thus represents a decomposition of the circulant  $X$  into  $m$ -paths. Moreover, the mapping  $T : \mathcal{P} \rightarrow \tilde{\mathcal{P}}, T(\rho^i(P_0)) = \rho^i(\tilde{P}_0)$ , is a bijection.

Unlike in the case of an  $m$ -cycle (Lemma 2.2.2), for an  $m$ -path  $P$ ,  $P(2)$  is not

$C_m$ -decomposable. However, each  $m$ -path  $P \in \mathcal{P}$  gives rise to a graph  $G_2(P)$ , closely related to  $P(2)$ , which is  $C_m$ -decomposable and which we define as follows. Let

$$P = x_0x_1 \dots x_sx_{s+1}x_{s+2} \dots x_{t-2}x_{t-1}x_t \dots x_{m-1}x_m$$

be an  $m$ -path in  $\mathcal{P}$  and

$$T(P) = x_mx_1 \dots x_sx_{t-1}x_{t-2} \dots x_{s+2}x_{s+1}x_t \dots x_{m-1}x_0$$

be the corresponding  $m$ -path in  $\tilde{\mathcal{P}}$ . We define  $G_2(P)$  to be the graph with vertex set  $V(G_2(P)) = \{x_i^j : i = 0, 1, \dots, m; j \in \mathbb{Z}_2\}$  and edge set  $E(G_2(P)) = \{x_{i_1}^j x_{i_2}^j : x_{i_1}x_{i_2} \in E(T(P)), j \in \mathbb{Z}_2\} \cup \{x_{i_1}^j x_{i_2}^{1-j} : x_{i_1}x_{i_2} \in E(P), j \in \mathbb{Z}_2\}$ ; that is,  $G_2(P)$  is obtained from  $P(2)$  by replacing each of the two copies of  $P$  by  $T(P)$ .

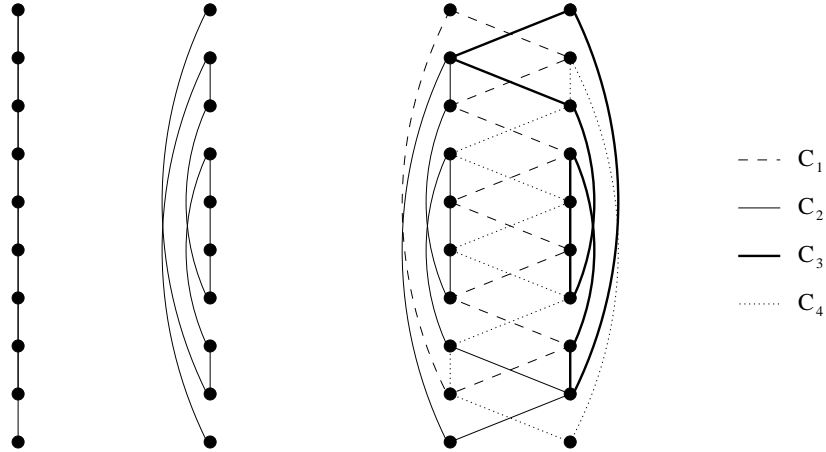


Figure 2: Lemma 2.2.1:  $m$ -paths  $P$  and  $T(P)$ , and a  $C_m$ -decomposition of  $G_2(P)$ .

It is not difficult to see that the following four  $m$ -cycles then form a decomposition of  $G_2(P)$ :

$$\begin{aligned} C_1 &= x_0^0x_1^1x_2^0x_3^1x_4^0 \dots x_{m-1}^0x_0^0, \\ C_2 &= x_1^0x_2^0 \dots x_s^0x_{t-1}^0x_{t-2}^0 \dots x_{s+2}^0x_{s+1}^0x_t^0 \dots x_{m-2}^0x_{m-1}^0x_m^0x_1^0, \\ C_3 &= x_0^1x_1^0x_2^1x_3^0 \dots x_s^1x_{t-1}^0x_{t-2}^1 \dots x_{s+2}^1x_{s+1}^0x_t^1 \dots x_{m-1}^1x_0^1, \text{ and} \\ C_4 &= x_1^1x_2^1x_3^0x_4^1 \dots x_{m-2}^0x_{m-1}^1x_m^1x_1^1. \end{aligned}$$

Since  $\{G_2(P) : P \in \mathcal{P}\}$  is a decomposition of  $X(2)$ , it now follows that the graph  $X(2)$  is  $C_m$ -decomposable.  $\square$

**Lemma 2.2.2** *Let  $G$  be a graph. If  $G$  is  $C_m$ -decomposable, then  $G(2)$  is  $C_m$ -decomposable.*

PROOF. It suffices to show that for an  $m$ -cycle  $C$ ,  $C(2)$  is  $C_m$ -decomposable. Let  $C = x_0x_1 \dots x_{m-1}x_0$ . Then  $E(C(2)) = \{x_i^{j_1}x_{i+1}^{j_2} : i \in Z_m; j_1, j_2 \in Z_2\}$ . It is not difficult to see that the following four  $m$ -cycles form a decomposition of  $C(2)$ :

$$\begin{aligned} C_1 &= x_0^0x_1^0 \dots x_{m-1}^0x_0^0, \\ C_2 &= x_0^0x_1^1x_2^1x_3^0x_4^1x_5^0 \dots x_{m-1}^1x_0^0, \\ C_3 &= x_0^1x_1^0x_2^1x_3^1 \dots x_{m-1}^1x_0^1, \text{ and} \\ C_4 &= x_0^1x_1^1x_2^0x_3^1 \dots x_{m-1}^0x_0^1. \end{aligned}$$

$\square$

We now discuss the parameters at play in the case that  $k = \frac{n}{2}$  is even. We let  $n = 2m + r$ , where  $r < m$  and  $r$  is even. Let  $d = \gcd(m, n) = \gcd(m, r)$  and let  $k = k'd$ ,  $m = m'd$ , and  $r = r'd$ . Notice that, since  $n(n-2) \equiv 0 \pmod{2m}$ ,  $d = 1$  implies  $n = 2m + 2$ , and  $m' = 1$  implies  $n = 2m$ , which are the two cases settled by Lemma 2.1.2. Hence we may assume that  $d \geq 3$  and  $m' \geq 3$ . Since  $n(n-2) \equiv 0 \pmod{2m}$ , we have  $k-1 \equiv 0 \pmod{m'}$ . Let  $k-1 = bm'$ . Since  $r \leq m-1$ , we have  $bm' = k-1 = m + \frac{r}{2} - 1 \leq \frac{3m-3}{2}$ . Since  $d$  is odd, it follows that  $b \leq \frac{3d-1}{2}$ .

We would now like to decompose  $K_k$  into  $m$ -paths and  $m$ -cycles. The number of edge lengths used in the  $m$ -paths from Lemma 2.2.1 is  $\frac{m+1}{2}$ . This leaves  $\frac{k}{2} - \frac{m+1}{2} = \frac{b-d}{2}m'$  edge lengths to be used in  $m$ -cycles, where  $c = \frac{b-d}{2}$  is an integer. As we shall see in the next lemma, this allows for a construction of  $c$  families of  $m$ -cycles, each using  $m'$  distinct edge lengths. Since  $b \leq \frac{3d-1}{2}$ , notice that  $c = \frac{b-d}{2} \leq \frac{d-1}{4}$ .

**Lemma 2.2.3** *Let  $k$  be even,  $m'$ ,  $k'$ , and  $c$  as defined above. There exists a subset  $S_c$  of the edge length set  $\{1, 2, \dots, \frac{k}{2} - 3\}$  with the following properties:*

1.  $1 \in S_c$  and  $2 \notin S_c$ ,

2.  $|S_c| = cm'$ , and

3.  $X(k; S_c)$  is  $C_m$ -decomposable.

PROOF. We prove the lemma by constructing  $c$  families of peripheral cycles in  $X(k; \{1, 2, \dots, \frac{k}{2} - 3\})$ . First assume  $m' \geq 5$ . Define a zig-zag  $(m' - 1)$ -path

$$P_{0,0} = u_0 u_{2+k'} u_{-2} u_{3+k'} u_{-3} \dots u_{-\frac{m'-1}{2}} u_{\frac{m'+1}{2}+k'} u_{k'-1}.$$

Notice that the edge length set of  $P_{0,0}$  is

$$L_0 = \left\{ \frac{m'+3}{2}, 2+k', 4+k', 5+k', \dots, m'+k' \right\}.$$

For  $j = 1, \dots, d-1$ , let  $P_{0,j} = \rho^{jk'}(P_{0,0})$ . Notice that, since  $\frac{m'-1}{2} + \frac{m'+1}{2} = m' \leq k' - 1$ , the  $(m' - 1)$ -paths  $P_{0,j}$  are pairwise vertex-disjoint. For  $i = 1, \dots, c - 1$ , obtain the zig-zag  $(m' - 1)$ -path  $P_{i,0}$  from  $P_{0,0}$  by adding  $2ik'$  to the subscripts of the even vertices. That is, let

$$P_{i,0} = u_0 u_{2+(2i+1)k'} u_{-2} u_{3+(2i+1)k'} u_{-3} \dots u_{-\frac{m'-1}{2}} u_{\frac{m'+1}{2}+(2i+1)k'} u_{k'-1}.$$

As before, let  $P_{i,j} = \rho^{jk'}(P_{i,0})$  for  $j = 1, \dots, d - 1$ . For  $i = 0, 1, \dots, c - 1$ , let  $\mathcal{P}_i = \{P_{i,j} : j = 0, \dots, d - 1\}$ . The set of edge lengths used by the family of  $(m' - 1)$ -paths  $\mathcal{P}_i$  is thus  $L_i = L_0 + 2ik'$ .

We now use the technique described on page 6 to connect each family of  $(m' - 1)$ -paths into an  $m$ -cycle using the auxiliary circulant  $X = X(d; \{1, 2, 4, 6, \dots, 2\lfloor \frac{c}{2} \rfloor\})$ . If  $c$  is even,  $X$  is first decomposed into circulants  $X(d; \{1\})$ ,  $X(d; \{2\})$ ,  $X(d; \{4, 6\})$ ,  $\dots$ ,  $X(d; \{c - 2, c\})$ , or  $X(d; \{1\})$ ,  $X(d; \{2\})$ ,  $X(d; \{4\})$ ,  $X(d; \{6, 8\})$ ,  $\dots$ ,  $X(d; \{c - 2, c\})$ , depending on whether  $\frac{c}{2}$  is odd or even. Since  $d$  is odd, the circulants  $X(d; \{2i, 2i+2\})$  are connected and hence decomposable into Hamilton cycles by Theorem 1.3.2. We thus have a Hamilton decomposition of  $X$ . Since the terminal vertex of each of the paths  $P_{i,0}$  is  $u_{k'-1}$ , an edge of length  $\ell$  in a  $d$ -cycle  $C$  gives rise to a connecting edge of length  $(\ell - 1)k' + 1$  or  $(\ell + 1)k' - 1$ , depending on the orientation of  $C$  we are using, as explained on page 6. Each of the  $d$ -cycles  $X(d; \{1\})$  and  $X(d; \{2\})$  will be used with only one orientation; namely, the orientation that gives rise to connecting edges of length 1 and  $k' + 1$ , respectively, whereas  $X(d; \{4\})$  is used with both orientations, thus giving rise to connecting edges of lengths  $3k' + 1$  and  $5k' - 1$ .

For the circulants  $X(d; \{2i, 2i + 2\})$  we have no choice; each is used to link up 4 families of  $(m' - 1)$ -paths, giving rise to connecting edges of lengths  $(2i - 1)k' + 1$ ,  $(2i + 1)k' - 1$ ,  $(2i + 1)k' + 1$ , and  $(2i + 3)k' - 1$ . The set of lengths of the connecting edges is thus

$$L_X = \{1, k' + 1\} \cup \{(2i - 1)k' + 1 : i = 2, \dots, \frac{c}{2}\} \cup \{(2i + 1)k' - 1 : i = 2, \dots, \frac{c}{2}\}.$$

If  $c$  is odd,  $X$  is first decomposed into  $X(d; \{1\})$ ,  $X(d; \{2, 4\})$ ,  $\dots$ ,  $X(d; \{c - 3, c - 1\})$ , or  $X(d; \{1\})$ ,  $X(d; \{2\})$ ,  $X(d; \{4, 6\})$ ,  $\dots$ ,  $X(d; \{c - 3, c - 1\})$ , depending on whether  $\frac{c-1}{2}$  is even or odd. The  $d$ -cycle  $X(d; \{1\})$  will be used only with the orientation that gives rise to connecting edges of length 1, whereas the  $d$ -cycle  $X(d; \{2\})$  is now used with both orientations. The set of connecting edge lengths is thus

$$L_X = \{1\} \cup \{(2i - 1)k' + 1 : i = 1, \dots, \frac{c-1}{2}\} \cup \{(2i + 1)k' - 1 : i = 1, \dots, \frac{c-1}{2}\}.$$

Notice that in both cases  $L_X$  and  $\bigcup_{i=0}^{c-1} L_i$  are disjoint. Let  $S_c = (\bigcup_{i=0}^{c-1} L_i) \cup L_X$ . If  $C_i$  is the  $m$ -cycle arising from the family  $\mathcal{P}_i$  of  $(m' - 1)$ -paths, then

$$\{\rho^j(C_i) : i = 0, \dots, c - 1; j = 0, \dots, k' - 1\}$$

is a  $C_m$ -decomposition of the circulant  $X(k; S_c)$ . The longest edge in  $X(k; S_c)$  has length  $m' + (2c - 1)k'$ . Since  $m' \leq k' - 1$ ,  $c \leq \frac{d-1}{4}$ , and  $m' \geq 5$ , we have  $m' + (2c - 1)k' \leq 2ck' - 1 \leq \frac{k}{2} - 4$ . It is now easy to see that the set  $S_c$  satisfies Conditions 1 - 3 of the lemma.

Now let  $m' = 3$  and hence  $k' = 4$ . Let

$$P_{0,0} = u_0 u_{2+k'} u_3 \quad \text{and} \quad P_{1,0} = u_0 u_{2+2k'} u_3,$$

and for  $i = 2, \dots, c - 1$ , obtain  $P_{i,0}$  from  $P_{1,0}$  by adding  $2(i - 1)k'$  to the subscript of the second vertex. The construction can now be completed as in Case  $m' \geq 5$ .  $\square$

We are now ready to prove the main result of this section.

**Lemma 2.2.4** *Let  $n$  be an even integer and  $m$  be an odd integer such that  $6 \leq 2m \leq n < 3m$  and  $n(n - 2) \equiv 0 \pmod{2m}$ . Then  $K_n - I$  is  $C_m$ -decomposable.*

PROOF.  $K_n - I$  is isomorphic to  $K_k(2)$ , where  $k = \frac{n}{2}$ . Notice that  $n(n-2) \equiv 0 \pmod{2m}$  implies  $k(k-1) \equiv 0 \pmod{2m}$ .

If  $k$  is odd, since  $k(k-1) \equiv 0 \pmod{2m}$  and  $k \geq m$ ,  $K_k$  is  $C_m$ -decomposable by the result of Alspach and Gavlas [1]. Hence  $K_k(2)$  is  $C_m$ -decomposable by Lemma 2.2.2.

We may now assume that  $k$  is even. Let  $d, k', m', r'$ , and  $c$  be defined as in the discussion on page 12. We have seen that the cases  $d = 1$  and  $m' = 1$  have already been settled by Lemma 2.1.2, so we may assume that  $d \geq 3$  and  $m' \geq 3$ . Let  $S_c$  be a set satisfying the conditions of Lemma 2.2.3. It is not difficult to see that the set  $S_p = \{1, \dots, \frac{k}{2}\} - S_c$  then satisfies the conditions of Lemma 2.2.1. We now decompose  $K_k$  into  $X(k; S_c)$  and  $X(k; S_p)$ . Since  $X(k; S_c)(2)$  is  $C_m$ -decomposable by Lemmas 2.2.3 and 2.2.2, and  $X(k; S_p)(2)$  is  $C_m$ -decomposable by Lemma 2.2.1,  $K_k(2)$  is  $C_m$ -decomposable.  $\square$

### 2.3 $C_m$ -decomposition of $K_n - I$ for $m \leq n < 2m$

Throughout this section it is assumed that  $n$  is an even integer and  $m$  is an odd integer such that  $3 \leq m \leq n < 2m$  and  $n(n-2) \equiv 0 \pmod{2m}$ .

We shall view the graph  $K_n$  minus a 1-factor as the join  $(K_{n-2} - I) \bowtie \overline{K_2}$  where the vertex set of  $K_{n-2}$  is  $\{u_0, u_1, \dots, u_{n-3}\}$ , the vertex set of  $\overline{K_2}$  is  $\{v, w\}$ , and the 1-factor of  $K_n$  consists of all the edges of diameter length  $\frac{n-2}{2}$  in  $K_{n-2}$  together with the edge  $vw$ . We refer to  $v$  and  $w$  as the central vertices and denote the diameter length  $\frac{n-2}{2}$  by  $D$ .

Let us now discuss the parameters. The remainder  $r = n - m$  is clearly odd and  $1 \leq r \leq m - 2$ . Let  $d = \gcd(m, n - 2)$ . Then  $d$  is odd and  $d = \gcd(m, r - 2)$  as well. Let  $n - 2 = dn'$ ,  $m = dm'$ , and  $r - 2 = dr'$ . We thus have  $D = \frac{dn'}{2}$ . Notice that the assumptions on  $m$  and  $n$  imply that  $d \geq 3$  and  $m' \geq 3$ .

Now if  $K_n - I$  is  $C_m$ -decomposable, the number of  $m$ -cycles is going to be  $\frac{n(n-2)}{2m} = \frac{n-2}{2} + \frac{1}{2}(r + \frac{r(r-2)}{m})$ , which must be an integer. We thus find that  $\frac{r(r-2)}{m}$  must be an odd integer. Hence there exists an odd integer  $c$  such that  $r = cm'$ . Notice that, since  $r \leq m - 2$ , we have  $c \leq d - \frac{2}{m'}$ . Since  $c$  and  $d$  are both odd integers,  $c \leq d - 2$ . The above expression for the number of  $m$ -cycles thus attains the form



$\frac{n(n-2)}{2m} = \frac{n-2}{2} + c\frac{n'}{2}$ . This suggests the following scheme:  $\frac{n-2}{2}$  central cycles will be generated by applying the rotation  $\rho^{2i}$ , for  $i = 0, \dots, \frac{n-2}{2} - 1$ , to a starter central cycle that contains both central vertices; this will take care of all the edges between  $K_{n-2}$  and  $\overline{K_2}$ . In addition,  $c\frac{n'}{2}$  peripheral cycles will be found by applying  $\rho^{2i}$ ,  $i = 0, \dots, \frac{n'}{2} - 1$ , to a solitary peripheral cycle, and by applying  $\rho^i$ ,  $i = 0, \dots, n' - 1$ , to each of  $\frac{c-1}{2}$  coupled peripheral cycles. Recall that, since  $n - 2$  is even, the permutation group  $\langle \rho^2 \rangle$  has two orbits on the set of edges of a fixed length as explained on page 4.

The following lemma establishes a basis for the construction of central cycles.

**Lemma 2.3.1** *Let  $L_0$  and  $L_C$  be two disjoint subsets of the edge length set  $\{1, \dots, D-1\}$  and let  $P$  be a zig-zag  $m'$ -path with edge length set  $L_0$ . Furthermore, let  $R_1$  and  $R_2$  be two vertex-disjoint paths in  $K_{n-2} - I$  with the following properties:*

1. *the length of every edge of  $R_1 \oplus R_2$  is in  $L_0 \cup L_C$ ,*
2.  *$R_1 \oplus R_2$  contains exactly one edge of each of the lengths in  $L_0$  and this edge belongs to the same orbit of  $\langle \rho^2 \rangle$  as the edge of the same length in  $P$ ,*
3.  *$R_1 \oplus R_2$  contains exactly two edges of each of the lengths in  $L_C$ , one from each of the two orbits of  $\langle \rho^2 \rangle$ ,*
4.  *$|E(R_1 \oplus R_2)| = m - 4$ , and*
5. *among the four vertices of degree 1 in  $R_1 \oplus R_2$  exactly two have odd subscripts.*

*Then  $(X(n-2; L_C) \oplus \langle \rho^2 \rangle(P)) \bowtie \overline{K_2}$  is  $C_m$ -decomposable.*

PROOF. Conditions 1 – 3 imply that  $\{\rho^{2i}(R_1 \oplus R_2) : i = 0, \dots, D-1\}$  is a partition of the edges of  $X(n-2; L_C) \oplus \langle \rho^2 \rangle(P)$ . Now let the endpoints of the path  $R_j$  be  $u_{s_j}$  and  $u_{t_j}$ , and let the vertices of  $\overline{K_2}$  be  $v$  and  $w$ . By Condition 5, without loss of generality, only two cases may arise: either  $s_1$  and  $t_1$  are odd and  $s_2$  and  $t_2$  are even, or  $s_1$  and  $s_2$  are odd and  $t_1$  and  $t_2$  are even. In both cases, if we define an  $m$ -cycle by  $C = vu_{s_1}R_1u_{t_1}wu_{s_2}R_2u_{t_2}v$ , it is not difficult to see that  $\{\rho^{2i}(C) : i = 0, \dots, D-1\}$  is a  $C_m$ -decomposition of  $(X(n-2; L_C) \oplus \langle \rho^2 \rangle(P)) \bowtie \overline{K_2}$ .  $\square$

In the next lemma we explain the details of the construction of central cycles. In Lemma 2.3.3 we then show that peripheral cycles can be found so that the remaining edge lengths satisfy the conditions of Lemma 2.3.2.

**Lemma 2.3.2** *Define a zig-zag  $m'$ -path  $P_{0,0}$  by*

$$P_{0,0} = u_0 u_1 u_{-\frac{r'+1}{2}} u_2 u_{-\frac{r'+3}{2}} \dots u_{\frac{m'-1}{2}} u_{-(\frac{n'}{2}-1)} u_{-\frac{n'}{2}+D}.$$

Furthermore, let  $L_C$  be a set satisfying the following conditions:

1.  $|L_C| = \frac{1}{2}(m - 4 - m')$ ,
2.  $L_C = L_B \cup L_A$ , where  $L_B = \{2, \dots, \frac{r'+1}{2}\}$ ,
3.  $D - 2 \in L_A \subseteq \{\frac{m'-1}{2} + \frac{n'}{2}, \dots, D - 2\}$ , and
4. if  $m' = 3$ ,  $L_A \subseteq \{n', \dots, D - 2\}$ .

Then  $(X(n - 2; L_C) \oplus \langle \rho^2 \rangle (\rho^{-1}(P_{0,0}))) \rtimes \overline{K_2}$  is  $C_m$ -decomposable.

PROOF. First observe that the edge length set of  $P_{0,0}$  is

$$L_0 = \{1, \frac{r'+3}{2}, \frac{r'+5}{2}, \dots, \frac{m'-3}{2} + \frac{n'}{2}, D - 1\} \quad (1)$$

so that  $L_0 \cap L_C = \emptyset$ . It thus makes sense to define the graph  $G = X(n - 2; L_C) \oplus \langle \rho^2 \rangle (\rho^{-1}(P_{0,0}))$ . We shall construct paths  $R_1$  and  $R_2$  that satisfy the conditions of Lemma 2.3.1 for this graph.

First we introduce the notation. Observe that the set  $L_B$  is empty for  $r' = 1$ . Consequently, the numbers  $B$  and  $B_1$ , and the path  $P_B$  below are defined only for  $r' \geq 3$ . Let

$$\begin{aligned} N &= |L_A| = \frac{1}{2}(m - 4 - m') - \frac{r'-1}{2} = \frac{1}{2}(m - m' - r' - 3), \\ L_A &= \{a_1, \dots, a_{N-1}, D - 2\}, \text{ where } \frac{m'-1}{2} + \frac{n'}{2} \leq a_1 < \dots < a_{N-1} < D - 2, \\ A_2 &= -\frac{n'}{2} + a_1 - a_2 + \dots + (-1)^N a_{N-1}, \\ A &= A_2 + (-1)^{N+1}(D - 2), \\ B &= -\frac{r'+1}{2} + \frac{r'-1}{2} - \frac{r'-3}{2} + \dots + (-1)^{\frac{r'-3}{2}} 3 + (-1)^{\frac{r'-1}{2}} 2, \text{ and} \\ B_1 &= B + (-1)^{\frac{r'+1}{2}}. \end{aligned}$$

Note that the numbers  $A$ ,  $A_2$ ,  $B$ , and  $B_1$ , evaluated in the integers, lie on the interval  $(-D, D)$ . Thus  $A < 0$  means the vertex  $u_A$  is in the set  $\{u_{D+1}, u_{D+2}, \dots, u_{2D-1}\}$  while  $A > 0$  means the vertex  $u_A$  is in the set  $\{u_1, u_2, \dots, u_{D-1}\}$ .

Let  $P_{A_2}$ ,  $P_A$ ,  $P_B$ , and  $Q$  be the following zig-zag paths:

$$\begin{aligned} P_{A_2} &= u_{-\frac{n'}{2}} u_{-\frac{n'}{2}+a_1} u_{-\frac{n'}{2}+a_1-a_2} \cdots u_{A_2}, \\ P_A &= P_{A_2} u_{A_2} u_A, \\ P_B &= u_B u_{B-(-1)\frac{r'-1}{2}} \cdots u_{-\frac{r'+1}{2}+\frac{r'-1}{2}} u_{-\frac{r'+1}{2}} u_0, \text{ and} \\ Q &= u_0 u_{-\frac{r'+3}{2}} u_1 u_{-\frac{r'+5}{2}} \cdots u_{\frac{m'-3}{2}} u_{-\frac{n'}{2}} \text{ so that} \\ \rho^{-1}(P_{0,0}) &= u_{-1} u_0 Q u_{-\frac{n'}{2}} u_{-\frac{n'}{2}+D-1}. \end{aligned}$$

To link these paths into paths  $R_1$  and  $R_2$ , we shall use an edge of length 1 and an edge of length  $D - 1$  from the same orbit of  $\langle \rho^2 \rangle$  as the edges of the same length in  $\rho^{-1}(P_{0,0})$ ; that is, from the orbit  $\langle \rho^2 \rangle(u_{-1} u_0)$  and  $\langle \rho^2 \rangle(u_{-\frac{n'}{2}} u_{-\frac{n'}{2}+D-1})$ , respectively. In some cases, two edges of length  $D - 2$  from distinct orbits of  $\langle \rho^2 \rangle$  will also be used as linking edges.

For ease of reference, let us list the edge length sets of the zig-zag paths  $P_{A_2}$ ,  $P_A$ ,  $P_B$ , and  $Q$ :

$$\begin{aligned} L(P_{A_2}) &= \{a_1, \dots, a_{N-1}\}, \\ L(P_A) &= L_A = \{a_1, \dots, a_{N-1}, D - 2\}, \\ L(P_B) &= L_B = \{2, 3, \dots, \frac{r'+1}{2}\}, \text{ and} \\ L(Q) &= \{\frac{r'+3}{2}, \dots, \frac{m'-3}{2} + \frac{n'}{2}\} = L_0 - \{1, D - 1\}. \end{aligned}$$

The paths  $R_1$  and  $R_2$  will be constructed in such a way that Conditions 1 – 3 of Lemma 2.3.1 are satisfied for the edge length sets  $L_0$  and  $L_C$ , and the  $m'$ -path  $P = \rho^{-1}(P_{0,0})$ . Notice that this implies that the total number of edges in  $R_1 \oplus R_2$  is  $|L_0| + 2|L_C| = m - 4$ , which satisfies Condition 4 of the same lemma.

The details of the construction of the paths  $R_1$  and  $R_2$  depend on the residue class of  $n'$  modulo 4, on whether  $r' \geq 3$  or  $r' = 1$ , and on whether  $A$  and  $N$  are odd or even (notice that  $N$  is odd if and only if  $A > 0$ ). Keep in mind that to be able to employ Lemma 2.3.1, we also need to make sure that Condition 5 is satisfied; that is, that exactly two of the four endpoints of the paths  $R_1$  and  $R_2$  have odd subscripts.

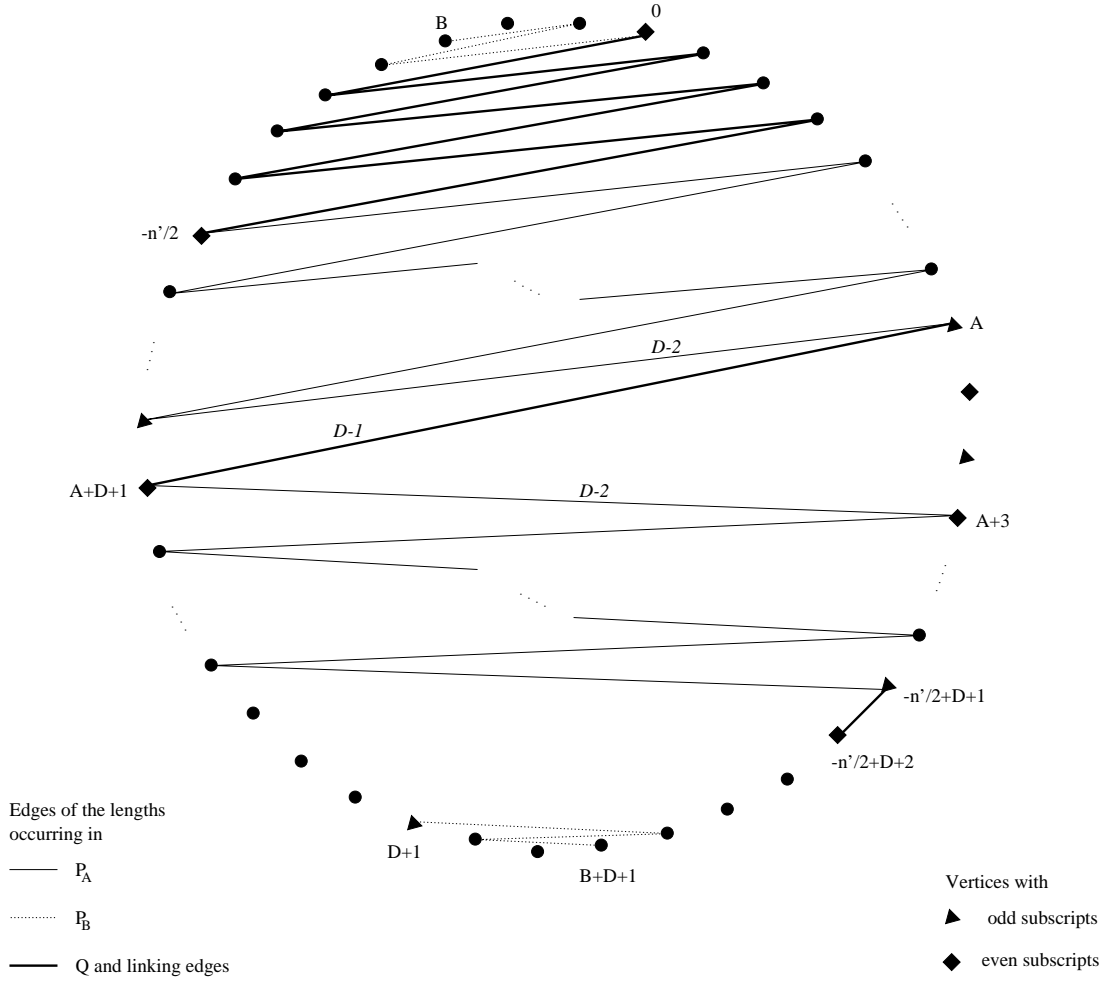


Figure 3: Lemma 2.3.2: the paths  $R_1$  and  $R_2$  for  $r' \geq 3$ ,  $n' \equiv 0 \pmod{4}$ ,  $A$  odd (subcase  $N$  odd shown).

We now describe the paths  $R_1$  and  $R_2$  for the various cases. First assume  $r' \geq 3$ .

1. Case  $n' \equiv 0 \pmod{4}$ . This implies  $\frac{n'}{2}$  and  $D$  are even.

1.1. Subcase  $A$  odd. Let

$$\begin{aligned}
 R_1 &= P_B Q P_A u_A u_{A+D+1} \rho^{D+1} (\overline{P_A}) u_{-\frac{n'}{2}+D+1} u_{-\frac{n'}{2}+D+2} \text{ and} \\
 R_2 &= \rho^{D+1} (P_B).
 \end{aligned}$$

Since  $r' \geq 3$ , it follows that  $m' \geq 5$  which in turn implies that  $R_1$  and  $R_2$  are indeed vertex-disjoint. The subscripts of the endpoints are:  $B$ ,  $B + D + 1$  (one odd, one even),  $D + 1$  (odd), and  $-\frac{n'}{2} + D + 2$  (even). The orbits of  $\langle \rho^2 \rangle$  containing the linking

edges are as follows:

$$\begin{aligned}\langle \rho^2 \rangle(u_A u_{A+D+1}) &= \langle \rho^2 \rangle(u_0 u_{D-1}) = \langle \rho^2 \rangle(u_{-\frac{n'}{2}} u_{-\frac{n'}{2}+D-1}) \text{ and} \\ \langle \rho^2 \rangle(u_{-\frac{n'}{2}+D+1} u_{-\frac{n'}{2}+D+2}) &= \langle \rho^2 \rangle(u_{-1} u_0).\end{aligned}$$

In the remaining cases we leave it to the reader to examine the subscripts of the endpoints of  $R_1$  and  $R_2$  and the orbits of the linking edges (that is, the edges of lengths 1 and  $D-1$ , and, in some cases,  $D-2$ ).

1.2. Subcase  $A$  even. Let

$$\begin{aligned}R_1 &= P_B Q P_A u_A u_{A+D-1} \rho^{D-1}(\overleftarrow{P_A}) u_{-\frac{n'}{2}+D-1} u_{-\frac{n'}{2}+D} \text{ and} \\ R_2 &= \rho^{D-1}(P_B).\end{aligned}$$

2. Case  $n' \equiv 2 \pmod{4}$ . This implies  $\frac{n'}{2}$  and  $D$  are odd.

2.1. Subcase  $N$  even. Hence  $A < 0$ .

If  $A$  is odd, let

$$\begin{aligned}R_1 &= P_B Q P_A u_A u_{A+D-1} u_{A+D} \rho^D(\overleftarrow{P_A}) \text{ and} \\ R_2 &= \rho^D(P_B).\end{aligned}$$

If  $A$  is even, let

$$\begin{aligned}R_1 &= P_B Q P_A u_A u_{A-1} u_{A+D} \rho^D(\overleftarrow{P_A}) \text{ and} \\ R_2 &= \rho^D(P_B).\end{aligned}$$

2.2. Subcase  $N$  odd. Hence  $A = A_2 + D - 2$  and  $A_2 < 0$ .

If  $A$  is even, let

$$\begin{aligned}R_1 &= \rho^{D-2}(\overleftarrow{P_B}) u_{B+D-2} u_B P_B Q P_{A_2} u_{A_2} u_{A_2+D-1} u_{A_2+D} \rho^D(\overleftarrow{P_{A_2}}) \text{ and} \\ R_2 &= u_{B_1} u_{B_1+D-2}.\end{aligned}$$

Observe that since  $n' \equiv 2 \pmod{4}$  and  $r' \geq 3$ ,  $m' \geq 7$ . Hence  $\rho^D(\overleftarrow{P_{A_2}})$  and  $\rho^{D-2}(P_B)$  are vertex-disjoint.

If  $A$  is odd, let

$$\begin{aligned}R_1 &= \rho^{D-2}(\overleftarrow{P_B}) u_{B+D-2} u_B P_B Q P_{A_2} u_{A_2} u_{A_2-1} u_{A_2+D-2} \rho^{D-2}(\overleftarrow{P_{A_2}}) \text{ and} \\ R_2 &= u_{B_1} u_{B_1+D-2}.\end{aligned}$$

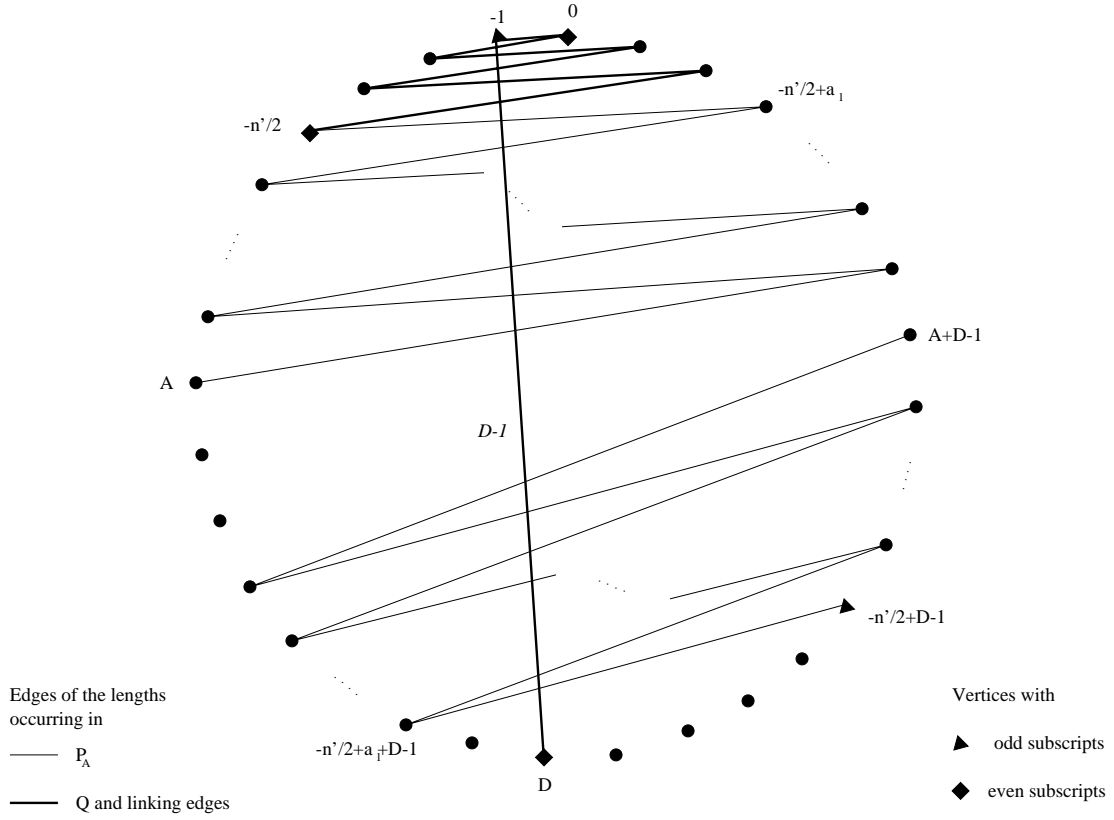


Figure 4: *Lemma 2.3.2:  $R_1$  and  $R_2$  for  $r' = 1$ ,  $n' \equiv 0(\text{mod } 4)$  (subcase  $A < 0$  shown here).*

Now let  $r' = 1$ .

1. Case  $n' \equiv 0(\text{mod } 4)$ . This implies  $\frac{n'}{2}$  and  $D$  are even. Let

$$\begin{aligned} R_1 &= u_D u_{-1} u_0 Q P_A \text{ and} \\ R_2 &= \rho^{D-1}(P_A). \end{aligned}$$

Since  $a_1 \geq m' = n' - 1$  for  $m' \geq 5$  and  $a_1 \geq n'$  for  $m' = 3$  by Conditions 3 and 4, we have  $-\frac{n'}{2} + a_1 \geq 2$ . Hence  $-\frac{n'}{2} + a_1 + D - 1 \geq D + 1 > D$  so that  $R_1$  and  $R_2$  are vertex-disjoint.

2. Case  $n' \equiv 2(\text{mod } 4)$ . This implies  $\frac{n'}{2}$  and  $D$  are odd. Let

$$\begin{aligned} R_1 &= u_{D-2} u_{-1} u_0 Q P_A \text{ and} \\ R_2 &= \rho^D(P_A). \end{aligned}$$

Since  $n' \geq 6$ , we have  $-\frac{n'}{2} + D \leq D - 3 < D - 2$  so that  $R_1$  and  $R_2$  are vertex-disjoint.

We have thus constructed the paths  $R_1$  and  $R_2$  that satisfy the conditions of Lemma 2.3.1. The graph  $(X(n-2; L_C) \oplus \langle \rho^2 \rangle (\rho^{-1}(P_{0,0}))) \rtimes \overline{K_2}$  is therefore  $C_m$ -decomposable.  $\square$

In the next lemma we show how to construct peripheral cycles.

**Lemma 2.3.3** *As in Lemma 2.3.2, let the zig-zag  $m'$ -path  $P_{0,0}$  be*

$$P_{0,0} = u_0 u_1 u_{-\frac{r'+1}{2}} u_2 u_{-\frac{r'+3}{2}} \dots u_{\frac{m'-1}{2}} u_{-(\frac{n'}{2}-1)} u_{\frac{d-1}{2}n'}$$

and denote its edge length set by  $L_0$ . There exists a set  $L_P$  with the following properties:

1.  $L_P \subseteq \{n' - 1, \dots, D - 3\}$ ,
2. if  $m' = 3$ ,  $n' - 1 \in L_P$ ,
3.  $|L_P| = \frac{c-1}{2}m'$ , and
4.  $X(n-2; L_P) \oplus \langle \rho^2 \rangle (P_{0,0})$  is  $C_m$ -decomposable.

PROOF. First we construct the solitary peripheral cycles. Observe that the edge length set  $L_0$  of the zig-zag path  $P_{0,0}$  as defined above is given in (1). Now let  $P_{0,j} = \rho^{jn'}(P_{0,0})$  for  $j = 1, \dots, d-1$ . Since the second longest edge in  $P_{0,0}$  has length  $\frac{m'-3}{2} + \frac{n'}{2} \leq n' - 2$ , the paths  $P_{0,j}$  are pairwise vertex-disjoint except for the endpoints. Since  $\gcd(d, \frac{d-1}{2}) = 1$ ,  $C_0 = \bigoplus_{j=0}^{d-1} P_{0,j}$  is an  $m$ -cycle. Hence  $\{\rho^{2i}(C_0) : i = 0, \dots, \frac{n'}{2} - 1\}$  is a  $C_m$ -decomposition of  $X(n-2; \emptyset) \oplus \langle \rho^2 \rangle (P_{0,0})$ .

Next we describe the coupled peripheral cycles. First assume that  $m' \geq 5$  and  $r' \geq 5$ . Define the zig-zag  $(m' - 1)$ -path  $P_{1,0}$  by

$$P_{1,0} = u_0 u_{2+n'} u_{-2} u_{3+n'} u_{-3} \dots u_{\frac{m'+3}{4}+n'} u_{-\frac{m'+7}{4}} u_{\frac{m'+7}{4}+n'} \dots u_{-\frac{m'+1}{2}} u_{\frac{m'+1}{2}+n'} u_{-1}$$

if  $m' \equiv 1 \pmod{4}$ , and by

$$P_{1,0} = u_0 u_{2+n'} u_{-2} u_{3+n'} u_{-3} \dots u_{\frac{m'+1}{4}+n'} u_{-\frac{m'+5}{4}} u_{\frac{m'+9}{4}+n'} u_{-\frac{m'+9}{4}} \dots u_{-\frac{m'+1}{2}} u_{\frac{m'+3}{2}+n'} u_{-1}$$

if  $m' \equiv 3 \pmod{4}$ . The edge length set of the path  $P_{1,0}$  is therefore

$$L_1 = \{2 + n', 4 + n', 5 + n', \dots, m' + 1 + n'\}$$

if  $m' \equiv 1 \pmod{4}$ , and

$$L_1 = \{2 + n', 4 + n', 5 + n', \dots, \frac{m'-1}{2} + n', \frac{m'+3}{2} + n', \dots, m' + 2 + n'\}$$

if  $m' \equiv 3 \pmod{4}$ . For  $j = 1, \dots, d-1$  let  $P_{1,j} = \rho^{jn'}(P_{1,0})$ . Since  $r' \geq 5$ , we have  $\frac{m'+3}{2} + \frac{m'+1}{2} < n'$ , implying that in both cases these paths are pairwise vertex-disjoint.

For  $i = 2, \dots, \frac{c-1}{2}$ , obtain the zig-zag  $(m' - 1)$ -path  $P_{i,0}$  from  $P_{1,0}$  by adding  $(i-1)n'$  to the subscripts of the even vertices. The paths in each of the families  $\mathcal{P}_i = \{\rho^{jn'}(P_{i,0}) : j = 0, \dots, d-1\}$  are thus pairwise vertex-disjoint. The edge length set of  $P_{i,0}$  is  $L_i = L_1 + (i-1)n'$ .

Again, we use the technique described on page 6 to connect each family of  $(m' - 1)$ -paths into an  $m$ -cycle. If  $\frac{c-1}{2}$  is even, use the auxiliary circulant  $X = X(d; \{2, 3, \dots, \frac{c+3}{4}\})$ , decomposed into circulants  $X(d; \{2, 3\})$ ,  $\dots$ ,  $X(d; \{\frac{c-1}{4}, \frac{c+3}{4}\})$ , or  $X(d; \{2\})$ ,  $X(d; \{3, 4\})$ ,  $\dots$ ,  $X(d; \{\frac{c-1}{4}, \frac{c+3}{4}\})$ , depending on whether  $\frac{c-1}{4}$  is even or odd. Taking each  $d$ -cycle in this decomposition with both orientations, thus connecting  $\frac{c-1}{2}$  families of  $(m' - 1)$ -paths altogether, we use precisely all the edges with their lengths in the set

$$L_X = \{\pm 1 + in' : i = 2, \dots, \frac{c+3}{4}\}.$$

If  $\frac{c-1}{2}$  is odd, use the auxiliary circulant  $X = X(d; \{1, 2, 3, \dots, \frac{c+1}{4}\})$  and decompose it into  $X(d; \{1\})$ ,  $X(d; \{2, 3\})$ ,  $\dots$ ,  $X(d; \{\frac{c-3}{4}, \frac{c+1}{4}\})$ , or  $X(d; \{1\})$ ,  $X(d; \{2\})$ ,  $X(d; \{3, 4\})$ ,  $\dots$ ,  $X(d; \{\frac{c-3}{4}, \frac{c+1}{4}\})$ , depending on whether  $\frac{c+1}{4}$  is odd or even. Obtain a Hamilton decomposition of  $X$  as before and then use each  $d$ -cycle in this decomposition with both orientations except for the  $d$ -cycle  $X(d; \{1\})$ , which we use only with the orientation resulting in connecting edges of length  $n' + 1$ . The set of lengths of the connecting edges is thus

$$L_X = \{n' + 1\} \cup \{\pm 1 + in' : i = 2, \dots, \frac{c+1}{4}\}.$$

Since  $r' \geq 5$ , the sets  $L_X$  and  $\bigcup_{i=1}^{\frac{c-1}{2}} L_i$  are disjoint in both cases. Hence the set  $L_P = (\bigcup_{i=1}^{\frac{c-1}{2}} L_i) \cup L_X$  has size  $\frac{c-1}{2}m'$ , satisfying Condition 3 of the lemma. If  $C_i$  is the



$m$ -cycle arising from the family  $\mathcal{P}_i$  of  $(m' - 1)$ -paths, then  $\{\rho^j(C_i) : i = 1, \dots, \frac{c-1}{2}, j = 0, \dots, n' - 1\}$  is a  $C_m$ -decomposition of the circulant  $X(n - 2; L_P)$ . Since the shortest edge of  $X(n - 2; L_P)$  has length at least  $n' + 1$  and the longest edge has length at most  $m' + 2 + \frac{c-1}{2}n' \leq \frac{c+1}{2}n' - 3 \leq \frac{d-1}{2}n' - 3$ ,  $L_P$  satisfies Condition 1 of the lemma. Since  $L_P$  and  $L_0$  are disjoint, it makes sense to define the graph  $X(n - 2; L_P) \oplus \langle \rho^2 \rangle(P_{0,0})$  and this graph is  $C_m$ -decomposable as shown above. Hence we have found a set  $L_P$  that satisfies all conditions of the lemma.

For the remaining cases, we highlight the differences from the case  $m' \geq 5, r' \geq 5$ , and let the reader do most of the checking.

Next, let  $m' \geq 5$  with  $r' \leq 3$ . Define the zig-zag  $(m' - 1)$ -path  $P_{1,0}$  by

$$P_{1,0} = u_0 u_{2+2n'} u_{-1} u_{3+2n'} u_{-2} \dots u_{-\frac{m'-3}{2}} u_{\frac{m'+1}{2}+2n'} u_{1+n'}.$$

Its edge length set is  $L_1 = \{\frac{m'-1}{2} + n', 2 + 2n', 3 + 2n', \dots, m' - 1 + 2n'\}$ . For  $i = 2, \dots, \frac{c-1}{2}$ , obtain the zig-zag  $(m' - 1)$ -path  $P_{i,0}$  from  $P_{1,0}$  by adding  $2(i - 1)n'$  to the subscripts of the even vertices, and  $P_{i,j}, j = 0, \dots, d - 1$ , in the usual way. The edge length set of  $P_{i,0}$  thus is  $L_i = L_1 + 2(i - 1)n'$ .

With the same auxiliary circulant and its Hamilton decomposition as before, the  $d$ -cycle  $X(d; \{1\})$  should now be taken only with the orientation that results in connecting edges of length  $2n' + 1$ . This results in the set of connecting edge lengths

$$L_X = \{-1 + in' : i = 1, \dots, \frac{c-1}{4}\} \cup \{1 + in' : i = 3, \dots, \frac{c+7}{4}\}$$

if  $\frac{c-1}{2}$  is even, and

$$L_X = \{-1 + in' : i = 1, \dots, \frac{c-3}{4}\} \cup \{1 + in' : i = 2, \dots, \frac{c+5}{4}\}$$

if  $\frac{c-1}{2}$  is odd. The shortest edge of  $X(n - 2; L_P)$ , where  $L_P = (\bigcup_{i=1}^{\frac{c-1}{2}} L_i) \cup L_X$ , now has length  $n' - 1$  while the longest edge has length  $m' - 1 + (c - 1)n'$ . Since  $c = \frac{r'd+2}{m'}$ , it is easy to show that  $c < \frac{d}{2}$  in all cases except for  $m' = 5, r' = 3$ . It follows that the longest edge in  $X(n - 2; L_P)$  has length  $m' - 1 + (c - 1)n' < n' - 2 + (\frac{d}{2} - 1)n' = D - 2$ , which implies that Conditions 1 - 4 are satisfied except when  $m' = 5$  and  $r' = 3$ .

Now let  $m' = 5$  and  $r' = 3$ . Hence  $n' = 8$ . Define  $P_{1,0} = u_0 u_{2+n'} u_{-3} u_{3+n'} u_{-1}$ . Its edge length set is  $L_1 = \{2 + n', 4 + n', 5 + n', 6 + n'\}$ . Obtain  $P_{i,0}$  from  $P_{1,0}$  by

adding  $(i-1)n'$  to the subscripts of the even vertices so that the edge length set of  $P_{i,0}$  is  $L_i = L_1 + (i-1)n'$ , and then obtain  $P_{i,j}$ ,  $j = 1, \dots, d-1$ , in the usual way. To link the paths  $P_{i,j}$  into an  $m$ -cycle, we now use precisely the same method as in the case  $m' \geq 5$ ,  $r' \geq 5$ . Again, let  $L_P = (\bigcup_{i=1}^{\frac{c-1}{2}} L_i) \cup L_X$  and observe that the shortest edge length in  $X(n-2; L_P)$  is at least  $n'+1$ , while the longest is  $6 + \frac{c-1}{2}n' < D-4$ .

Finally, let  $m' = 3$  and hence  $n' = 4$ . Let the zig-zag 2-paths  $P_{1,0}$  and  $P_{1,0}^*$  be

$$P_{1,0} = u_0 u_{2+2n'} u_{-1} \quad \text{and} \quad P_{1,0}^* = u_0 u_{1+3n'} u_{-1}.$$

Their edge length sets are

$$L_1 = \{2 + 2n', -1 + 3n'\} \quad \text{and} \quad L_1^* = \{1 + 3n', 2 + 3n'\}.$$

For  $i = 2, \dots, \lceil \frac{c-1}{4} \rceil$  obtain  $P_{i,0}$  from  $P_{1,0}$  by adding  $2(i-1)n'$  to the subscript of the second vertex, and for  $i = 2, \dots, \lfloor \frac{c-1}{4} \rfloor$  obtain  $P_{i,0}^*$  from  $P_{1,0}^*$  in the same way. Denote the edge length sets of the zig-zag paths  $P_{i,0}$  and  $P_{i,0}^*$  by  $L_i$  and  $L_i^*$ , respectively. Then let  $\mathcal{P}_i = \{\rho^{jn'}(P_{i,0}) : j = 0, \dots, d-1\}$  and  $\mathcal{P}_i^* = \{\rho^{jn'}(P_{i,0}^*) : j = 0, \dots, d-1\}$ .

We now use the auxiliary circulant  $X = X(d; \{1, 2, 4, \dots, 2\lfloor \frac{c-3}{4} \rfloor\})$  to choose the linking edges. If  $\frac{c-1}{2}$  is even, decompose  $X$  into  $X(d; \{1, 2\})$ ,  $X(d; \{4, 6\})$ ,  $\dots$ ,  $X(d; \{\frac{c-7}{2}, \frac{c-5}{2}\})$ , or  $X(d; \{1\})$ ,  $X(d; \{2, 4\})$ ,  $\dots$ ,  $X(d; \{\frac{c-7}{2}, \frac{c-5}{2}\})$ , depending on whether  $\frac{c-5}{4}$  is odd or even. Using every  $d$ -cycle in the Hamilton decomposition of  $X$  thus obtained with both orientations results in the set of lengths of the connecting edges

$$L_X = \{n' - 1, n' + 1\} \cup \{\pm 1 + 2in' : i = 1, \dots, \frac{c-5}{4}\}.$$

If  $\frac{c-1}{2}$  is odd, decompose  $X$  into  $X(d; \{1\})$ ,  $X(d; \{2, 4\})$ ,  $\dots$ ,  $X(d; \{\frac{c-5}{2}, \frac{c-3}{2}\})$ , or  $X(d; \{1\})$ ,  $X(d; \{2\})$ ,  $X(d; \{4, 6\})$ ,  $\dots$ ,  $X(d; \{\frac{c-5}{2}, \frac{c-3}{2}\})$ , depending on whether  $\frac{c-3}{4}$  is even or odd. Take the  $d$ -cycle  $X(d; \{1\})$  with the orientation resulting in connecting edges of length  $n' - 1$  and all other cycles in the Hamilton decomposition of  $X$  with both orientations. The set of lengths of the connecting edges is thus

$$L_X = \{n' - 1\} \cup \{\pm 1 + 2in' : i = 1, \dots, \frac{c-3}{4}\}.$$

Observe that in both cases  $L_X$  and  $(\bigcup_{i=1}^{\lceil \frac{c-1}{4} \rceil} L_i) \cup (\bigcup_{i=1}^{\lfloor \frac{c-1}{4} \rfloor} L_i^*)$  are disjoint. Let  $L_P$  be their union. The set  $L_P$  thus satisfies Conditions 2 and 3 of the lemma and

$X(n-2; L_P)$  is  $C_m$ -decomposable. The least element of  $L_P$  is clearly  $n' - 1$ , while the greatest is at most

$$-1 + \frac{c+3}{2}n' = -1 + \frac{d+11}{6}n' \leq \frac{d-1}{2}n' - 1 = D - 3$$

since  $c = \frac{d+2}{3}$  and  $d \geq 7$ . Hence Conditions 1 and 4 are satisfied as well.

This proves the lemma for all cases.  $\square$

Finally, we show that the collection of central cycles from Lemma 2.3.2 and the collection of peripheral cycles from Lemma 2.3.3 form a  $C_m$ -decomposition of  $K_n - I$ .

**Lemma 2.3.4** *Let  $n$  be an even integer and  $m$  be an odd integer such that  $3 \leq m \leq n < 2m$  and  $n(n-2) \equiv 0 \pmod{2m}$ . Then  $K_n - I$  is  $C_m$ -decomposable.*

PROOF. Define the parameters  $r, d, m', n', r', c$ , and  $D$  as on page 15. As we have seen,  $d \geq 3$  and  $m' \geq 3$ . Observe that  $K_n - I$  is isomorphic to  $X(n-2; L) \rtimes \overline{K_2}$ , where  $L = \{1, \dots, \frac{dn'}{2} - 1\}$ . Let the zig-zag  $m'$ -path  $P_{0,0}$  be defined as in Lemma 2.3.3, let its edge length set be  $L_0$ , and let  $L_P$  be an edge length set satisfying the conditions of Lemma 2.3.3. It is not difficult to see that Conditions 1 – 4 of Lemma 2.3.2 are then satisfied for the set  $L_C = L - (L_P \cup L_0)$ . The graph  $X(n-2; L) \rtimes \overline{K_2}$  can now be partitioned into  $(X(n-2; L_C) \oplus \langle \rho^2 \rangle(P_{0,0})) \rtimes \overline{K_2}$ , which is  $C_m$ -decomposable by Lemma 2.3.2, and  $X(n-2; L_P) \oplus \langle \rho^2 \rangle(P_{0,0})$ , which is  $C_m$ -decomposable by Lemma 2.3.3. Hence  $K_n - I$  is  $C_m$ -decomposable.  $\square$

## 3 Decomposition of $K_n$ into cycles of even length

### 3.1 Proof of Theorem 1.1.2

As in Part 2, we first present the framework for the proof of the main result of this part, that is, Theorem 1.1.2, and then give the details of the construction in subsequent sections. The crucial step in the proof of Theorem 1.1.2 is the following result by Rodger [9].

**Theorem 3.1.1** [9] *Let  $n$  be an odd integer and  $m$  be an even integer such that  $3 \leq m \leq n$ . If  $K_n$  is  $C_m$ -decomposable then, for all positive integers  $q$ ,  $K_{n+2mq}$  is  $C_m$ -decomposable as well.*

PROOF OF THEOREM 1.1.2. Assume that  $n \geq m$  satisfies the necessary condition  $n(n-1) \equiv 0 \pmod{2m}$  and write  $n = q \cdot 2m + p$ , where  $m \leq p < 3m$ . Hence  $p$  satisfies the necessary condition  $p(p-1) \equiv 0 \pmod{2m}$  as well. By Theorem 3.1.1, it is now sufficient to prove that  $K_p$  is  $C_m$ -decomposable. This is shown in Lemma 3.3.11 for  $m \leq p < 2m$ , and in Lemma 3.2.2 for  $2m \leq p < 3m$ .  $\square$

### 3.2 $C_m$ -decomposition of $K_n$ for $2m \leq n < 3m$

Throughout this section, let  $m$  and  $n$  be positive integers satisfying the necessary conditions for the existence of a  $C_m$ -decomposition of  $K_n$ , where  $m$  is even and  $2m \leq n < 3m$ . A  $C_m$ -decomposition of  $K_n = X(n; \{1, 2, \dots, \frac{n-1}{2}\})$  will be presented in the next two lemmas. That is, in Lemma 3.2.2 we construct peripheral cycles and show that the remaining edge lengths satisfy the conditions for the construction of central cycles of Lemma 3.2.1.

**Lemma 3.2.1** *Let  $L_C = \{a_1, a_2, \dots, a_m\}$ , where  $a_1 < a_2 < \dots < a_m$ , be a subset of the edge length set  $\{1, 2, \dots, \frac{n-1}{2}\}$  and let  $S_i = a_1 - a_2 + \dots + (-1)^{i+1}a_i$  for  $i = 1, \dots, m$ . Let the following conditions be satisfied:*

1.  $a_m = \frac{n-1}{2}$ ,
2. *there exists an even integer  $k$  such that  $a_{i+1} - a_i = 1$  for all  $i = k+1, \dots, m-1$ , and*
3.  $a_{k+1} - 2S_k \leq \frac{n+1}{2}$ .

*Then  $X(n; L_C)$  is  $C_m$ -decomposable.*

PROOF. We show that the above conditions allow for the construction of a zig-zag  $m$ -cycle with edge length set  $L_C$  by finding an odd integer  $j$ , where  $k+1 \leq j \leq m-1$ , such that

$$a_1 - a_2 + \dots - a_{j-1} + a_{j+1} - \dots + (a_m + \delta) = a_j, \quad (2)$$

where  $\delta$  is either 0 or 1, whichever makes the number  $S_m - \delta = S_k - \frac{m-k}{2} - \delta$  even. Let  $x = \frac{1}{2}(j - k - 1)$ . Expressing  $a_1 - a_2 + \cdots - a_{j-1}$  as  $S_k - x$  and  $a_{j+1} - \cdots + a_m$  as  $a_j + \frac{m-k}{2} - x$ , equation (2) can now be rewritten as  $S_k + \frac{m-k}{2} - 2x + \delta = 0$ . We can see that there exists an odd integer  $j$ ,  $k + 1 \leq j \leq m - 1$ , satisfying (2) if and only if  $x = \frac{1}{2}(S_k + \frac{m-k}{2} + \delta)$  is an integer in the range  $0 \leq x \leq \frac{1}{2}(m - k - 2)$ . Since  $S_k$  is negative,  $x$  is in the appropriate range if and only if  $S_k + \frac{m-k}{2} + \delta \geq 0$ . This condition is equivalent to  $a_{k+1} - 2S_k - 2\delta \leq \frac{n+1}{2}$  since  $m - k = \frac{n-1}{2} - (a_{k+1} - 1)$ . Furthermore,  $x$  is an integer since  $S_k - \frac{m-k}{2} - \delta$  is even. Since  $a_{k+1} - 2S_k \leq \frac{n+1}{2}$  by Condition 3 of the lemma, we thus have an odd integer  $j$  in the range  $k + 1 \leq j \leq m - 1$  satisfying (2). This implies that

$C = u_0 u_{a_1} u_{a_1 - a_2} \cdots u_{a_1 - a_2 + \cdots - a_{j-1}} u_{a_1 - a_2 + \cdots - a_{j-1} + a_{j+1}} \cdots u_{a_1 - a_2 + \cdots - a_{j-1} + a_{j+1} + \cdots + a_m + \delta} u_0$  is an  $m$ -cycle. Since  $a_m + \delta = \frac{n-1}{2} = a_m$  in  $\mathbb{Z}_n$ , the edge length set of  $C$  is precisely  $L_C$  and  $\{\rho^i(C) : i = 0, \dots, n - 1\}$  is a  $C_m$ -decomposition of  $X(n; L_C)$ .  $\square$

**Lemma 3.2.2** *Let  $n$  be an odd integer and  $m$  be an even integer such that  $6 \leq 2m \leq n < 3m$  and  $n(n - 1) \equiv 0 \pmod{2m}$ . Then  $K_n$  is  $C_m$ -decomposable.*

PROOF. We shall prove the lemma by constructing peripheral cycles for the various cases. In each case we need to show that the remaining edge lengths satisfy the conditions of Lemma 3.2.1 and hence can be used up in central cycles.

But first, let us have a quick look at the parameters involved. Let  $r = n - 2m$  and  $d = \gcd(m, n) = \gcd(m, r)$ . Since  $K_{2m+1}$  is well-known to be  $C_m$ -decomposable [7, 10], we may assume that  $r \geq 3$  and  $d \geq 3$ . Let  $n = n'd$ ,  $m = m'd$ , and  $r = r'd$ . Since  $2m|n(n - 1)$ , there exists a positive integer  $c$  such that  $r - 1 = 2cm'$ . Since  $r < m$ , we have  $2cm' < m'd - 1$ , whence  $c \leq \frac{d-1}{2}$ . The construction of peripheral cycles now splits into two large cases:  $c \leq \frac{d-1}{4}$  and  $c > \frac{d-1}{4}$ .

CASE  $c \leq \frac{d-1}{4}$ . The following inequalities will be useful in proving that  $a_{k+1}$  and  $S_k$  of Lemma 3.2.1 satisfy the inequality  $a_{k+1} - 2S_k \leq \frac{n+1}{2}$  for the different constructions and parameter values:

$$2m' < n', \tag{3}$$

$$2(c - 1)n' \leq 2\left(\frac{d-1}{4} - 1\right)n' = \frac{n+1}{2} - \frac{5}{2}n' - \frac{1}{2} < \frac{n+1}{2} - 5m' - \frac{1}{2}, \text{ and} \tag{4}$$

$$2cn' \leq \frac{d-1}{2}n' = \frac{n+1}{2} - \frac{n'}{2} - \frac{1}{2} < \frac{n+1}{2}. \tag{5}$$

SUBCASE  $m' \equiv 2 \pmod{4}$ ,  $m' > 2$ . Let

$$P_{0,0} = u_0 u_1 u_{-2} u_2 \dots u_{-\frac{m'+2}{4}} u_{\frac{m'+6}{4}} \dots u_{-\frac{m'}{2}} u_{\frac{m'}{2}+1} u_{n'}.$$

Since  $n' - (\frac{m'}{2} + 1) > m' + 1$ ,  $P_{0,0}$  uses precisely  $m'$  distinct edge lengths. Furthermore, since  $m' + 1 < n'$ , the sets of internal vertices of the paths  $P_{0,j} = \rho^{jn'}(P_{0,0})$ , for  $j = 0, \dots, d-1$ , are pairwise disjoint. Hence  $C_0 = \bigoplus_{j=0}^{d-1} P_{0,j}$  is an  $m$ -cycle. For  $i = 1, \dots, c-1$  obtain  $P_{i,0}$  from  $P_{0,0}$  by adding  $in'$  to the subscripts of the even vertices, and for  $j = 0, \dots, d-1$ , obtain  $P_{i,j}$  in the usual way. By the observation above,  $C_i = \bigoplus_{j=0}^{d-1} P_{i,j}$  is an  $m$ -cycle for every  $i$ .

Let  $L_i$  denote the edge length set of the zig-zag  $m'$ -path  $P_{i,0}$ . We have

$$L_0 = \{1, 3, \dots, \frac{m'}{2}, \frac{m'}{2} + 2, \dots, m' + 1, n' - \frac{m'}{2} - 1\}$$

and, for  $i = 1, \dots, c-1$ ,

$$L_i = \{1 + in', 3 + in', \dots, \frac{m'}{2} + in', \frac{m'}{2} + 1 + (i-1)n', \frac{m'}{2} + 2 + in', \dots, m' + 1 + in'\}.$$

Clearly, these sets are pairwise disjoint. Hence  $\{\rho^j(C_i) : i = 0, \dots, c-1, j = 0, \dots, n' - 1\}$  is a  $C_m$ -decomposition of  $X(n; L_P)$ , where  $L_P = \bigcup_{i=0}^{c-1} L_i$ .

We proceed to show that the set  $L_C = \{1, \dots, \frac{n-1}{2}\} - L_P$  satisfies the conditions of Lemma 3.2.1 and hence that  $X(n; L_C)$  is  $C_m$ -decomposable as well. Clearly  $|L_C| = \frac{n-1}{2} - cm' = m$ . Let  $L_C = \{a_1, \dots, a_m\}$ , where  $a_1 < \dots < a_m$ . Notice that  $\max(L_P) < cn' < \frac{n-1}{2}$  so that Condition 1 of Lemma 3.2.1 is satisfied. We shall find the least integer  $k$  satisfying Condition (2) and show that the corresponding edge length  $a_{k+1}$  and the alternating sum  $S_k$  satisfy Condition 3 of the same lemma.

For  $i = 0, \dots, c-2$  let  $A_i$  denote the alternating sum of the edge lengths in  $L_C \cap [in' + 1, (i+1)n']$  and let  $A_{c-1}$  denote the alternating sum of the edge lengths in  $L_C \cap [(c-1)n' + 1, \max(L_P)]$ . If  $c \geq 2$  we have

$$A_0 = 2 - (m' + 2) + \dots - (n' - \frac{m'}{2} - 2) + (n' - \frac{m'}{2}) - \dots - n' = -\frac{1}{2}n' - \frac{1}{2}m' + \frac{3}{2},$$

$$A_i = (2 + in') - (m' + 2 + in') + \dots + (n' + in') = \frac{1}{2}n' - \frac{1}{2}m' + \frac{3}{2} + in'$$

for  $i = 1, \dots, c-2$ , and

$$A_{c-1} = (2 + (c-1)n') - (\frac{m'}{2} + 1 + (c-1)n') = -\frac{1}{2}m' + 1.$$

Keep in mind that the index  $k$  from Lemma 3.2.1 has to be even (that is,  $S_k$  must be negative) so that  $S_k$  and  $a_{k+1}$  depend on whether  $c$  is odd or even.

If  $c$  is even, we have

$$\begin{aligned} S_k &= A_0 + A_1 - A_2 + \cdots - A_{c-2} + A_{c-1} = \\ &= \left(-\frac{1}{2}n' - \frac{1}{2}m' + \frac{3}{2}\right) - \frac{c-2}{2}n' + \left(-\frac{1}{2}m' + 1\right) = \\ &= -\frac{c-1}{2}n' - m' + \frac{5}{2} \end{aligned}$$

and  $a_{k+1} = m' + 2 + (c-1)n'$ . Hence, by (4),

$$a_{k+1} - 2S_k = 2(c-1)n' + 3m' - 3 < \frac{n+1}{2} - 2m' - \frac{7}{2} < \frac{n+1}{2}.$$

If  $c$  is odd and  $c \geq 3$ , since  $k$  has to be even, we take

$$S_k = A_0 + A_1 - A_2 + \cdots + A_{c-2} - A_{c-1} - (m' + 2 + (c-1)n') = -\frac{c-1}{2}n' - \frac{3}{2}m'$$

and  $a_{k+1} = m' + 3 + (c-1)n'$ . Then, using (4) and  $m' \geq 6$ ,

$$a_{k+1} - 2S_k = 2(c-1)n' + 4m' + 3 < \frac{n+1}{2} - m' + \frac{5}{2} < \frac{n+1}{2}.$$

If  $c = 1$ ,

$$\begin{aligned} S_k &= 2 - \left(\frac{m'}{2} + 1\right) + (m' + 2) - \cdots + \left(n' - \frac{m'}{2} - 2\right) - \left(n' - \frac{m'}{2}\right) \\ &= -\frac{1}{2}n' + \frac{1}{4}m' + 1 \end{aligned}$$

and  $a_{k+1} = n' - \frac{1}{2}m' + 1$ . Hence

$$a_{k+1} - 2S_k = 2n' - m' - 1 = 2cn' - m' - 1 < \frac{n+1}{2} - m' - 1 < \frac{n+1}{2}.$$

In the remaining cases, we give pointers to the construction of peripheral cycles and state the values of  $a_{k+1}$  and  $S_k$ , but leave most of the verification to the reader.

SUBCASE  $m' \equiv 0 \pmod{4}$ . We now let

$$P_{0,0} = u_0 u_1 u_{-1} \cdots u_{-\frac{m'}{4}} u_{\frac{m'}{4}+2} \cdots u_{-(\frac{m'}{2}-1)} u_{\frac{m'}{2}+1} u_{n'}$$

and create the zig-zag  $m'$ -paths  $P_{i,j}$  and the  $m$ -cycles  $C_i = \bigoplus_{j=0}^{d-1} P_{i,j}$  as before. The edge length sets are now

$$L_0 = \{1, 2, \dots, \frac{m'}{2}, \frac{m'}{2} + 2, \dots, m', n' - \frac{m'}{2} - 1\}$$

and, for  $i = 1, \dots, c - 1$ ,

$$L_i = \{1 + in', 2 + in', \dots, \frac{m'}{2} + in', \frac{m'}{2} + 1 + (i - 1)n', \frac{m'}{2} + 2 + in', \dots, m' + in'\}.$$

Continuing as in the previous case we obtain the following results.

If  $c$  is even, then  $S_k = -\frac{c-1}{2}n' - \frac{1}{2}$  and  $a_{k+1} = m' + 2 + (c - 1)n'$ .

If  $c$  is odd and  $c \geq 3$ , then  $S_k = -\frac{c-1}{2}n' + \frac{1}{2}m' - 1$  and  $a_{k+1} = m' + 1 + (c - 1)n'$ .

If  $c = 1$ , then  $S_k = -\frac{1}{2}n' + \frac{1}{4}m' + \frac{3}{2}$  and  $a_{k+1} = n' - \frac{1}{2}m'$ .

SUBCASE  $m' = 2$ . This case, as tends to happen for small values of the parameter  $m'$ , requires a different construction. Observe that  $r' = 1$ ,  $n' = 5$ , and  $c = \frac{d-1}{4}$ . Let

$$P_{0,0} = u_0 u_1 u_5, \quad \text{and} \quad P_{0,0}^* = u_0 u_2 u_5,$$

and for  $i \geq 1$ ,

$$P_{i,0} = u_0 u_{1+10i} u_5 \quad \text{and} \quad P_{i,0}^* = u_0 u_{2+10i} u_5.$$

Note that the  $P_{i,0}$  will be used for  $i = 0, \dots, \lceil \frac{c}{2} \rceil - 1$ , whereas  $P_{i,0}^*$  will be used only for  $i = 0, \dots, \lfloor \frac{c}{2} \rfloor - 1$ . The paths  $P_{i,j}$  and  $P_{i,j}^*$ , and the  $m$ -cycles  $C_i = \bigoplus_{j=0}^{d-1} P_{i,j}$  and  $C_i^* = \bigoplus_{j=0}^{d-1} P_{i,j}^*$  are created as before.

The edge length sets of the zig-zag paths  $P_{i,0}$  and  $P_{i,0}^*$  are

$$L_0 = \{1, 4\}, \quad \text{and} \quad L_0^* = \{2, 3\},$$

and for  $i \geq 1$ ,

$$L_i = \{1 + 10i, 6 + 10(i - 1)\} \quad \text{and} \quad L_i^* = \{2 + 10i, 7 + 10(i - 1)\}.$$

We let  $L_P = (\bigcup_{i=0}^{\lceil \frac{c}{2} \rceil - 1} L_i) \cup (\bigcup_{i=0}^{\lfloor \frac{c}{2} \rfloor - 1} L_i^*)$  and conclude that  $X(n; L_P)$  is  $C_m$ -decomposable.

We now continue to verify the conditions of Lemma 3.2.1 for the set  $L_C = \{1, \dots, \frac{n-1}{2}\} - L_P$  as before and obtain the following results.

If  $c$  even and  $c > 2$ , then  $S_k = -\frac{5}{2}c + 6$  and  $a_{k+1} = 3 + 10(\frac{c}{2} - 1)$ .

If  $c = 2$ , then  $S_k = 0$  and  $a_{k+1} = 5$ .

If  $c$  odd and  $c > 1$ , then  $S_k = -\frac{5}{2}(c - 1)$  and  $a_{k+1} = 3 + 10\frac{c-1}{2}$ .

If  $c = 1$ , then  $S_k = -1$  and  $a_{k+1} = 5$ .

CASE  $c > \frac{d-1}{4}$ . The following inequalities will be useful in proving that  $a_{k+1} - 2S_k \leq \frac{n+1}{2}$ :



$$cn' \leq \frac{d-1}{2}n' = \frac{n+1}{2} - \frac{1}{2}n' - \frac{1}{2} = \frac{n+1}{2} - m' - \frac{1}{2}r' - \frac{1}{2} < \frac{n+1}{2} \text{ and} \quad (6)$$

$$r' \geq \frac{m'}{2} + 1. \quad (7)$$

The latter follows from  $r'd - 1 = 2cm' \geq 2\frac{d+1}{4}m'$ .

SUBCASE  $m' \equiv 2 \pmod{4}$ . Since the case  $m' = 2$  has been settled, we may assume that  $m' \geq 6$ . Furthermore, since  $r' \geq \frac{m'}{2} + 1$  by (7) and  $r'$  is odd,  $r' \geq \frac{m'}{2} + 2$  throughout this case. In particular,  $r' \geq 5$  and  $n' \geq 17$ .

Let  $t = n' - m' - 3$  and let

$$P_{0,0} = u_0 u_1 u_{-2} u_2 \dots u_{-\frac{m'+2}{4}} u_{\frac{m'+10}{4}} \dots u_{-\frac{m'}{2}} u_{\frac{m'}{2}+2} u_{n'}. \quad (8)$$

Obtain  $P_{0,0}^*$  from  $P_{0,0}$  by adding  $t$  to the subscripts of the even vertices. That is, let

$$P_{0,0}^* = u_0 u_{1+t} u_{-2} u_{2+t} \dots u_{-\frac{m'+2}{4}} u_{\frac{m'+10}{4}+t} \dots u_{-\frac{m'}{2}} u_{\frac{m'}{2}+2+t} u_{n'}. \quad (9)$$

Since  $n' - (\frac{m'}{2} + 2) > m' + 2$  and  $n' - (\frac{m'}{2} + 2 + t) < 1 + t$ , each of the paths  $P_{0,0}$  and  $P_{0,0}^*$  uses precisely  $m'$  distinct edge lengths. As usual, we let  $P_{0,j} = \rho^{jn'}(P_{0,0})$  and  $P_{0,j}^* = \rho^{jn'}(P_{0,0}^*)$ . Since  $m' + 2 < n'$  and  $m' + 2 + t = n' - 1 < n'$ , the sets of internal vertices of the  $m$ -paths  $P_{0,j}$  and the sets of internal vertices of the  $m$ -paths  $P_{0,j}^*$  are pairwise disjoint.

Note that  $n' - (\frac{m'}{2} + 2) = \frac{m'}{2} + 1 + t$  and  $n' - (\frac{m'}{2} + 2 + t) = \frac{m'}{2} + 1$ . The following are the edge length sets of the zig-zag  $m'$ -paths  $P_{0,j}$  and  $P_{0,j}^*$ :

$$L_0 = \{1, 3, \dots, \frac{m'}{2}, \frac{m'}{2} + 3, \dots, m' + 2, \frac{m'}{2} + 1 + t\}$$

and

$$L_0^* = \{\frac{m'}{2} + 1, 1 + t, 3 + t, \dots, \frac{m'}{2} + t, \frac{m'}{2} + 3 + t, \dots, m' + 2 + t\}.$$

Since  $r' \geq 5$ , we have  $m' + 2 < 1 + t$  so that these two sets are disjoint.

For  $i = 1, \dots, \lfloor \frac{r'}{2} \rfloor - 2$ , we obtain  $P_{i,0}$  from  $P_{0,0}$  and  $P_{i,0}^*$  from  $P_{0,0}^*$  by adding  $in'$  to the subscripts of the even vertices. The edge length sets of these zig-zag  $m'$ -paths are

$$L_i = \{1 + in', 3 + in', \dots, \frac{m'}{2} + in', \frac{m'}{2} + 2 + (i-1)n', \frac{m'}{2} + 3 + in', \dots, m' + 2 + in'\}$$

and  $L_i^* = L_i + t$ . Finally, let

$$P_{\lceil \frac{c}{2} \rceil - 1, 0} = u_0 u_{3 + (\lceil \frac{c}{2} \rceil - 1)n'} u_{-1} \cdots u_{-(\frac{m'}{2} - 1)} u_{\frac{m'}{2} + 2 + (\lceil \frac{c}{2} \rceil - 1)n'} u_{n'} \quad (10)$$

and obtain  $P_{\lceil \frac{c}{2} \rceil - 1, 0}^*$  from  $P_{\lceil \frac{c}{2} \rceil - 1, 0}$  by adding  $t$  to the subscripts of the even vertices. The edge length sets of  $P_{\lceil \frac{c}{2} \rceil - 1, 0}$  and  $P_{\lceil \frac{c}{2} \rceil - 1, 0}^*$  are

$$L_{\lceil \frac{c}{2} \rceil - 1} = \left\{ \frac{m'}{2} + 2 + (\lceil \frac{c}{2} \rceil - 2)n', 3 + (\lceil \frac{c}{2} \rceil - 1)n', \dots, m' + 1 + (\lceil \frac{c}{2} \rceil - 1)n' \right\}$$

and  $L_{\lceil \frac{c}{2} \rceil - 1}^* = L_{\lceil \frac{c}{2} \rceil - 1} + t$ .

Since the  $L_i$  and  $L_i^*$  are pairwise disjoint, the zig-zag  $m'$ -paths  $P_{i,0}$ , for  $i = 0, \dots, \lceil \frac{c}{2} \rceil - 1$ , and  $P_{i,0}^*$ , for  $i = 0, \dots, \lfloor \frac{c}{2} \rfloor - 1$ , generate a  $C_m$ -decomposition of  $X(n; L_P)$ , where  $L_P = (\bigcup_{i=0}^{\lceil \frac{c}{2} \rceil - 1} L_i) \cup (\bigcup_{i=0}^{\lfloor \frac{c}{2} \rfloor - 1} L_i^*)$ .

We now follow the procedure described in Case  $c \leq \frac{d-1}{4}$ , Subcase  $m' \equiv 2 \pmod{4}$ , to show that the set  $L_C = \{1, \dots, \frac{n-1}{2}\} - L_P$  satisfies the conditions of Lemma 3.2.1. The following results are obtained. We leave it to the reader to check that  $a_{k+1} - 2S_k \leq \frac{n+1}{2}$  for each of these subcases using (6) and (7).

If  $c \equiv 0 \pmod{4}$ , then  $S_k = -\frac{c}{4}n' + 1$  and  $a_{k+1} = \frac{c}{2}n' - 1$ .

If  $c \equiv 1 \pmod{4}$  and  $c \geq 5$ , then  $S_k = -\frac{c-1}{4}n' - \frac{1}{2}m' - 4$  and  $a_{k+1} = m' + 3 + \frac{c-1}{2}n'$ .

If  $c \equiv 2 \pmod{4}$  and  $c \geq 6$ , then  $S_k = -\frac{c}{4}n' - \frac{7}{2}$  and  $a_{k+1} = \frac{c}{2}n'$ .

If  $c = 2$ , using  $C_0$  and  $C_0^*$  arising from the paths in (8) and (9), we obtain  $S_k = -\frac{1}{2}n' + \frac{5}{2}$  and  $a_{k+1} = n'$ .

If  $c \equiv 3 \pmod{4}$  and  $c \geq 7$ , then  $S_k = -\frac{c-1}{4}n' - \frac{1}{2}m' - \frac{3}{2}$  and  $a_{k+1} = m' + 2 + \frac{c-1}{2}n'$ .

If  $c = 3$ , then  $S_k = -\frac{1}{2}n' - \frac{1}{2}m' - \frac{3}{2}$  and  $a_{k+1} = m' + 2 + n'$ .

For  $c = 1$ , using the zig-zag path  $P_{0,0}$  as defined in (8) does not yield the desired result. Instead, we use  $P_{\lceil \frac{c}{2} \rceil - 1, 0}$  as defined in (10) with  $c = 1$  to obtain  $S_k = -\frac{1}{2}n' + \frac{3}{4}m' + 1$  and  $a_{k+1} = n' - \frac{m'}{2} - 1$ .

SUBCASE  $m' \equiv 0 \pmod{4}$ ,  $m' \geq 8$ . We now have  $r' \geq 5$  and  $n' \geq 21$ . This construction is very similar to the previous one. We start with

$$P_{0,0} = u_0 u_1 u_{-1} \cdots u_{-\frac{m'}{4}} u_{\frac{m'}{4} + 3} \cdots u_{-(\frac{m'}{2} - 1)} u_{\frac{m'}{2} + 2} u_{n'} \quad (11)$$

and obtain  $P_{0,0}^*$  from  $P_{0,0}$  by adding  $t = n' - m' - 3$  to the subscripts of the even vertices. The edge length sets of  $P_{0,0}$  and  $P_{0,0}^*$  are thus

$$L_0 = \left\{ 1, 2, \dots, \frac{m'}{2}, \frac{m'}{2} + 3, \dots, m' + 1, \frac{m'}{2} + 1 + t \right\}$$

and

$$L_0^* = \left\{ \frac{m'}{2} + 1, 1 + t, 2 + t, \dots, \frac{m'}{2} + t, \frac{m'}{2} + 3 + t, \dots, m' + 1 + t \right\}.$$

For  $i = 1, \dots, \lceil \frac{c}{2} \rceil - 2$ , we obtain  $P_{i,0}$  from  $P_{0,0}$ , and  $P_{i,0}^*$  from  $P_{0,0}^*$ , by adding  $in'$  to the subscripts of the even vertices as in the previous case. The edge length sets of the zig-zag  $m'$ -paths  $P_{i,0}$  and  $P_{i,0}^*$  are therefore

$$L_i = \left\{ 1 + in', 2 + in', \dots, \frac{m'}{2} + in', \frac{m'}{2} + 2 + (i-1)n', \frac{m'}{2} + 3 + in', \dots, m' + 1 + in' \right\}$$

and  $L_i^* = L_i + t$ . Finally, let

$$P_{\lceil \frac{c}{2} \rceil - 1, 0} = u_0 u_{2+(\lceil \frac{c}{2} \rceil - 1)n'} u_{-1} \cdots u_{-(\frac{m'}{2} - 1)} u_{\frac{m'}{2} + 1 + (\lceil \frac{c}{2} \rceil - 1)n'} u_{n'}$$

and obtain  $P_{\lceil \frac{c}{2} \rceil - 1, 0}^*$  from  $P_{\lceil \frac{c}{2} \rceil - 1, 0}$  by adding  $t$  to the subscripts of the even vertices. The edge length sets of these zig-zag  $m'$ -paths are then

$$L_{\lceil \frac{c}{2} \rceil - 1} = \left\{ \frac{m'}{2} + 1 + (\lceil \frac{c}{2} \rceil - 2)n', 2 + (\lceil \frac{c}{2} \rceil - 1)n', \dots, m' + (\lceil \frac{c}{2} \rceil - 1)n' \right\}$$

and  $L_{\lceil \frac{c}{2} \rceil - 1}^* = L_{\lceil \frac{c}{2} \rceil - 1} + t$ .

Continuing as in the previous cases we find the following values for the edge length  $a_{k+1}$  and the alternating sum  $S_k$ .

If  $c \equiv 0 \pmod{4}$  and  $c \geq 8$ , then  $S_k = -\frac{c}{4}n' + 3$  and  $a_{k+1} = \frac{c}{2}n' - 1$ .

If  $c \equiv 1 \pmod{4}$  and  $c \geq 5$ , then  $S_k = -\frac{c-1}{4}n' - \frac{1}{2}m' - 1$  and  $a_{k+1} = m' + 1 + \frac{c-1}{2}n'$ .

If  $c \equiv 2 \pmod{4}$  and  $c \geq 6$ , then  $S_k = -\frac{c}{4}n' - \frac{1}{2}$  and  $a_{k+1} = \frac{c}{2}n' - 2$ .

If  $c \equiv 3 \pmod{4}$  and  $c \geq 7$ , then  $S_k = -\frac{c-1}{4}n' - \frac{1}{2}m' + \frac{1}{2}$  and  $a_{k+1} = m' + 2 + \frac{c-1}{2}n'$ .

If  $c = 1$ , then  $S_k = -\frac{1}{2}n' + \frac{3}{4}m' - \frac{1}{2}$  and  $a_{k+1} = n' - \frac{m'}{2}$ , using  $P_{0,0}$  from (11).

If  $c = 2$ , then  $S_k = -\frac{1}{2}n' + \frac{5}{2}$  and  $a_{k+1} = n'$ , using  $P_{0,0}$  and  $P_{0,0}^*$  from (11).

If  $c = 3$ , using  $P_{i,0}$  arising from (11) for  $i = 0, 1$  and  $P_{0,0}^*$ , we obtain  $S_k = -\frac{1}{2}n' + \frac{1}{2}m' - \frac{3}{2}$  and  $a_{k+1} = m' + 2 + n'$ .

If  $c = 4$ , we use  $P_{i,0}$  and  $P_{i,0}^*$  arising from (11) for  $i = 0, 1$ . Thus  $S_k = -n' + 1$  and  $a_{k+1} = 2n' - 1$ .

SUBCASE  $m' = 4$ . Again, a separate construction is required for the smallest value of  $m'$ . Notice that  $r' = 3$  by (7) and hence  $n' = 11$  and  $c = \frac{3d-1}{8}$ . The latter implies  $c \equiv 1 \pmod{3}$ . Let

$$P_{0,0} = u_0 u_{-1} u_{-5} u_2 u_{11} \quad \text{and} \quad P_{0,0}^* = u_0 u_{-2} u_{-7} u_1 u_{11}.$$

The edge length sets of these two zig-zag 4-paths are

$$L_0 = \{1, 4, 7, 9\} \quad \text{and} \quad L_0^* = \{2, 5, 8, 10\}.$$

For  $i = 1, \dots, \lceil \frac{c}{2} \rceil - 1$  let

$$P_{i,0} = u_0 u_{1+11i} u_{-1} u_{3+11i} u_{11},$$

and for  $i = 1, \dots, \lfloor \frac{c}{2} \rfloor - 1$ ,

$$P_{i,0}^* = u_0 u_{5+11i} u_{-2} u_{6+11i} u_{11}.$$

The edge length sets of these zig-zag 4-paths are

$$L_i = \{1 + 11i, 2 + 11i, 3 + 11(i - 1), 4 + 11i\}$$

and

$$L_i^* = \{5 + 11i, 6 + 11(i - 1), 7 + 11i, 8 + 11i\}.$$

After completing the construction in the way described earlier and then examining the alternating sums of the leftover edge lengths, we determine the following.

If  $c \equiv 0 \pmod{4}$ , then  $S_k = -11\frac{c}{4} + 5$  and  $a_{k+1} = 10 + 11(\frac{c}{2} - 1)$ .

If  $c \equiv 1 \pmod{4}$  and  $c \geq 5$ , then  $S_k = -11\frac{c-1}{4} + 1$  and  $a_{k+1} = 5 + 11\frac{c-1}{2}$ .

If  $c \equiv 2 \pmod{4}$  and  $c \geq 6$ , then  $S_k = -11\frac{c-2}{4} - 2$  and  $a_{k+1} = 9 + 11(\frac{c}{2} - 1)$ .

If  $c \equiv 3 \pmod{4}$  and  $c \geq 7$ , then  $S_k = -11\frac{c-3}{4} - 5$  and  $a_{k+1} = 6 + 11\frac{c-1}{2}$ .

If  $c = 1$ , then  $S_k = -4$  and  $a_{k+1} = 11$ , so that Condition 3 of Lemma 3.2.1 is not satisfied. However, it is possible to find a central  $m$ -cycle that uses the remaining edge lengths; one such central  $m$ -cycle is given by the equality  $2 - 3 + 5 - 6 + 8 - 10 + 12 - 13 + 14 - 15 + 17 = 11$ .

We have shown that in all cases  $X(n; L_P)$  is  $C_m$ -decomposable and that the set of the remaining edge lengths  $L_C = \{1, \dots, \frac{n-1}{2}\} - L_P$  satisfies the conditions of Lemma 3.2.1 so that  $X(n; L_C)$  is  $C_m$ -decomposable as well.

Hence  $K_n$  is  $C_m$ -decomposable. □

### 3.3 $C_m$ -decomposition of $K_n$ for $m \leq n < 2m$

Throughout this section it is assumed that  $n$  is an odd integer and  $m$  is an even integer such that  $3 \leq m \leq n < 2m$  and  $n(n-1) \equiv 0 \pmod{2m}$ . We shall view the graph  $K_n$  as the join  $K_{n-1} \bowtie K_1$ , where the vertex set of  $K_{n-1}$  is  $\{u_0, u_1, \dots, u_{n-2}\}$  and the vertex of  $K_1$  (that is, the central vertex) is  $w$ . First we discuss the parameters. Let  $r = n - m$  and  $d = \gcd(m, n - 1) = \gcd(m, r - 1)$ . Note that  $r$  is odd and  $d$  is even. Let  $n - 1 = dn'$ ,  $m = dm'$ , and  $r - 1 = dr'$ . Observe that  $m' = 1$  implies  $m = n - 1$ , contradicting the assumption that  $n(n - 1) \equiv 0 \pmod{2m}$ . Hence  $m' > 1$  and  $r > 1$ .

If  $K_n$  is  $C_m$ -decomposable, then the number of  $m$ -cycles is going to be  $\frac{n(n-1)}{2m} = \frac{n-1}{2} + \frac{1}{2}(r + \frac{r(r-1)}{m})$ , and this must be an integer. Since  $r$  is odd,  $\frac{r(r-1)}{m} = \frac{rr'}{m'}$  must be an odd integer. Hence there exists an odd integer  $c$  such that  $r = cm'$ . This in turn implies that  $m'$  is odd, and since  $\frac{r(r-1)}{m} = cr'$  is odd,  $r'$  is odd as well. Therefore  $n' = m' + r'$  is even and  $n - 1 = dn' \equiv 0 \pmod{4}$ . Hence the diameter length  $\frac{n-1}{2} = \frac{dn'}{2}$ , denoted by  $D$ , is even. Furthermore, since  $r \leq m - 1$ , we have  $c \leq d - \frac{1}{m'}$ . Hence  $c \leq d - 1$ .

The expression for the number of  $m$ -cycles now attains the form  $\frac{n(n-1)}{2m} = \frac{n-1}{2} + \frac{1}{2}(cm' + cr') = \frac{n-1}{2} + c\frac{n'}{2}$ . Since this expression is of the same form as the expression for the number of  $m$ -cycles in the case  $n$  even,  $m$  odd, we try to follow the example of the construction in Section 2.3. Thus  $\frac{n-1}{2}$  central cycles will be generated by applying the rotation  $\rho^{2i}$ ,  $i = 0, \dots, \frac{n-1}{2} - 1$ , to a central cycle that contains the central vertex; this will take care of the edges between  $K_{n-1}$  and  $K_1$ . In this case, however, the construction is further complicated by the presence of the edges of diameter length  $\frac{n-1}{2}$ . These have to be taken care of in two different ways, depending on whether  $c < \frac{d}{2}$  or  $c \geq \frac{d}{2}$ . For  $c \geq \frac{d}{2}$  a new type of a cycle, called a diameter cycle, is introduced. The generating diameter cycle contains two edges of diameter length, one from each of the two orbits of  $\langle \rho^2 \rangle$ . Applying  $\rho^{2i}$ ,  $i = 0, \dots, \frac{n-1}{4} - 1$ , to this cycle gives us  $\frac{n-1}{4}$  diameter cycles which use up all the edges of the diameter length. The remaining  $c\frac{n'}{2} - \frac{n-1}{4} = (c - \frac{d}{2})\frac{n'}{2}$  cycles are peripheral, similar to those of Section 2.3.

For  $c < \frac{d}{2}$ , however, the number of cycles is not large enough to include  $\frac{n-1}{4}$  diameter cycles on top of the  $\frac{n-1}{2}$  central cycles. So edges of diameter length have

to be used up in central cycles. We shall construct two generating central cycles that each contain one edge of diameter length, so that the two edges are from distinct orbits of  $\langle \rho^2 \rangle$ . Applying  $\rho^{2i}$ ,  $i = 0, \dots, \frac{n-1}{4} - 1$  to each of them results in the  $\frac{n-1}{2}$  central cycles we aimed for. Having only one edge of diameter length in a cycle, however, means that the rest of the edge lengths represented in the cycle cannot be paired up. We shall explain shortly how this problem can be solved.

### 3.3.1 The case $c < \frac{d}{2}$

The construction for this case involves three types of cycles. The central cycles (Lemmas 3.3.1 and 3.3.2) are generated by two starter cycles and contain the edges of diameter length. The coupled peripheral cycles (Lemma 3.3.3) are similar to those of Section 2.3, whereas the solitary peripheral cycles (Lemmas 3.3.4 and 3.3.5) are quite different from any other peripheral cycles.

We begin with the construction of central cycles. Since  $c < \frac{d}{2}$ , observe that  $d \geq 4$ . In addition, we have  $dr' + 1 = r = cm' < \frac{d}{2}m'$ , whence  $r' < \frac{m'}{2} - \frac{1}{d}$ . Since  $m'$  is odd, we have  $r' \leq \frac{m'-1}{2}$ .

**Lemma 3.3.1** *Let  $L_C$  be a subset of the edge length set  $\{1, \dots, D\}$  with the following properties:*

1.  $|L_C| = \frac{m}{2} - 1$ ,
2.  $\{1, 4, D\} \subseteq L_C$ , and
3.  $2, 3, 5 \notin L_C$ .

*Then  $(X(n-1; L_C) \oplus \langle \rho^2 \rangle(u_0u_2)) \rtimes K_1$  is  $C_m$ -decomposable.*

PROOF. Let  $L_C = \{1, 4, a_3, \dots, a_{\frac{m}{2}-2}, D\}$ , where  $6 \leq a_3 < \dots < a_{\frac{m}{2}-2} < D$ . Define the zig-zag  $(\frac{m}{2} - 3)$ -path  $P$  by

$$P = u_0u_4u_{4-a_3}u_{4-a_3+a_4} \dots u_A,$$

where  $A = 4 - a_3 + a_4 - \dots + (-1)^{\frac{m}{2}-2}a_{\frac{m}{2}-2}$ . Furthermore, let

$$T = Pu_Au_{A+D}\rho^D(\overleftarrow{P}).$$

Observe that  $T$  contains one edge of diameter length and two diametrically opposed edges of each of the lengths  $4, a_3, \dots, a_{\frac{m}{2}-2}$ . Since  $D$  is even, these two diametrically opposed edges belong to the same orbit of  $\langle \rho^2 \rangle$ . Now define the paths  $R_1$  and  $R_2$  by

$$\begin{aligned} R_1 &= u_1 u_2 u_0 T u_D u_{D+1} \text{ and} \\ R_2 &= u_0 u_1 \rho(T) u_{D+1} u_{D+2} u_D. \end{aligned}$$

Observe that  $|E(R_1)| = |E(R_2)| = m - 2$ . Together,  $R_1$  and  $R_2$  contain exactly four edges of each of the lengths  $1, 4, a_3, \dots, a_{\frac{m}{2}-2}$  (that is, two pairs of diametrically opposed edges, each pair from a different orbit of  $\langle \rho^2 \rangle$ ), exactly two edges of diameter length (from distinct orbits of  $\langle \rho^2 \rangle$ ), and a pair of diametrically opposed edges of length 2 from the orbit  $\langle \rho^2 \rangle(u_0 u_2)$ . Therefore  $\{\rho^{2i}(R_j) : j = 1, 2, i = 0, \dots, \frac{D}{2} - 1\}$  is a decomposition of  $X(n-1; L_C) \oplus \langle \rho^2 \rangle(u_0 u_2)$  into  $(m-2)$ -paths.

Define the central  $m$ -cycles  $C_1^C$  and  $C_2^C$  by

$$C_1^C = w u_1 R_1 u_{D+1} w \quad \text{and} \quad C_2^C = w u_0 R_2 u_D w.$$

Since the endpoints of the paths  $R_1$  and  $R_2$  form two diametrically opposed pairs from distinct orbits of  $\langle \rho^2 \rangle$ ,  $\{\rho^{2i}(C_j^C) : j = 1, 2, i = 0, \dots, \frac{D}{2} - 1\}$  is a  $C_m$ -decomposition of  $(X(n-1; L_C) \oplus \langle \rho^2 \rangle(u_0 u_2)) \rtimes K_1$ .  $\square$

**Lemma 3.3.2** *Let  $L_C$  be a subset of the edge length set  $\{1, \dots, D\}$  with the following properties:*

1.  $|L_C| = \frac{m}{2} - 2$ ,
2.  $\{1, 6, D\} \subseteq L_C$ , and
3.  $2, 3, 4, 5, 7 \notin L_C$ .

*Then  $(X(n-1; L_C) \oplus \langle \rho^2 \rangle(\{u_1 u_3, u_1 u_4, u_1 u_6\})) \rtimes K_1$  is  $C_m$ -decomposable.*

PROOF. Let  $L_C = \{1, 6, a_3, \dots, a_{\frac{m}{2}-3}, D\}$ , where  $8 \leq a_3 < \dots < a_{\frac{m}{2}-3} < D$ . Define the zig-zag  $(\frac{m}{2} - 4)$ -path  $P$  by

$$P = u_0 u_6 u_{6-a_3} u_{6-a_3+a_4} \dots u_A,$$

where  $A = 6 - a_3 + a_4 - \dots + (-1)^{\frac{m}{2}-3} a_{\frac{m}{2}-3}$ , and let

$$T = Pu_A u_{A+D} \rho^D(\overline{P}).$$

Define the paths  $R_1$  and  $R_2$  by

$$\begin{aligned} R_1 &= u_1 u_4 u_{-1} u_0 T u_D u_{D-1} u_{D+1} \text{ and} \\ R_2 &= u_0 u_1 u_{-1} \rho^{-1}(T) u_{D-1} u_{D+4} u_{D+1} u_D, \end{aligned}$$

and the  $m$ -cycles  $C_1^C$  and  $C_2^C$  by

$$C_1^C = w u_1 R_1 u_{D+1} w \quad \text{and} \quad C_2^C = w u_0 R_2 u_D w.$$

We now leave it to the reader to verify that  $\{\rho^{2i}(C_j^C) : j = 1, 2, i = 0, \dots, \frac{D}{2} - 1\}$  is a  $C_m$ -decomposition of  $(X(n-1; L_C) \oplus \langle \rho^2 \rangle(\{u_1 u_3, u_1 u_4, u_1 u_6\})) \rtimes K_1$ .  $\square$

In the next lemma we construct the  $\frac{c-1}{2}$  families of coupled peripheral cycles that represent a  $C_m$ -decomposition of a circulant  $X(n-1; L_{CP})$ .

**Lemma 3.3.3** *There exists a set  $L_{CP}$  with the following properties:*

1.  $L_{CP} \subseteq \{n' + 1, \dots, D - n'\}$ ,
2. if  $m' = 3$  and  $c > 1$ , then  $n' + 1 \in L_{CP}$ ,
3.  $|L_{CP}| = \frac{c-1}{2} m'$ , and
4.  $X(n-1; L_{CP})$  is  $C_m$ -decomposable.

PROOF. Throughout this proof let  $a = \frac{c-1}{2}$ . Since  $c \leq \frac{d}{2} - 1$ , observe that  $a \leq \frac{d}{4} - 1$ . First we assume that  $m' \geq 5$ . Define the zig-zag  $(m' - 1)$ -path  $P_{1,0}$  by

$$P_{1,0} = u_0 u_{2+2n'} u_{-1} u_{3+2n'} \dots u_{-\frac{m'-3}{2}} u_{\frac{m'+1}{2}+2n'} u_{1+n'}.$$

Notice that this is precisely the same  $(m' - 1)$ -path as in Case  $m' \geq 5$ ,  $r' \leq 3$  of Lemma 2.3.3. Its edge length set is

$$L_1 = \left\{ \frac{m'-1}{2} + n', 2 + 2n', 3 + 2n', \dots, m' - 1 + 2n' \right\}.$$



For  $i = 2, \dots, a$ , obtain the zig-zag  $(m' - 1)$ -path  $P_{i,0}$  from  $P_{1,0}$  by adding  $2(i - 1)n'$  to the subscripts of the even vertices. The edge length set of the zig-zag path  $P_{i,j} = \rho^{jn'}(P_{i,0})$  is then  $L_i = L_1 + 2(i - 1)n'$ .

As usual, we use the method described on page 6 to connect each family of  $(m' - 1)$ -paths into an  $m$ -cycle. Since the terminal vertex of each of the paths  $P_{i,0}$  is  $u_{n'+1}$ , an edge of length  $\ell$  in the auxiliary circulant results in connecting edges of length  $(\ell - 1)n' - 1$  or  $(\ell + 1)n' + 1$ , depending on the orientation of the cycle.

We first assume that  $d \geq 20$ . Define a number  $p$  as follows. If  $\frac{d}{2}$  is odd, let  $p = \frac{d}{2} - 4$ . If  $\frac{d}{2}$  is even and  $\frac{d}{4}$  is odd, let  $p = \frac{d}{4} + 2$ . If  $\frac{d}{2}$  and  $\frac{d}{4}$  are both even, let  $p = \frac{d}{4} + 1$ . Observe that in all cases  $\gcd(d, p) = 1$ . Since  $a \leq \frac{d}{4} - 1$ , we have  $\frac{a}{2} + 3 \leq p \leq \frac{d}{2} - 3$ .

If  $a \equiv 1 \pmod{4}$ , take the auxiliary circulant  $X = X(d; \{1, 3, 4, \dots, \frac{a+3}{2}\})$  and decompose it into  $X(d; \{1\})$ ,  $X(d; \{3, 4\})$ ,  $\dots$ ,  $X(d; \{\frac{a+1}{2}, \frac{a+3}{2}\})$ . Use the  $d$ -cycle  $X(d; \{1\})$  with the orientation resulting in connecting edges of length  $2n' + 1$ . The set of lengths of all connecting edges is thus

$$L_X = \{-1 + in' : i = 2, \dots, \frac{a+1}{2}\} \cup \{1 + in' : i = 2, 4, 5, \dots, \frac{a+5}{2}\}.$$

If  $a \equiv 2 \pmod{4}$ , take  $X(d; \{3, 4\}) \oplus \dots \oplus X(d; \{\frac{a}{2}, \frac{a}{2} + 1\}) \oplus X(d; \{p\})$  and use the  $d$ -cycle  $X(d; \{p\})$  with both orientations. Hence

$$L_X = \{-1 + in' : i = 2, \dots, \frac{a}{2}, p - 1\} \cup \{1 + in' : i = 4, \dots, \frac{a}{2} + 2, p + 1\}.$$

If  $a \equiv 3 \pmod{4}$ , take  $X(d; \{1\}) \oplus X(d; \{3, 4\}) \oplus \dots \oplus X(d; \{\frac{a-1}{2}, \frac{a+1}{2}\}) \oplus X(d; \{p\})$ . Use  $X(d; \{p\})$  with both orientations and  $X(d; \{1\})$  only with the orientation resulting in connecting edges of length  $2n' + 1$ . Hence

$$L_X = \{-1 + in' : i = 2, \dots, \frac{a-1}{2}, p - 1\} \cup \{1 + in' : i = 2, 4, 5, \dots, \frac{a+3}{2}, p + 1\}.$$

If  $a \equiv 0 \pmod{4}$ , take  $X(d; \{3, 4\}) \oplus \dots \oplus X(d; \{\frac{a}{2} + 1, \frac{a}{2} + 2\})$ . Hence

$$L_X = \{-1 + in' : i = 2, \dots, \frac{a}{2} + 1\} \cup \{1 + in' : i = 4, \dots, \frac{a}{2} + 3\}.$$

In all cases  $\min(L_X) = 2n' - 1$  and  $\max(L_X) \leq (\frac{d}{2} - 2)n' + 1 < D - n'$ .

Now let's look at the small values of  $d$ ; that is, the case  $d \leq 18$  and therefore  $a \leq 3$ . If  $a = 0$ , the lemma is vacuous. If  $a = 1$ , then  $d \geq 8$ . We take  $X(d; \{1\})$  with

the orientation resulting in  $L_X = \{2n' + 1\}$ . If  $a = 2$ , then  $d \in \{12, 14, 16, 18\}$ . Take  $X(d; \{1\}) \oplus X(d; \{5\})$  with the orientations resulting in  $L_X = \{2n' + 1, 4n' - 1\}$ . If  $a = 3$ , then  $d \in \{16, 18\}$ . Take the same auxiliary circulant as before except that the cycle  $X(d; \{5\})$  should now be taken with both orientations so that  $L_X = \{2n' + 1, 4n' - 1, 6n' + 1\}$ . In all cases  $\min(L_X) = 2n' + 1$  and  $\max(L_X) < (\frac{d}{2} - 1)n' = D - n'$ .

Observe that the zig-zag  $(m' - 1)$ -paths  $P_{i,j}$  defined above use no edges of lengths  $\pm 1$  modulo  $n'$ . Hence  $L_X$  and  $\bigcup_{i=1}^a L_i$  are disjoint, the set  $L_{CP} = (\bigcup_{i=1}^a L_i) \cup L_X$  has size  $am' = \frac{c-1}{2}m'$ , and  $X(n-1; L_{CP})$  is  $C_m$ -decomposable. Since  $\max(L_X) < D - n'$  and the longest edge in the zig-zag paths has length  $m' - 1 + 2an' \leq n' - 2 + (\frac{d}{2} - 2)n' < D - n'$ , the set  $L_{CP}$  satisfies Conditions 1 – 4 of the lemma.

Now let  $m' = 3$  so that  $n' = 4$ . This construction will be similar to that of Lemma 2.3.3 for  $m' = 3$ . Let the zig-zag 2-paths  $P_{1,0}$  and  $P_{1,0}^*$  be

$$P_{1,0} = u_0 u_{2+n'} u_{-1} \quad \text{and} \quad P_{1,0}^* = u_0 u_{1+2n'} u_{-1}.$$

Their edge length sets are

$$L_1 = \{2 + n', -1 + 2n'\} \quad \text{and} \quad L_1^* = \{1 + 2n', 2 + 2n'\}.$$

For  $i = 2, \dots, \lceil \frac{a}{2} \rceil$  obtain  $P_{i,0}$  from  $P_{1,0}$  by adding  $2(i-1)n'$  to the subscript of the second vertex, and for  $i = 2, \dots, \lfloor \frac{a}{2} \rfloor$  obtain  $P_{i,0}^*$  from  $P_{1,0}^*$  in the same way. Then let  $\mathcal{P}_i = \{\rho^{jn'}(P_{i,0}) : j = 0, \dots, d-1\}$  and  $\mathcal{P}_i^* = \{\rho^{jn'}(P_{i,0}^*) : j = 0, \dots, d-1\}$ .

The auxiliary circulant to be used for linking the 2-paths into  $m$ -cycles is  $X = X(d; \{1, 3, 5, \dots, 2\lfloor \frac{a}{2} \rfloor + 1\})$ . Since  $3c = r = d + 1$ , we observe that  $a = \frac{d-2}{6}$ . Hence  $d \equiv 2 \pmod{6}$  and  $\gcd(d, 3) = 1$ . In addition, if  $a$  is even, then  $\gcd(d, a+1) = 1$ .

If  $a$  is even, decompose  $X$  into  $X(d; \{1\})$ ,  $X(d; \{3\})$ ,  $X(d; \{5, 7\})$ ,  $\dots$ ,  $X(d; \{a-1, a+1\})$ , or  $X(d; \{1\})$ ,  $X(d; \{3\})$ ,  $X(d; \{5, 7\})$ ,  $\dots$ ,  $X(d; \{a-3, a-1\})$ ,  $X(d; \{a+1\})$ , depending on whether  $\frac{a}{2}$  is odd or even, respectively. Now use the  $d$ -cycles  $X(d; \{1\})$  and  $X(d; \{3\})$  with only one orientation; namely, the one resulting in connecting edges of lengths  $n' + 1$  and  $3n' + 1$ , respectively. Use the  $d$ -cycle  $X(d; \{a+1\})$  with both orientations. The set of connecting edge lengths is thus

$$L_X = \{n' + 1, 3n' + 1\} \cup \{\pm 1 + (2i+1)n' : i = 2, \dots, \frac{a}{2}\}.$$

If  $a$  is odd, take  $X(d; \{1\}) \oplus X(d; \{3\}) \oplus X(d; \{5, 7\}) \oplus \dots \oplus X(d; \{a-2, a\})$ , or  $X(d; \{1\}) \oplus X(d; \{3, 5\}) \oplus \dots \oplus X(d; \{a-2, a\})$ , depending on whether  $a \equiv 3 \pmod{4}$

or  $a \equiv 1 \pmod{4}$ , respectively. Now use the  $d$ -cycle  $X(d; \{1\})$  only with the orientation resulting in connecting edges of length  $n' + 1$ , and  $X(d; \{3\})$  with both orientations. Hence

$$L_X = \{n' + 1\} \cup \{\pm 1 + (2i + 1)n' : i = 1, \dots, \frac{a-1}{2}\}.$$

It is easy to see that in both cases  $L_X$  and  $(\bigcup_{i=1}^{\lceil \frac{a}{2} \rceil} L_i) \cup (\bigcup_{i=1}^{\lfloor \frac{a}{2} \rfloor} L_i^*)$  are disjoint. Let  $L_{CP}$  be their union. The set  $L_{CP}$  thus satisfies Conditions 2 – 4 of the lemma. The least element of  $L_{CP}$  is clearly  $n' + 1$ , while the greatest is at most

$$1 + (a + 1)n' = 1 + \frac{d+4}{6}n' = \frac{1}{3}(2d + 11) < 2d - 4 = D - n'$$

since  $a = \frac{d-2}{6}$ ,  $n' = 4$ , and  $d \geq 8$ . Hence Condition 1 is satisfied as well.

This proves the lemma for all cases.  $\square$

In the next two lemmas we shall construct solitary peripheral cycles and combine them with the coupled peripheral cycles just described. An important role in the construction of these unusual peripheral cycles will be played by 4-paths of the form  $u_x u_{y+s} u_{x-s} u_y u_{x-2s}$ , denoted by  $Q_s(x, y)$ . If  $s$  is odd, observe that  $Q_s(x, y)$  contains a pair of edges of each of the two lengths  $|y - x + s|$  and  $|y - x + 2s|$ , and the two edges in each pair belong to distinct orbits of  $\langle \rho^2 \rangle$ .

**Lemma 3.3.4** *Let one of the following hold:  $m' \equiv 3 \pmod{8}$  and  $n' \equiv 2 \pmod{4}$ ;  $m' \equiv 7 \pmod{8}$  and  $n' \equiv 0 \pmod{4}$ ;  $m' \equiv 1 \pmod{8}$  and  $n' \equiv 2 \pmod{4}$  and  $m' \geq 17$ ;  $m' \equiv 5 \pmod{8}$  and  $n' \equiv 0 \pmod{4}$ ;  $m' = 13$  and  $n' = 14$ ;  $m' = 7$  and  $n' = 10$ ;  $m' = 5$ ; or  $m' = 3$ . Then there exists a set  $L_P \subseteq \{1, \dots, D\}$  such that*

1.  $|L_P| = \frac{r-1}{2}$ ,
2.  $\{3, 5\} \subseteq L_P$ ,
3.  $1, 2, 4, D \notin L_P$ , and
4.  $X(n - 1; L_P) \oplus \langle \rho^2 \rangle(u_1 u_3)$  is  $C_m$ -decomposable.

PROOF. First let us assume that  $m' \geq 7$  and either  $m' \equiv 3 \pmod{8}$  and  $n' \equiv 2 \pmod{4}$ , or  $m' \equiv 7 \pmod{8}$  and  $n' \equiv 0 \pmod{4}$ . Let

$$\begin{aligned} Z &= \frac{1}{2}\left(\frac{m'-5}{2} + n' + 5\right) && \text{and} \\ x &= -\frac{m'-5}{2} + Z = \frac{1}{4}(2n' - m' + 15). \end{aligned}$$

Notice that in both cases  $Z$  is odd and  $x$  is even. Define the  $m'$ -path  $P_{0,0}$  by

$$P_{0,0} = u_0 u_3 u_1 u_4 u_{-1} Q_1(-1, 6) Q_1(-3, 8) \dots Q_1\left(-\frac{m'-9}{2}, \frac{m'+1}{2}\right) u_{-\frac{m'-5}{2}} u_{x+D} u_{n'+5} u_{n'}$$

if  $x > \frac{m'+1}{2}$ , and

$$\begin{aligned} P_{0,0} &= u_0 u_3 u_1 u_4 u_{-1} Q_1(-1, 6) Q_1(-3, 8) \dots Q_1(-(x-7), x-2) Q_1(-(x-5), x+1) \\ &\quad Q_1(-(x-3), x+3) \dots Q_1\left(-\frac{m'-9}{2}, \frac{m'+3}{2}\right) u_{-\frac{m'-5}{2}} u_{x+D} u_{n'+5} u_{n'} \end{aligned}$$

if  $x \leq \frac{m'+1}{2}$ . For  $j = 1, \dots, d-1$ , let  $P_{0,j} = \rho^{jn'}(P_{0,0})$ . Since  $\frac{m'+5}{2} < n' - \frac{m'-5}{2}$ , the paths  $P_{0,j}$  and  $P_{0,j+1}$  are in both cases vertex-disjoint except for the endpoints. Since  $u_x$  does not belong to  $P_{0,0}$ , the paths  $P_{0,j}$  and  $P_{0,j+\frac{d}{2}}$  are vertex-disjoint, and since  $x = \frac{1}{4}(2n' - m' + 15) < n' - \frac{m'-5}{2}$ ,  $u_x$  does not belong to  $P_{0,1}$  so that  $P_{0,j+1}$  and  $P_{0,j+\frac{d}{2}}$  are vertex-disjoint. Hence  $C_0 = \bigcup_{j=0}^{d-1} P_{0,j}$  is an  $m$ -cycle.

Observe that  $P_{0,0}$  contains exactly two edges of each of the lengths in the set  $L_0$ , where

$$L_0 = \{3, 5, 8, 9, 12, 13, \dots, m' - 3, m' - 2, D - Z\}$$

if  $x > \frac{m'+1}{2}$ , and

$$L_0 = \{3, 5, 8, 9, 12, 13, \dots, 2x - 8, 2x - 7, 2x - 3, 2x - 2, \dots, m' - 2, m' - 1, D - Z\}$$

if  $x \leq \frac{m'+1}{2}$ . Moreover, the two edges of the same length belong to distinct orbits of  $\langle \rho^2 \rangle$ . In addition,  $P_{0,0}$  contains an edge of length 2 from the orbit  $\langle \rho^2 \rangle(u_1 u_3)$ . Hence  $\{\rho^{2i}(C_0) : i = 0, \dots, \frac{n'}{2} - 1\}$  is a  $C_m$ -decomposition of  $X(n-1; L_0) \oplus \langle \rho^2 \rangle(u_1 u_3)$ .

If  $c = 1$ , let  $L_P = L_0$ . Since  $|L_0| = \frac{m'-1}{2} = \frac{r-1}{2}$ , it is easy to see that Conditions 1-4 are satisfied. We may thus assume that  $c \geq 3$ .

Now let  $L_{CP}$  be a set satisfying the conditions of Lemma 3.3.3 and let  $L_P = L_0 \cup L_{CP}$ . Since the maximum element of  $L_0$  is

$$D - Z = D - \frac{1}{2}\left(\frac{m'-5}{2} + n' + 5\right) > D - \left(\frac{m'}{2} + \frac{n'}{2}\right) > D - n',$$

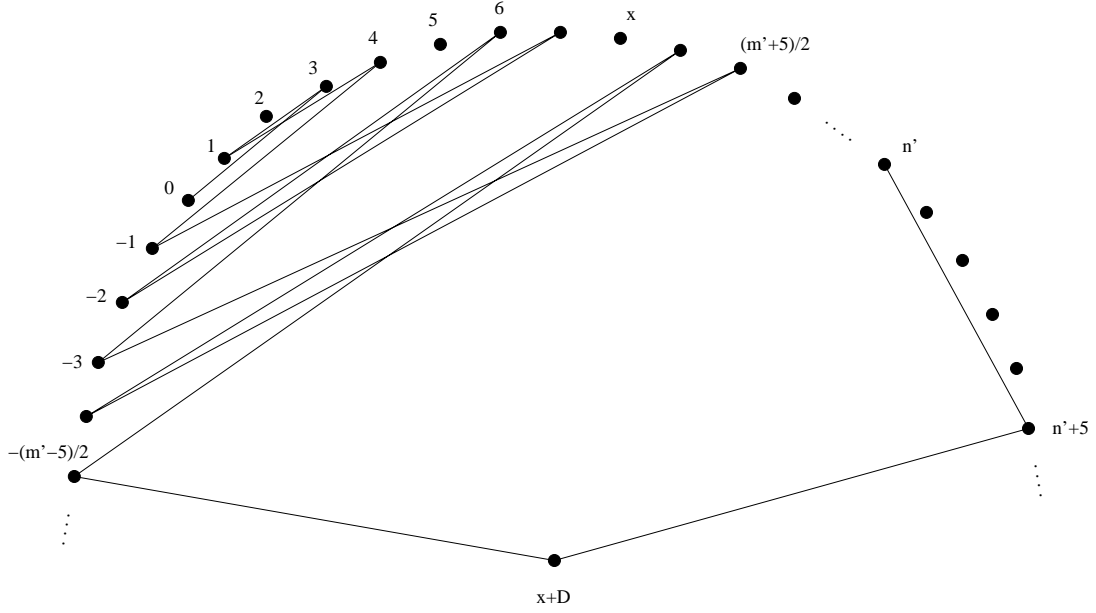


Figure 5: Lemma 3.3.4: the  $m'$ -path  $P_{0,0}$  with  $x \leq \frac{m'+1}{2}$  for  $m' \equiv 3 \pmod{8}$  and  $n' \equiv 2 \pmod{4}$ , or  $m' \equiv 7 \pmod{8}$  and  $n' \equiv 0 \pmod{4}$ .

the sets  $L_0$  and  $L_{CP}$  are disjoint. In addition,

$$|L_P| = |L_0| + |L_{CP}| = \frac{m'-1}{2} + \frac{c-1}{2}m' = \frac{cm'-1}{2} = \frac{r-1}{2}.$$

Since  $X(n-1; L_{CP})$  is  $C_m$ -decomposable by Lemma 3.3.3, it is now easy to see that the set  $L_P$  satisfies Conditions 1 – 4.

Now assume that either  $m' \equiv 1 \pmod{8}$  and  $n' \equiv 2 \pmod{4}$  and  $m' \geq 17$ , or  $m' \equiv 5 \pmod{8}$  and  $n' \equiv 0 \pmod{4}$ . Notice that, since  $r' \leq \frac{m'-1}{2}$  and  $r'$  is odd, the last condition implies  $m' \geq 13$ . Let

$$\begin{aligned} Z &= \frac{1}{2}(n' - 5 - \frac{m'-3}{2}), \\ x &= \frac{m'-3}{2} + Z = \frac{1}{4}(2n' + m' - 13), \quad \text{and} \\ y &= n' - x. \end{aligned}$$

Notice that in both cases  $Z$  is odd while  $x$  and  $y$  are even. Expressing  $n'$  in terms of  $m'$  and  $r'$ , and using  $r' \geq 1$  and  $m' \geq 13$ , we can see that  $Z \geq 3$ . Define the

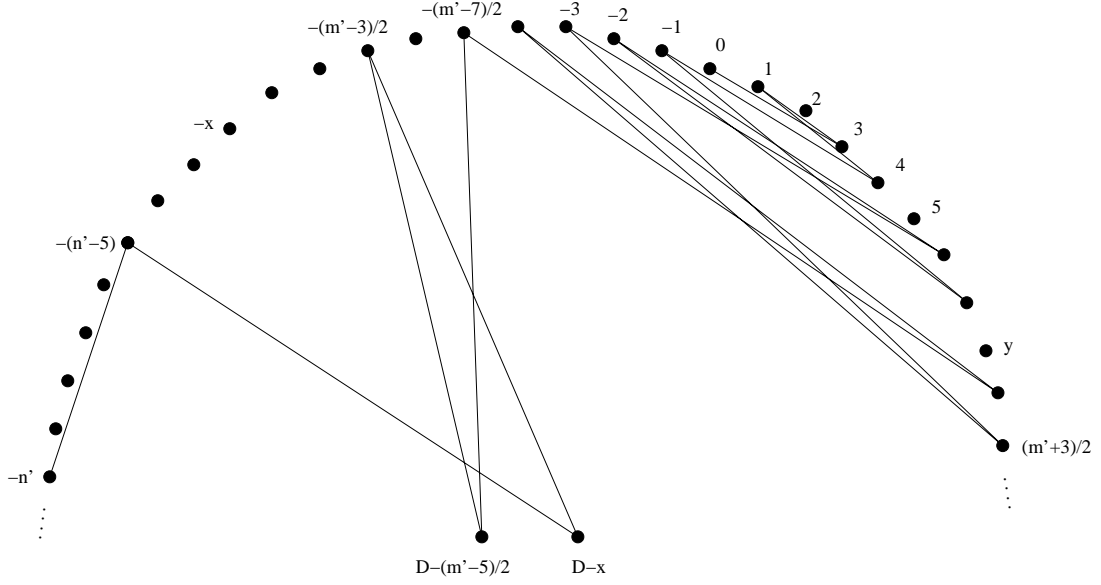


Figure 6: Lemma 3.3.4: the  $m'$ -path  $P_{0,0}$  with  $y \leq \frac{m'-1}{2}$  for  $m' \equiv 1 \pmod{8}$  and  $n' \equiv 2 \pmod{4}$  and  $m' \geq 17$ , or  $m' \equiv 5 \pmod{8}$  and  $n' \equiv 0 \pmod{4}$ .

$m'$ -path  $P_{0,0}$  by

$$P_{0,0} = u_0 u_3 u_1 u_4 u_{-1} Q_1(-1, 6) Q_1(-3, 8) \dots \\ \dots Q_1\left(-\frac{m'-11}{2}, \frac{m'-1}{2}\right) u_{-\frac{m'-7}{2}} u_{D-\frac{m'-5}{2}} u_{-\frac{m'-3}{2}} u_{D-x} u_{-(n'-5)} u_{-n'}$$

if  $y > \frac{m'-1}{2}$ , and by

$$P_{0,0} = u_0 u_3 u_1 u_4 u_{-1} Q_1(-1, 6) Q_1(-3, 8) \dots Q_1(-(y-7), y-2) \\ Q_1(-(y-5), y+1) Q_1(-(y-3), y+3) \dots \\ \dots Q_1\left(-\frac{m'-11}{2}, \frac{m'+1}{2}\right) u_{-\frac{m'-7}{2}} u_{D-\frac{m'-5}{2}} u_{-\frac{m'-3}{2}} u_{D-x} u_{-(n'-5)} u_{-n'}$$

if  $y \leq \frac{m'-1}{2}$ . For  $j = 1, \dots, d-1$ , let  $P_{0,j} = \rho^{jn'}(P_{0,0})$ . Since  $\frac{m'+3}{2} < n' - \frac{m'-3}{2}$ , the paths  $P_{0,j}$  and  $P_{0,j+1}$  are in both cases vertex-disjoint except for the endpoints. Since  $u_{-\frac{m'-5}{2}}$  and  $u_{-x}$  do not belong to  $P_{0,0}$ , the paths  $P_{0,j}$  and  $P_{0,j+\frac{d}{2}}$  are vertex-disjoint, and since  $u_{n'-\frac{m'-5}{2}}$  and  $u_y$  do not belong to  $P_{0,0}$ , the paths  $P_{0,j}$  and  $P_{0,j+\frac{d}{2}+1}$  are vertex-disjoint. Hence  $C_0 = \bigcup_{j=0}^{d-1} P_{0,j}$  is an  $m$ -cycle.

Observe that  $P_{0,0}$  contains exactly two edges of each of the lengths in the set  $L_0$ , where

$$L_0 = \{3, 5, 8, 9, 12, 13, \dots, m' - 5, m' - 4, D - Z, D - 1\}$$

if  $y > \frac{m'-1}{2}$ , and

$$L_0 = \{3, 5, 8, 9, 12, 13, \dots, 2y-8, 2y-7, 2y-3, 2y-2, \dots, m'-4, m'-3, D-Z, D-1\}$$

if  $y \leq \frac{m'-1}{2}$ . The two edges of the same length belong to distinct orbits of  $\langle \rho^2 \rangle$ . In addition,  $P_{0,0}$  contains an edge of length 2 from the orbit  $\langle \rho^2 \rangle(u_1 u_3)$ . The proof is now completed as in the previous case by observing that the second largest element of  $L_0$  is  $D - Z = D - \frac{1}{2}(n' - 5 - \frac{m'-3}{2}) > D - n'$ .

Now let us consider the case  $m' = 13$ ,  $n' = 14$ . We let

$$P_{0,0} = u_0 u_3 u_1 u_4 u_{-1} u_{-6} \overleftarrow{Q}_1(-4, 6) Q_5(-4, D - 3),$$

where  $\overleftarrow{Q}_1(-4, 6)$  is, of course, the 4-path  $u_{-6} u_6 u_{-5} u_7 u_{-4}$  and  $Q_5(-4, D - 3)$  is the 4-path  $u_{-4} u_{D+2} u_{-9} u_{D-3} u_{-14}$ . With  $L_0 = \{3, 5, 11, 12, D - 11, D - 6\}$  the proof can now be completed as before.

For  $m' = 7$  and  $n' = 10$ , let  $P_{0,0} = u_0 u_3 u_1 u_4 u_{-1} u_7 u_2 u_{10}$  so that  $L_0 = \{3, 5, 8\}$ .

For  $m' = 5$  (and hence  $n' = 6$ ) let  $P_{0,0} = u_0 u_3 u_1 u_4 u_{-1} u_{-6}$  so that  $L_0 = \{3, 5\}$  and complete the proof as before.

Finally, let  $m' = 3$  and hence  $n' = 4$ . Let  $P_{0,0} = u_0 u_3 u_1 u_4$ . With  $L_0 = \{3\}$ , the proof can now be completed as before except that the edge length 5 has to come from the set  $L_{CP}$ . Fortunately, a set  $L_{CP}$  satisfying the conditions of Lemma 3.3.3 for  $m' = 3$  does contain 5 whenever  $c \geq 3$ . Since  $c = \frac{d+1}{3}$  is odd and  $d \geq 4$ , we indeed have  $c \geq 3$ . Hence  $5 \in L_P$  and Conditions 1 - 4 are met for the case  $m' = 3$  as well.

This proves the lemma for all cases. □

**Lemma 3.3.5** *Let one of the following hold:  $m' \equiv 3 \pmod{8}$  and  $n' \equiv 0 \pmod{4}$  and  $m' \geq 11$ ;  $m' \equiv 7 \pmod{8}$  and  $n' \equiv 2 \pmod{4}$  and  $m' \geq 15$ ;  $m' \equiv 1 \pmod{8}$  and  $n' \equiv 0 \pmod{4}$  and  $m' \geq 17$ ;  $m' \equiv 5 \pmod{8}$  and  $n' \equiv 2 \pmod{4}$  and  $m' \geq 21$ ;  $m' = 13$  and  $n' = 18$ ; or  $m' = 9$ . Then there exists a set  $L_P \subseteq \{1, \dots, D\}$  such that*

1.  $|L_P| = \frac{r-3}{2}$ ,
2.  $\{4, 7\} \subseteq L_P$ ,
3.  $1, 2, 3, 5, 6, D \notin L_P$ , and
4.  $X(n-1; L_P) \oplus \langle \rho^2 \rangle (\{u_0u_2, u_0u_3, u_0u_5\})$  is  $C_m$ -decomposable.

PROOF. First assume that  $m' \equiv 3 \pmod{8}$  and  $n' \equiv 0 \pmod{4}$  and  $m' \geq 11$ , or  $m' \equiv 7 \pmod{8}$  and  $n' \equiv 2 \pmod{4}$  and  $m' \geq 15$ . Let

$$\begin{aligned} Z &= \frac{1}{2}(\frac{m'-1}{2} + n' + 5) && \text{and} \\ x &= -\frac{m'-1}{2} + Z = \frac{1}{4}(2n' - m' + 11). \end{aligned}$$

Notice that in both cases  $Z$  is odd and  $x$  is even. Define the  $m'$ -path  $P_{0,0}$  by

$$\begin{aligned} P_{0,0} &= u_0u_2u_{-2}u_1u_{-3}Q_1(-3, 3)Q_1(-5, 6)Q_1(-7, 8) \dots \\ &\dots Q_1(-\frac{m'-5}{2}, \frac{m'-3}{2})u_{-\frac{m'-1}{2}}u_{x+D}u_{n'+5}u_{n'} \end{aligned}$$

if  $x > \frac{m'-3}{2}$ , and

$$\begin{aligned} P_{0,0} &= u_0u_2u_{-2}u_1u_{-3}Q_1(-3, 3)Q_1(-5, 6)Q_1(-7, 8) \dots \\ &\dots Q_1(-(x-3), x-2)Q_1(-(x-1), x+1)Q_1(-(x+1), x+3) \dots \\ &\dots Q_1(-\frac{m'-5}{2}, \frac{m'-1}{2})u_{-\frac{m'-1}{2}}u_{x+D}u_{n'+5}u_{n'} \end{aligned}$$

if  $x \leq \frac{m'-3}{2}$ . For  $j = 1, \dots, d-1$ , let  $P_{0,j} = \rho^{jn'}(P_{0,0})$ . It is not difficult to see that these paths are pairwise vertex-disjoint except for the endpoints so that  $C_0 = \bigcup_{j=0}^{d-1} P_{0,j}$  is an  $m$ -cycle.

Observe that  $P_{0,0}$  contains exactly two edges of each of the lengths in the set  $L_0$ , where

$$L_0 = \{4, 7, 8, 12, 13, 16, 17, \dots, m' - 3, m' - 2, D - Z\}$$

if  $x > \frac{m'-3}{2}$ , and

$$L_0 = \{4, 7, 8, 12, 13, 16, 17, \dots, 2x-4, 2x-3, 2x+1, 2x+2, \dots, m' - 2, m' - 1, D - Z\}$$

if  $x \leq \frac{m'-3}{2}$ . The two edges of the same length belong to distinct orbits of  $\langle \rho^2 \rangle$ . In addition,  $P_{0,0}$  contains one edge of length 2 from the orbit  $\langle \rho^2 \rangle(u_0u_2)$ , one edge



of length 3 from the orbit  $\langle \rho^2 \rangle(u_0 u_3)$ , and one edge of length 5 from the orbit  $\langle \rho^2 \rangle(u_0 u_5)$ . Hence  $\{\rho^{2i}(C_0) : i = 0, \dots, \frac{n'}{2} - 1\}$  is a  $C_m$ -decomposition of  $X(n-1; L_0) \oplus \langle \rho^2 \rangle(\{u_0 u_2, u_0 u_3, u_0 u_5\})$ .

If  $c = 1$ , let  $L_P = L_0$ . Since  $|L_0| = \frac{m'-3}{2} = \frac{r-3}{2}$ , it is easy to see that Conditions 1 – 4 are satisfied. We may thus assume that  $c \geq 3$ .

Now let  $L_{CP}$  be a set satisfying the conditions of Lemma 3.3.3 and let  $L_P = L_0 \cup L_{CP}$ . Since the maximum element of  $L_0$  is

$$D - Z = D - \frac{1}{2}(\frac{m'-1}{2} + n' + 5) \geq D - (\frac{m'}{2} + \frac{n'}{2}) > D - n',$$

the sets  $L_0$  and  $L_{CP}$  are disjoint. Now

$$|L_P| = |L_0| + |L_{CP}| = \frac{m'-3}{2} + \frac{c-1}{2}m' = \frac{cm'-3}{2} = \frac{r-3}{2}.$$

Since  $X(n-1; L_{CP})$  is  $C_m$ -decomposable by Lemma 3.3.3, it is now easy to see that the set  $L_P$  satisfies Conditions 1 – 4.

Next assume that  $m' \equiv 1 \pmod{8}$  and  $n' \equiv 0 \pmod{4}$  and  $m' \geq 17$ , or  $m' \equiv 5 \pmod{8}$  and  $n' \equiv 2 \pmod{4}$  and  $m' \geq 21$ , or  $m' = 13$  and  $n' = 18$ . Let

$$\begin{aligned} Z &= \frac{1}{2}(n' - 5 - \frac{m'+1}{2}), \\ x &= \frac{m'+1}{2} + Z, \quad \text{and} \\ y &= n' - x. \end{aligned}$$

Notice that in all cases  $Z$  is odd while  $x$  and  $y$  are even. Since  $m' + 2r' \geq 23$ , we have  $Z \geq 3$ . Define the  $m'$ -path  $P_{0,0}$  by

$$\begin{aligned} P_{0,0} &= u_0 u_2 u_{-2} u_1 u_{-3} Q_1(-3, 3) Q_1(-5, 6) Q_1(-7, 8) \dots \\ &\quad \dots Q_1(-\frac{m'-7}{2}, \frac{m'-5}{2}) u_{-\frac{m'-3}{2}} u_{D-\frac{m'-1}{2}} u_{-\frac{m'+1}{2}} u_{D-x} u_{-(n'-5)} u_{-n'} \end{aligned}$$

if  $y > \frac{m'-5}{2}$ , and by

$$\begin{aligned} P_{0,0} &= u_0 u_2 u_{-2} u_1 u_{-3} Q_1(-3, 3) Q_1(-5, 6) Q_1(-7, 8) \dots \\ &\quad \dots Q_1(-(y-3), y-2) Q_1(-(y-1), y+1) Q_1(-(y+1), y+3) \dots \\ &\quad \dots Q_1(-\frac{m'-7}{2}, \frac{m'-3}{2}) u_{-\frac{m'-3}{2}} u_{D-\frac{m'-1}{2}} u_{-\frac{m'+1}{2}} u_{D-x} u_{-(n'-5)} u_{-n'} \end{aligned}$$

if  $y \leq \frac{m'-5}{2}$ . Again, it is not difficult to see that  $C_0 = \bigcup_{j=0}^{d-1} P_{0,j}$  is an  $m$ -cycle.

Observe that  $P_{0,0}$  contains exactly two edges of each of the lengths in the set  $L_0$ , where

$$L_0 = \{4, 7, 8, 12, 13, 16, 17, \dots, m' - 5, m' - 4, D - Z, D - 1\}$$

if  $y > \frac{m'-5}{2}$ , and

$$L_0 = \{4, 7, 8, 12, 13, \dots, 2y - 4, 2y - 3, 2y + 1, 2y + 2, \dots, m' - 4, m' - 3, D - Z, D - 1\}$$

if  $y \leq \frac{m'-5}{2}$ . The two edges of the same length belong to distinct orbits of  $\langle \rho^2 \rangle$ . In addition,  $P_{0,0}$  contains one edge of length 2 from the orbit  $\langle \rho^2 \rangle(u_0u_2)$ , one edge of length 3 from the orbit  $\langle \rho^2 \rangle(u_0u_3)$ , and one edge of length 5 from the orbit  $\langle \rho^2 \rangle(u_0u_5)$ . The proof can now be completed as in the previous case observing that the second largest element of  $L_0$  is  $D - Z = D - \frac{1}{2}(n' - 5 - \frac{m'+1}{2}) > D - n'$ .

Finally, let  $m' = 9$  and, since  $\gcd(m', n') = 1$ ,  $n' = 10$ . Let

$$P_{0,0} = u_0u_2u_{-2}u_1u_{-3}Q_1(-3, 3)u_{-5}u_{-10}.$$

With  $L_0 = \{4, 7, 8\}$  the proof is completed as before.

This proves the lemma for all cases. □

### 3.3.2 The case $c \geq \frac{d}{2}$

First observe that  $c \geq \frac{d}{2}$  implies  $r' = \frac{cm'-1}{d} \geq \frac{m'}{2} - \frac{1}{d} \geq \frac{m'-1}{2}$ . If  $d > 2$ , then in fact  $r' \geq \frac{m'+1}{2}$ . Two slightly different constructions are needed depending on whether  $\frac{d}{2}$  is odd or even. The following lemma provides the basis for the construction of central cycles and diameter cycles in both cases.

**Lemma 3.3.6** *Let  $L_0$ ,  $L_D$ , and  $L$  be pairwise disjoint subsets of the edge length set  $\{1, \dots, D - 1\}$ , and let  $P$ ,  $P_D$ , and  $R$  be paths in  $X(n - 1; \{1, \dots, D - 1\})$  with the following properties:*

1.  $P$  is a zig-zag path with edge length set  $L_0$ ,
2.  $P_D$  is a basic zig-zag path with edge length set  $L_D$ ,
3.  $|E(P_D)| = |L_D| = \frac{m}{2} - 1$ ,

4. exactly one of the two endpoints of  $P_D$  has an odd subscript,
5. the length of every edge of  $R$  is in  $L \cup L_0 \cup L_D$ ,
6.  $R$  contains exactly one edge of each of the lengths in  $L_0 \cup L_D$  and this edge belongs to the same orbit of  $\langle \rho^2 \rangle$  as the edge of the same length in  $P$  or  $P_D$ ,
7.  $R$  contains exactly two edges of each of the lengths in  $L$ , one from each of the two orbits of  $\langle \rho^2 \rangle$ ,
8.  $|E(R)| = m - 2$ , and
9. exactly one of the two endpoints of  $R$  has an odd subscript.

Then  $(X(n-1; L \cup L_D \cup \{D\}) \oplus \langle \rho^2 \rangle(P)) \rtimes K_1$  is  $C_m$ -decomposable.

PROOF. First we describe the central cycles. Conditions 5 – 8 imply that  $\{\rho^{2i}(R) : i = 0, \dots, \frac{n-1}{2} - 1\}$  is a partition of the edge set of  $X(n-1; L) \oplus \langle \rho^2 \rangle(P) \oplus \langle \rho^2 \rangle(P_D)$  into  $(m-2)$ -paths. Let  $u_{s_R}$  and  $u_{t_R}$  be, respectively, the initial and the terminal vertex of the path  $R$ . Define an  $m$ -cycle  $C_C = wu_{s_R}Ru_{t_R}w$ . Conditions 8 and 9 now imply that  $\{\rho^{2i}(C_C) : i = 0, \dots, \frac{n-1}{2} - 1\}$  is a  $C_m$ -decomposition of  $(X(n-1; L) \oplus \langle \rho^2 \rangle(P) \oplus \langle \rho^2 \rangle(P_D)) \rtimes K_1$ .

Next we describe the diameter cycles. Let  $u_{s_D}$  and  $u_{t_D}$  be, respectively, the initial and the terminal vertex of the path  $\rho(P_D)$ . Define the cycle  $C_D$  by

$$C_D = \rho(P_D)u_{t_D}u_{t_D+D}\rho^{D+1}(\overleftarrow{P_D})u_{s_D+D}u_{s_D}.$$

Since  $P_D$  is a basic zig-zag path without edges of diameter length, the paths  $\rho(P_D)$  and  $\rho^{D+1}(\overleftarrow{P_D})$  are vertex-disjoint. Hence, by Condition 3,  $C_D$  is an  $m$ -cycle and since  $D$  is even, Condition 4 implies that the two edges of diameter length in  $C_D$  belong to distinct orbits of  $\langle \rho^2 \rangle$ . On the other hand, the edges in  $\rho^{D+1}(\overleftarrow{P_D})$  belong to the same orbit of  $\langle \rho^2 \rangle$  as the corresponding edges in  $\rho(P_D)$  and are diametrically opposed to them. Hence  $\{\rho^{2i}(C_D) : i = 0, \dots, \frac{n-1}{4} - 1\}$  is a  $C_m$ -decomposition of  $X(n-1; \{D\}) \oplus \langle \rho^2 \rangle(\rho(P_D))$ .

We thus have a  $C_m$ -decomposition of  $(X(n-1; L \cup L_D \cup \{D\}) \oplus \langle \rho^2 \rangle(P)) \rtimes K_1$ .

□

We continue with the details of the construction, beginning with peripheral cycles for  $\frac{d}{2}$  odd. Since the number of peripheral cycles is  $(c - \frac{d}{2})\frac{n'}{2}$ , only coupled peripheral cycles are needed in this case.

**Lemma 3.3.7** *Let  $\frac{d}{2}$  be odd. There exists a set  $L_{CP}$  with the following properties:*

1.  $L_{CP} \subseteq \{n' + 1, \dots, D - 4\}$ ,
2.  $|L_{CP}| = \frac{1}{2}(c - \frac{d}{2})m'$ , and
3.  $X(n - 1; L_{CP})$  is  $C_m$ -decomposable.

PROOF. Denote  $\frac{1}{2}(c - \frac{d}{2})$  by  $a$  and observe that  $a$  is an integer. Notice that  $d = 2$  implies  $c = 1$  and hence  $a = 0$ . In this case the lemma is trivially true. Now  $m' = 3$  implies  $r' = 1$  and  $\frac{d+1}{3} = c \geq \frac{d}{2}$ , whence  $d = 2$  and  $c = 1$ . We may thus assume that  $d \geq 6$  and  $m' \geq 5$ .

Since  $r' \geq \frac{m'+1}{2}$ , we have  $r' \geq 3$ . We shall use the construction of Lemma 3.3.3 (Case  $m' \geq 5$ ) with a few modifications. We ask the reader to refer to the proof of Lemma 3.3.3 since only an outline and any differences will be presented here.

First of all,  $a = \frac{c-1}{2}$  of Lemma 3.3.3 is replaced by  $a = \frac{1}{2}(c - \frac{d}{2})$ . Since  $c \leq d - 1$ , we have  $a \leq \frac{d-2}{4}$ . As in Lemma 3.3.3, the basis for the construction is the zig-zag  $(m' - 1)$ -path

$$P_{1,0} = u_0 u_{2+2n'} u_{-1} u_{3+2n'} \dots u_{-\frac{m'-3}{2}} u_{\frac{m'+1}{2}+2n'} u_{1+n'}$$

with edge length set

$$L_1 = \{\frac{m'-1}{2} + n', 2 + 2n', 3 + 2n', \dots, m' - 1 + 2n'\},$$

and for  $i = 2, \dots, a$  we obtain the zig-zag  $(m' - 1)$ -paths  $P_{i,0}$  with edge length sets  $L_i = L_1 + 2(i - 1)n'$  by adding  $2(i - 1)n'$  to the subscripts of the even vertices in  $P_{1,0}$ .

For  $d \geq 14$ , let  $p = \frac{d}{2} - 2$ . Since  $\frac{d}{2}$  is odd,  $\gcd(d, p) = 1$ . Since  $a \leq \frac{d-2}{4}$ , we have  $\frac{a}{2} + 3 \leq p$ . With this value of  $p$  we now define the set  $L_X$  of connecting edge lengths for the various values of  $a$  modulo 4 as in Lemma 3.3.3. Notice that in all cases  $\min(L_X) = 2n' - 1$  and  $\max(L_X) \leq (\frac{d}{2} - 1)n' + 1 < D - 4$ .

For  $d \leq 10$  we have  $a \leq 2$ . The connecting edge lengths are now chosen in the following way. If  $a = 1$ , use the  $d$ -cycle  $X(d; \{1\})$  with the orientation resulting in  $L_X = \{2n' + 1\}$ . If  $a = 2$ , we must have  $d = 10$ . Now we can use the  $d$ -cycle  $X(d; \{3\})$  with both orientations so that  $L_X = \{2n' - 1, 4n' + 1\}$ . In these two cases  $\min(L_X) \geq 2n' - 1$  and  $\max(L_X) \leq (\frac{d}{2} - 1)n' + 1 < D - 4$ .

Since  $L_X$  and  $\bigcup_{i=1}^a L_i$  are disjoint,  $L_{CP} = (\bigcup_{i=1}^a L_i) \cup L_X$  has size  $am'$  and  $X(n - 1; L_{CP})$  is  $C_m$ -decomposable. Since the longest edge length in the zig-zag paths is  $m' - 1 + 2an' \leq n' - 4 + \frac{d-2}{2}n' = D - 4$ , the set  $L_{CP}$  satisfies Conditions 1 – 3 of the lemma.  $\square$

Next we describe how to construct central cycles for the case  $\frac{d}{2}$  odd.

**Lemma 3.3.8** *Let  $\frac{d}{2}$  be odd and let  $L_C = \{1, \dots, D\} - L_{CP}$ , where  $L_{CP}$  is a set satisfying the conditions of Lemma 3.3.7. Then  $X(n - 1; L_C) \bowtie K_1$  is  $C_m$ -decomposable.*

PROOF. As in the proof of the previous lemma, let  $a = \frac{1}{2}(c - \frac{d}{2})$ .

If  $m' = 3$ , we have seen that  $r' = 1$ ,  $c = 1$ ,  $d = 2$ , and  $a = 0$ . Since the set  $L_{CP}$  of Lemma 3.3.7 is now empty,  $L_C = \{1, 2, 3, 4\}$ . We now apply Lemma 3.3.6 with  $L_0 = \emptyset$ ,  $L_D = \{1, 2\}$ ,  $L = \{3\}$ ,  $P_D = u_0u_1u_{-1}$ , and  $R = u_3u_0u_1u_{-1}u_2$  to see that  $X(n - 1; L_C) \bowtie K_1$  is  $C_m$ -decomposable.

We may now assume that  $m' \geq 5$  and hence  $r' \geq 3$ . If  $m' = 5$ , we have  $r' = 3$  and hence  $d = \frac{5c-1}{3} \geq 8$ . This implies that either  $m' \geq 7$  or  $d \geq 10$ , whence  $m \geq 14$ .

Condition 1 of Lemma 3.3.7 implies that  $\{D - 2, D - 1, D\} \subseteq L_C$ . Let  $L_C$  be the disjoint union of sets  $L_B$ ,  $L_A$ , and  $\{D - 1, D\}$  with the properties that  $\max(L_B) < \min(L_A)$  and  $|L_A| = \frac{m}{2} - 1$ . That is,  $L_A$  is the set of the  $\frac{m}{2} - 1$  largest members of  $L_C - \{D - 1, D\}$  and  $L_B$  is the remainder. Since  $m \geq 14$ ,

$$\begin{aligned} |L_B| &= |L_C| - (|L_A| + 2) = \frac{dn'}{2} - \frac{1}{2}(c - \frac{d}{2})m' - (\frac{m}{2} + 1) \\ &= \frac{1}{2}(n - 1 - r + \frac{m}{2} - m - 2) = \frac{1}{4}(m - 6) \geq 2. \end{aligned} \tag{12}$$

Hence the sets  $L_B$  and  $L_A$  are well defined. Let us introduce the rest of the notation for this proof:

$$\begin{aligned} N &= |L_B|, \\ M &= |L_B| + |L_A| = N + \frac{m}{2} - 1, \end{aligned}$$

$$\begin{aligned}
L_B &= \{1, a_2, \dots, a_N\}, \text{ where } 2 \leq a_2 < \dots < a_N, \\
L_A &= \{a_{N+1}, \dots, a_{M-1}, D-2\}, \text{ where } a_N < a_{N+1} < \dots < a_{M-1} < D-2, \\
B &= -a_2 + a_3 - \dots + (-1)^{N-1}a_N, \\
A_2 &= B + (-1)^N a_{N+1} + (-1)^{N+1}a_{N+2} + \dots + (-1)^{M-2}a_{M-1}, \quad \text{and} \\
A &= A_2 + (-1)^{M-1}(D-2).
\end{aligned}$$

Define the following zig-zag paths:

$$\begin{aligned}
P_{B_1} &= u_0 u_{-a_2} u_{-a_2+a_3} \dots u_B, \\
P_B &= u_{-1} u_0 P_{B_1}, \\
P_{A_2} &= u_B u_{B+(-1)^N a_{N+1}} u_{B+(-1)^N a_{N+1}+(-1)^{N+1} a_{N+2}} \dots u_{A_2}, \quad \text{and} \\
P_A &= P_{A_2} u_{A_2} u_A.
\end{aligned}$$

Since  $n'$  is even, each of the  $a$  families of coupled peripheral cycles in Lemma 3.3.7 uses an even number of odd edge lengths. Hence  $L_C$  contains an odd number of odd edge lengths if and only if  $\frac{D}{2}$  is odd. From the way  $A$  is defined it now follows that  $A$  is odd if and only if  $\frac{D}{2}$  is odd.

Since  $M = |L_C - \{D-1, D\}| = D - am' - 2$  and  $m'$  is odd, we have  $A > 0$  (evaluated in the integers) if and only if  $a$  is odd. Note that  $A > 0$  means that  $u_A$  is in the set  $\{u_1, u_2, \dots, u_{D-1}\}$  while  $A < 0$  means that  $u_A$  is in the set  $\{u_{D+1}, u_{D+2}, \dots, u_{2D-1}\}$ .

We are now ready to define the paths  $P$ ,  $P_D$ , and  $R$  that satisfy the conditions of Lemma 3.3.6. The details of the construction depend on whether  $A-B$ ,  $a = \frac{1}{2}(c - \frac{d}{2})$ , and  $\frac{D}{2}$  are odd or even. In all cases, however, the set  $L_0$  of Lemma 3.3.6 will be empty so that  $P$  is an empty path. Observe that, since  $\frac{d}{2}$  is odd,  $M - N = \frac{m}{2} - 1$  is even. Hence  $A$  and  $B$  are either both positive or both negative.

1. Case  $A-B$  odd. Let  $L = L_B \cup \{D-1\}$  and let  $P_D = P_A$  so that  $L_D = L_A$ . Notice that Conditions 2 – 4 of Lemma 3.3.6 are satisfied for the path  $P_D$ . We shall define the path  $R$  to meet Conditions 5 – 7. Since  $|L| = \frac{m-2}{4}$  by (12) and  $|L_D| = \frac{m}{2} - 1$ , Conditions 5 – 7 imply Condition 8. The only things that require verification are the endpoints of the path  $R$  and the orbits of the linking edges, that is, the edges of length  $D-1$ , and in some of the cases, the edges of length 1.

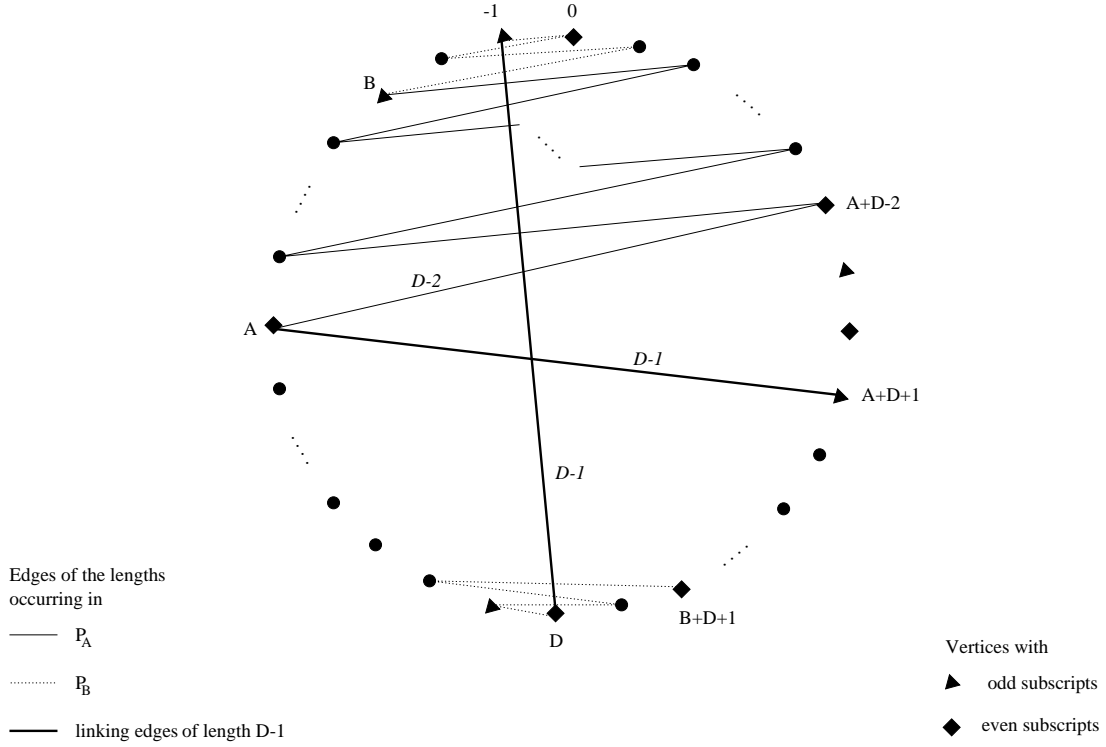


Figure 7: Lemma 3.3.8: the path  $R$  for Case 1.1.1 ( $A - B$  odd,  $a$  and  $\frac{D}{2}$  even).

1.1. Subcase  $a$  even. Hence  $A < 0$  and  $B < 0$ .

1.1.1. Subcase  $\frac{D}{2}$  even. Thus  $A$  is even and  $B$  is odd. Let

$$R = \rho^{D+1}(\overleftarrow{P_B})u_D u_{-1} P_B P_A u_A u_{A+D+1}.$$

The subscripts of the endpoints of  $R$  are  $B + D + 1$  (even) and  $A + D + 1$  (odd). The orbits of  $\langle \rho^2 \rangle$  containing the linking edges are:

$$\begin{aligned} \langle \rho^2 \rangle(u_D u_{-1}) &= \langle \rho^2 \rangle(u_0 u_{D-1}) & \text{and} \\ \langle \rho^2 \rangle(u_A u_{A+D+1}) &= \langle \rho^2 \rangle(u_1 u_D). \end{aligned}$$

In the remaining cases we leave it to the reader to examine the endpoints of the path  $R$  and the orbits of the linking edges.

1.1.2. Subcase  $\frac{D}{2}$  odd. Thus  $A$  is odd and  $B$  is even. Let

$$R = \rho^{D+1}(\overleftarrow{P_{B_1}})u_{D+1} u_0 P_{B_1} P_A u_A u_{A+D+1} u_{A+D} u_{A+D-1}.$$

1.2. Subcase  $a$  odd. Hence  $A > 0$  and  $B > 0$ .

1.2.1. Subcase  $\frac{D}{2}$  even. Thus  $A$  is even and  $B$  is odd. Let

$$R = \rho^{D-1}(\overleftarrow{P_B})u_{D-2}u_{-1}P_B P_A u_A u_{A+D-1}.$$

1.2.2. Subcase  $\frac{D}{2}$  odd. Thus  $A$  is odd and  $B$  is even. Let

$$R = \rho^{D-1}(\overleftarrow{P_{B_1}})u_{D-1}u_0 P_{B_1} P_A u_A u_{A+D-1} u_{A+D} u_{A+D+1}.$$

2. Case  $A - B$  even. Since the subscripts of the endpoints of  $P_A$  are both even or both odd in this case, we replace the edge of length  $D - 2$  in  $P_A$  by an edge of length  $D - 1$ . That is, let  $L = L_B \cup \{D - 2\}$  and let  $P_D = P_{A_2} u_{A_2} u_{A_2+(-1)^{M-1}(D-1)}$  so that  $L_D = (L_A - \{D - 2\}) \cup \{D - 1\}$ . Conditions 2 - 4 of Lemma 3.3.6 are thus satisfied. Again, we construct the path  $R$  so that Conditions 5 - 7 are met, and Condition 8 follows automatically. The reader is asked to check the subscripts of the endpoints of  $R$ , the orbits of the linking edges (in this case, these are the edges of length  $D - 2$  and 1), and the orbit of the “misplaced” edge of  $P_D$  of length  $D - 1$ .

2.1. Subcase  $a$  even. Hence  $A < 0$ ,  $B < 0$ , and  $A_2 = A + (D - 2) > 0$ .

2.1.1. Subcase  $\frac{D}{2}$  even. Thus  $A$ ,  $A_2$ , and  $B$  are all even. Let

$$R = \rho^{D+1}(\overleftarrow{P_{B_1}})u_{D+1}u_{-1}u_0 P_{B_1} P_{A_2} u_{A_2-(D-2)} u_{A_2+3} u_{A_2+2}.$$

2.1.2. Subcase  $\frac{D}{2}$  odd. Thus  $A$ ,  $A_2$ , and  $B$  are all odd. Let

$$R = \rho^{D-1}(\overleftarrow{P_{B_1}})u_{D-1}u_{D-2}u_0 P_{B_1} P_{A_2} u_{A_2-(D-2)} u_{A_2+3} u_{A_2+2}.$$

Observe that since  $\frac{m}{2} - 1 \geq 6$ , we have  $B + D - 1 > A + D + 1$ . Hence the vertices of  $R$  do not overlap and the path is well defined.

2.2. Subcase  $a$  odd. Thus  $A > 0$ ,  $B > 0$ , and  $A_2 = A - (D - 2) < 0$ .

2.2.1. Subcase  $\frac{D}{2}$  even. Thus  $A$ ,  $A_2$ , and  $B$  are all even. Let

$$R = \rho^{D-3}(\overleftarrow{P_{B_1}})u_{D-3}u_{-1}u_0 P_{B_1} P_{A_2} u_{A_2+(D-2)} u_{A_2-3} u_{A_2-4}.$$



2.2.2. Subcase  $\frac{D}{2}$  odd. Hence  $A$ ,  $A_2$ , and  $B$  are all odd. Let

$$R = \rho^{D-1}(\overleftarrow{P_{B_1}})u_{D-1}u_{D-2}u_0P_{B_1}P_{A_2}u_{A_2}u_{A_2+(D-2)}u_{A_2-3}u_{A_2-4}.$$

Since this covers all cases, it now follows from Lemma 3.3.6 that  $X(n-1; L_C) \boxtimes K_1$  is  $C_m$ -decomposable.  $\square$

We continue with the case  $\frac{d}{2}$  even. Since  $c = \frac{dr'+1}{m'} \geq \frac{d}{2}$  and  $d \geq 4$ , we have  $m' \geq 5$  and  $r' \geq 3$ . Since  $c - \frac{d}{2}$  is now odd, two types of peripheral cycles are needed: coupled peripheral cycles, which will be similar to those of the case  $\frac{d}{2}$  odd, and solitary peripheral cycles. They are constructed as follows.

**Lemma 3.3.9** *Let  $\frac{d}{2}$  be even. Define the zig-zag  $m'$ -path  $P_{0,0}$  by*

$$P_{0,0} = u_0 u_1 u_{-1} u_2 u_{-2} \dots u_{\frac{m'-1}{2}} u_{-\frac{m'-1}{2}} u_{(\frac{d}{2}-1)n'}.$$

*There exists a set  $L_{CP}$  with the following properties:*

1.  $L_{CP} \subseteq \{n'+1, \dots, D-n'\}$ ,
2.  $|L_{CP}| = \frac{1}{2}(c - \frac{d}{2} - 1)m'$ , and
3.  $X(n-1; L_{CP}) \oplus \langle \rho^2 \rangle(P_{0,0})$  is  $C_m$ -decomposable.

PROOF. Let  $a = \frac{1}{2}(c - \frac{d}{2} - 1)$  and observe that  $a$  is an integer. First we construct the solitary  $m$ -cycles. The edge length set of  $P_{0,0}$  is

$$L_0 = \{1, 2, \dots, m' - 1, \frac{m'-1}{2} + (\frac{d}{2} - 1)n'\}.$$

Let  $P_{0,j} = \rho^{jn'}(P_{0,0})$  for  $j = 1, \dots, d-1$ . Since  $\gcd(d, \frac{d}{2} - 1) = 1$ ,  $C_0 = \bigoplus_{j=0}^{d-1} P_{0,j}$  is an  $m$ -cycle and  $\{\rho^{2i}(C_0) : i = 0, \dots, \frac{n'}{2} - 1\}$  is a  $C_m$ -decomposition of  $X(n-1; \emptyset) \oplus \langle \rho^2 \rangle(P_{0,0})$ .

For coupled peripheral cycles we again use the construction of Lemma 3.3.3 for  $m' \geq 5$  with minor modifications. First of all, we replace  $a = \frac{c-1}{2}$  of Lemma 3.3.3 by  $a = \frac{1}{2}(c - \frac{d}{2} - 1)$ . Observe that since  $c \leq d-1$ , we have  $a = \frac{1}{2}(c - \frac{d}{2} - 1) \leq \frac{d}{4} - 1$ , which is the same upper bound for  $a$  as in Lemma 3.3.3. We may now closely follow the construction of Lemma 3.3.3 for  $m' \geq 5$ , discarding the cases with  $\frac{d}{2}$  odd.

The result is a set  $L_{CP}$  satisfying Conditions 1 and 2 of this lemma and with the property that  $X(n-1; L_{CP})$  is  $C_m$ -decomposable. Since  $L_0$  and  $L_{CP}$  are disjoint and  $X(n-1; \emptyset) \oplus \langle \rho^2 \rangle (P_{0,0})$  is  $C_m$ -decomposable, Condition 3 is satisfied as well.  $\square$

And lastly, the details of the construction of central cycles in the case  $\frac{d}{2}$  even. This construction is an extension of the one in the case  $\frac{d}{2}$  odd (Lemma 3.3.8), further complicated by the need to accommodate the leftover orbit of  $\langle \rho^2 \rangle$  that contains the path  $\rho^{-1}(P_{0,0})$ .

**Lemma 3.3.10** *Let  $\frac{d}{2}$  be even. Let the zig-zag  $m'$ -path  $P_{0,0}$  with edge length set  $L_0$  be defined as in Lemma 3.3.9, and let  $L_{CP}$  be a set satisfying the conditions of Lemma 3.3.9. Let  $L_C = \{1, \dots, D\} - (L_0 \cup L_{CP})$ . Then  $(X(n-1; L_C) \oplus \langle \rho^2 \rangle (\rho^{-1}(P_{0,0}))) \rtimes K_1$  is  $C_m$ -decomposable.*

PROOF. Since the largest member of  $L_0 \cup L_{CP}$  is

$$\frac{m'-1}{2} + (\frac{d}{2} - 1)n' \leq \frac{n'-4}{2} + (\frac{d}{2} - 1)n' < D - 2,$$

$\{D-2, D-1, D\} \subseteq L_C$ . Let  $L_C$  be the disjoint union of sets  $L_B$ ,  $L_A$ , and  $\{D-1, D\}$  with the properties that  $\max(L_B) < \min(L_A)$  and  $|L_A| = \frac{m}{2} - 1$ . Thus

$$\begin{aligned} |L_B| &= |L_C| - (|L_A| + 2) = \frac{dn'}{2} - \frac{1}{2}(c - \frac{d}{2} + 1)m' - (\frac{m}{2} + 1) \\ &= \frac{1}{2}(n-1-r + \frac{m}{2} - m' - m - 2) = \frac{1}{4}(m - 2m' - 6) \geq \frac{1}{4}(\frac{m}{2} - 6) \geq 1 \end{aligned} \quad (13)$$

since  $d \geq 4$  and  $m' \geq 5$ . Hence the sets  $L_B$  and  $L_A$  are well defined.

The notation we shall use will be similar to that of Lemma 3.3.8:

$$\begin{aligned} a &= \frac{1}{2}(c - \frac{d}{2} - 1), \\ Z &= \frac{m'-1}{2} + (\frac{d}{2} - 1)n', \\ N &= |L_B|, \\ M &= |L_B| + |L_A| = N + \frac{m}{2} - 1, \\ L_B &= \{a_1, a_2, \dots, a_N\}, \text{ where } m' = a_1 < a_2 < \dots < a_N, \\ L_A &= \{a_{N+1}, \dots, a_{M-1}, D-2\}, \text{ where } a_N < a_{N+1} < \dots < a_{M-1} < D-2, \\ B &= -\frac{m'+1}{2} + a_1 - a_2 + a_3 - \dots + (-1)^{N-1}a_N, \end{aligned}$$

$$\begin{aligned}
A_2 &= B + (-1)^N a_{N+1} + (-1)^{N+1} a_{N+2} + \cdots + (-1)^{M-2} a_{M-1}, & \text{and} \\
A &= A_2 + (-1)^{M-1} (D - 2).
\end{aligned}$$

Define the following zig-zag paths:

$$\begin{aligned}
P_B &= u_{-\frac{m'+1}{2}} u_{-\frac{m'+1}{2}+a_1} u_{-\frac{m'+1}{2}+a_1-a_2} \cdots u_B, \\
P_{A_2} &= u_B u_{B+(-1)^N a_{N+1}} u_{B+(-1)^N a_{N+1}+(-1)^{N+1} a_{N+2}} \cdots u_{A_2}, \\
P_A &= P_{A_2} u_{A_2} u_A, \\
Q_6 &= u_2 u_{-4} u_3 u_{-5} \cdots u_{\frac{m'-3}{2}} u_{-\frac{m'+1}{2}}, \\
Q_5 &= u_{-3} u_2 Q_6, \\
Q_4 &= u_1 u_{-3} Q_5, \\
Q_3 &= u_{-2} u_1 Q_4, \\
Q_2 &= u_0 u_{-2} Q_3, & \text{and} \\
Q_1 &= u_{-1} u_0 Q_2 & \text{so that} \\
\rho^{-1}(P_{0,0}) &= Q_1 u_{-\frac{m'+1}{2}} u_{-\frac{m'+1}{2}+Z}.
\end{aligned}$$

We proceed to define the paths  $P$ ,  $P_D$ , and  $R$  that satisfy the conditions of Lemma 3.3.6. The details of the construction depend on the residue class of  $m'$  modulo 4, and on whether  $A - B$  and  $a$  are odd or even. In all cases, however, we let  $P = \rho^{-1}(P_{0,0})$ .

Observe that since  $\frac{d}{2}$  is even,  $\frac{D}{2}$  is even and  $M - N = \frac{m}{2} - 1$  is odd. The latter implies that  $A$  and  $B$  are opposite in sign. Hence  $A_2$  and  $B$  have the same sign and

$$|A_2 - B| \geq \frac{1}{2} \left( \frac{m}{2} - 1 - 1 \right) = \frac{m}{4} - 1. \quad (14)$$

The second inequality that will prove useful in showing that  $R$  is indeed a path is obtained as follows. Since  $n \leq 2m - 1$  and  $n'$  is even,  $n' \leq 2m' - 2$ . For  $d \geq 8$  we thus have

$$\frac{m}{4} - n' \geq \frac{d}{4} m' - (2m' - 2) = \left( \frac{d}{4} - 2 \right) m' + 2 \geq 2. \quad (15)$$

Since  $\frac{d}{2}$  is even,  $D - 2 \equiv 2 \pmod{4}$ . Hence the number of odd elements of  $\{1, \dots, D - 2\}$  is odd. Since  $n'$  is even, every family of peripheral cycles, including solitary peripheral cycles, uses an even number of odd edge lengths. Hence  $L_C -$

$\{D - 1, D\}$  contains an odd number of odd edge lengths. This implies that the alternating sum  $A$  is even if  $m' \equiv 1 \pmod{4}$ , and odd if  $m' \equiv 3 \pmod{4}$ . Since  $M = |L_C - \{D - 1, D\}| = D - 2 - (a + 1)m'$ ,  $A$  is positive (evaluated in the integers) if and only if  $a$  is even.

Assume now that  $d = 4$ . We have  $c = 3$  and hence  $a = 0$ . Since  $3m' = 4r' + 1$ , we must have  $m' \equiv 3 \pmod{4}$ . Evaluate the alternating sum  $A$  for this case:

$$\begin{aligned} A &= -\frac{m'+1}{2} + m' - (m' + 1) + \cdots - \left(\frac{m'-3}{2} + n'\right) + \left(\frac{m'+1}{2} + n'\right) - \cdots + (n' - 2 + n') \\ &= n' - 1. \end{aligned}$$

Hence  $A$  is odd and positive and, consequently,  $B$  is negative. Since  $m' \equiv 3 \pmod{4}$ , we have

$$N = |L_B| = 2n' - 2 - m' - \left(\frac{m}{2} - 1\right) = 2n' - 3m' - 1 \equiv 2 \pmod{4}.$$

Since  $\frac{m}{2} - 1 = 2m' - 1 > n'$ ,  $L_B$  consists of  $N$  consecutive integers starting with  $m'$ . This implies that  $a_N = 2n' - 2m' - 2$ . On the other hand, since  $m'$  is odd and  $N \equiv 2 \pmod{4}$ ,  $L_B$  therefore contains an odd number of odd edge lengths. From the definition of  $B$  we can now see that  $B$  is odd. The construction for  $d = 4$  will thus be carried out in Subcase 2.2.1. For the other seven subcases we may assume that  $d \geq 8$ .

1. Case  $A - B$  odd. Let  $L = L_B \cup \{D - 1\}$  and  $P_D = P_A$  so that  $L_D = L_A$ . Notice that Conditions 2 - 4 of Lemma 3.3.6 are satisfied for the path  $P_D$ . We define a path  $R$  to meet Conditions 5 - 7. Since  $|L| = \frac{1}{4}(m - 2m' - 2)$  by (13), Conditions 5 - 7 imply

$$|E(R)| = 2|L| + |L_0| + |L_A| = \frac{1}{2}(m - 2m' - 2) + m' + \frac{m}{2} - 1 = m - 2,$$

so that Condition 8 is satisfied as well. In each subcase we have to verify that the endpoints of  $R$  satisfy Condition 9 and that the vertices of  $R$  do not overlap. Note that except for the alternating sums  $A$ ,  $A_2$ , and  $B$ , which are evaluated in the integers, all calculations involving subscripts of the vertices are carried out modulo  $2D$ . We should also check to see that the linking edges (that is, the edges of lengths 1,  $D - 1$ ,  $Z$ , and possibly others) belong to the appropriate orbits of  $\langle \rho^2 \rangle$ .

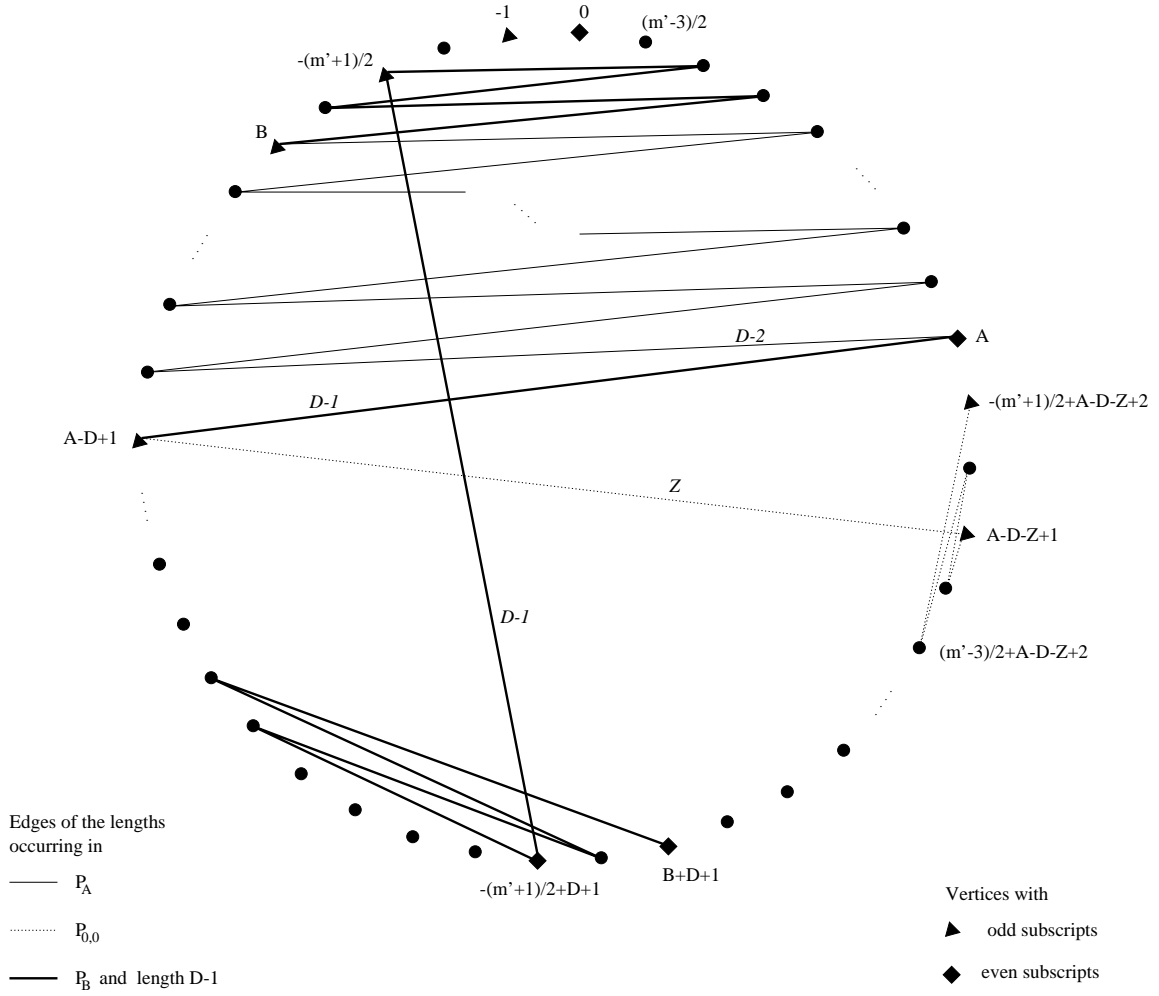


Figure 8: Lemma 3.3.10: the path  $R$  for Case 1.1.1 ( $A - B$  odd,  $m' \equiv 1 \pmod{4}$ , and  $a$  even).

1.1. Subcase  $m' \equiv 1 \pmod{4}$ . Hence  $Z$  is even.

1.1.1. Subcase  $a$  even. Thus  $A > 0$  is even and  $B < 0$  is odd. Let

$$R = \rho^{D+1}(\overleftarrow{P_B})u_{-\frac{m'+1}{2}+D+1}u_{-\frac{m'+1}{2}}P_B P_A u_A u_{A-D+1}u_{A-D-Z+1}\rho^{A-D-Z+2}(Q_1).$$

Note that the second to last vertex in  $Q_1$  is  $u_{\frac{m'-3}{2}}$  and that by (14),  $B - (A - D + 1) = B - (A_2 - 1) \geq \frac{m}{4}$ . Now, since  $r' \geq 3$ ,

$$\begin{aligned} \left(-\frac{m'+1}{2} + A - D - Z + 2\right) - A &= -\frac{m'+1}{2} - D + 2 - \left(D - n' + \frac{m'-1}{2}\right) \\ &= n' - m' + 2 \geq 5, \end{aligned}$$

and by inequality (15),

$$\begin{aligned}
(B + D + 1) - \left(\frac{m'-3}{2} + A - D - Z + 2\right) &= (B - (A - D + 1)) \\
&\quad - \frac{m'-3}{2} + D + \left(D - n' + \frac{m'-1}{2}\right) \\
&\geq \frac{m}{4} - n' + 1 \geq 3.
\end{aligned}$$

Hence the vertices of  $R$  do not overlap and the path is well defined.

The subscripts of the endpoints of  $R$  are  $B+D+1$  (even) and  $-\frac{m'+1}{2}+A-D-Z+2$  (odd). The orbits of  $\langle \rho^2 \rangle$  containing the linking edges are:

$$\begin{aligned}
\langle \rho^2 \rangle(u_{-\frac{m'+1}{2}+D+1}u_{-\frac{m'+1}{2}}) &= \langle \rho^2 \rangle(u_0u_{D-1}), \\
\langle \rho^2 \rangle(u_Au_{A-D+1}) &= \langle \rho^2 \rangle(u_{-1}u_{D-2}), \quad \text{and} \\
\langle \rho^2 \rangle(u_{A-D+1}u_{A-D-Z+1}) &= \langle \rho^2 \rangle(u_{-\frac{m'+1}{2}}u_{-\frac{m'+1}{2}+Z}).
\end{aligned}$$

In the remaining subcases, most of the verification is left to the reader.

1.1.2. Subcase  $a$  odd. Thus  $A < 0$  is even and  $B > 0$  is odd. Let

$$R = \rho^{D-1}(\overleftarrow{P}_B)u_{-\frac{m'+1}{2}+D-1}u_{-\frac{m'+1}{2}}P_BP_Au_Au_{A+D-1}u_{A+D+Z-1}\rho^{A+D+Z}(Q_1).$$

1.2. Subcase  $m' \equiv 3 \pmod{4}$ . Hence  $Z$  is odd.

1.2.1. Subcase  $a$  even. Thus  $A > 0$  is odd and  $B < 0$  is even. Let

$$R = \rho^{D+1}(\overleftarrow{P}_B)u_{-\frac{m'+1}{2}+D+1}u_{-\frac{m'+1}{2}}P_BP_Au_Au_{A-D+1}u_{A-D}u_{A-D-Z}\rho^{A-D-Z}(Q_2).$$

1.2.2. Subcase  $a$  odd. Thus  $A < 0$  is odd and  $B > 0$  is even. Let

$$R = \rho^{D-1}(\overleftarrow{P}_B)u_{-\frac{m'+1}{2}+D-1}u_{-\frac{m'+1}{2}}P_BP_Au_Au_{A+D-1}u_{A+D+Z-1}\rho^{A+D+Z}(Q_1).$$

2. Case  $A - B$  even. Since the subscripts of the endpoints of  $P_A$  are both even or both odd in this case, to obtain the path  $P_D$ , we replace the edge of length  $D - 2$  in  $P_A$  by an edge of length  $D - 1$ . That is, let  $L = L_B \cup \{D - 2\}$  and let  $P_D = P_{A_2}u_{A_2}u_{A_2+(-1)^{M-1}(D-1)}$  so that  $L_D = (L_A - \{D - 2\}) \cup \{D - 1\}$ .

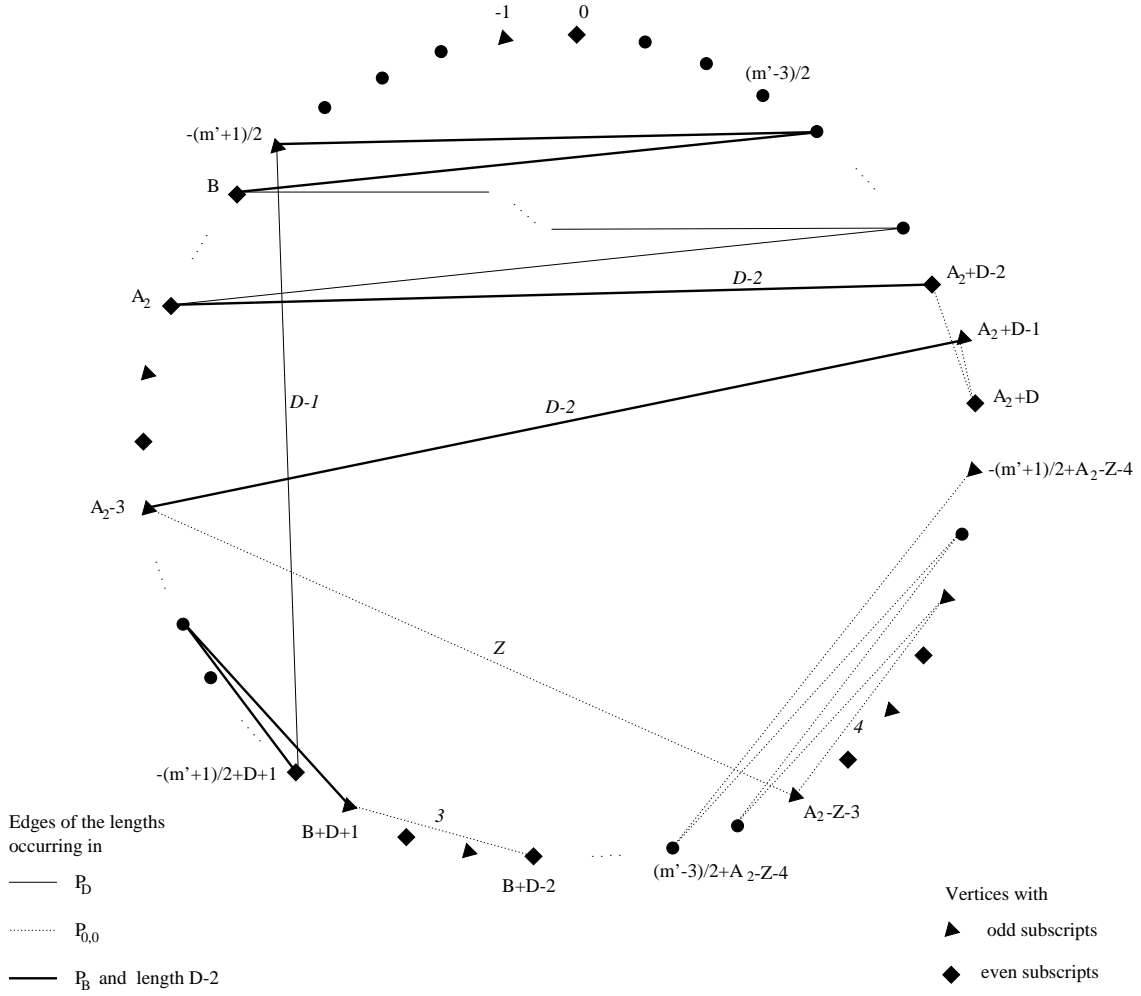


Figure 9: Lemma 3.3.10: the path  $R$  for Case 2.1.1 ( $A - B$  even,  $m' \equiv 1 \pmod{4}$ ,  $a$  even),  $r' \geq 5$ .

2.1. Subcase  $m' \equiv 1 \pmod{4}$ . Hence  $Z$  is even.

2.1.1. Subcase  $a$  even. Thus  $A > 0$ ,  $A_2 < 0$ , and  $B < 0$  are all even. First we assume that  $r' \geq 5$ . Let

$$R = u_{B+D-2} u_{B+D+1} \rho^{D+1} (\overleftarrow{P_B}) u_{-\frac{m'+1}{2}+D+1} u_{-\frac{m'+1}{2}} P_B P_{A_2} u_{A_2} u_{A_2+D-2} u_{A_2+D} u_{A_2+D-1} u_{A_2-3} u_{A_2-Z-3} \rho^{A_2-Z-4} (Q_4).$$

We have

$$\begin{aligned} \left(-\frac{m'+1}{2} + A_2 - Z - 4\right) - (A_2 + D) &= -\frac{m'+1}{2} - 4 - \left(D - n' + \frac{m'-1}{2}\right) - D \\ &= n' - m' - 4 \geq 1 \end{aligned}$$

since  $r' \geq 5$ , and by (14) and (15),

$$\begin{aligned} (B + D - 2) - \left(\frac{m'-3}{2} + A_2 - Z - 4\right) &= (B - A_2) \\ &\quad + D - 2 - \frac{m'-3}{2} + 4 + (D - n' + \frac{m'-1}{2}) \\ &\geq \frac{m}{4} - 1 - n' + 3 \geq 4. \end{aligned}$$

Hence the vertices of  $R$  do not overlap and the path is well defined.

Now let  $r' = 3$ . Since  $r' \geq \frac{m'+1}{2}$ , we have  $m' = 5$  and hence  $n' = 8$ . Now let

$$\begin{aligned} R &= \rho^{D+1}(\overleftarrow{P}_B)u_{-\frac{m'+1}{2}+D+1}u_{-\frac{m'+1}{2}}P_B P_{A_2} \\ &\quad u_{A_2}u_{A_2+D-2}u_{A_2+D}u_{A_2+D-1}u_{A_2-3}u_{A_2-Z-3}\rho^{A_2-Z}(\overleftarrow{Q}_3). \end{aligned}$$

2.1.2. Subcase  $a$  odd. Thus  $A < 0$ ,  $A_2 > 0$ , and  $B > 0$  are all even. Let

$$\begin{aligned} R &= \rho^{D-3}(\overleftarrow{P}_B)u_{-\frac{m'+1}{2}+D-3}u_{-\frac{m'+1}{2}+D-1}u_{-\frac{m'+1}{2}}P_B P_{A_2}u_{A_2}u_{A_2-D+2} \\ &\quad u_{A_2-D+1}u_{A_2+3}u_{A_2+Z+3}\rho^{A_2+Z+2}(Q_4)u_{-\frac{m'+1}{2}+A_2+Z+2}u_{-\frac{m'+1}{2}+A_2+Z-1}. \end{aligned}$$

2.2. Subcase  $m' \equiv 3 \pmod{4}$ . Hence  $Z$  is odd.

2.2.1. Subcase  $a$  even. Thus  $A > 0$ ,  $A_2 < 0$ , and  $B < 0$  are all odd.

If  $d \geq 8$  and  $r' \geq 7$ , let

$$\begin{aligned} R &= u_{B+D-4}u_{B+D+1}\rho^{D+1}(\overleftarrow{P}_B)u_{-\frac{m'+1}{2}+D+1}u_{-\frac{m'+1}{2}}P_B P_{A_2} \\ &\quad u_{A_2}u_{A_2+D-2}u_{A_2+D+2}u_{A_2+D-1}u_{A_2+D+1}u_{A_2-1}u_{A_2-2}u_{A_2-Z-2}\rho^{A_2-Z-4}(Q_6). \end{aligned}$$

If  $d \geq 8$  and  $r' \leq 5$ , notice that the only possibility is  $r' = 5$ ,  $m' = 7$ , and hence  $n' = 12$ . In this case, we take

$$\begin{aligned} R &= \rho^{D+1}(\overleftarrow{P}_B)u_{-\frac{m'+1}{2}+D+1}u_{-\frac{m'+1}{2}}P_B P_{A_2} \\ &\quad u_{A_2}u_{A_2+D-2}u_{A_2+D+2}u_{A_2+D-1}u_{A_2+D+1}u_{A_2-1}u_{A_2-2}u_{A_2-Z-2}\rho^{A_2-Z+2}(\overleftarrow{Q}_5). \end{aligned}$$

If  $d = 4$ , let

$$R = \rho^{D-B-2}(P_B)u_{D-2}u_0Q_2P_B P_{A_2}u_{A_2}u_{A_2+D-2}u_{A_2-3}u_{A_2-4}u_{A_2-Z-4}.$$



In this case, recall that  $A = n' - 1$  so that  $A_2 = A - (D - 2) = n' - 1 - (2n' - 2) = 3n' + 1$ . Now  $a_N = 2n' - 2m' - 2 \leq n' - 4$  since  $2m' \geq n' + 2$ . On the other hand,

$$(A_2 - 4) - (D - 2) = (3n' - 3) - (2n' - 2) = n' - 1 > a_N.$$

Since, in addition,

$$\begin{aligned} (D - 2) - (A_2 - Z - 4) &= (2n' - 2) - (3n' + 1 - 4 - (n' + \frac{m'-1}{2})) \\ &= \frac{m'-1}{2} + 1 \geq 4, \end{aligned}$$

the vertices of  $R$  do not overlap and the path is well defined.

2.2.2. Subcase  $a$  odd. Thus  $A < 0$ ,  $A_2 > 0$ , and  $B > 0$  are all odd. Let

$$\begin{aligned} R &= \rho^{D-1}(\overleftarrow{P}_B)u_{-\frac{m'+1}{2}+D-1}u_{-\frac{m'+1}{2}}P_B P_{A_2}u_{A_2}u_{A_2-D+2}u_{A_2-D-1}u_{A_2+1} \\ &\quad u_{A_2+3}u_{A_2+Z+3}\rho^{A_2+Z+2}(Q_4)u_{-\frac{m'+1}{2}+A_2+Z+2}u_{-\frac{m'+1}{2}+A_2+Z+1}. \end{aligned}$$

We have shown that Conditions 1 – 9 of Lemma 3.3.6 are satisfied for the paths  $P$ ,  $P_D$ , and  $R$  in all cases. Therefore  $(X(n-1; L_C) \oplus \langle \rho^2 \rangle (\rho^{-1}(P_{0,0})) \rtimes K_1$  is  $C_m$ -decomposable.  $\square$

### 3.3.3 Conclusion

Finally, the many pieces of the construction for the case  $m \leq n < 2m$  are put together.

**Lemma 3.3.11** *Let  $n$  be an odd integer and  $m$  be an even integer such that  $3 \leq m \leq n < 2m$  and  $n(n-1) \equiv 0 \pmod{2m}$ . Then  $K_n$  is  $C_m$ -decomposable.*

PROOF. Define the parameters  $r$ ,  $d$ ,  $m'$ ,  $n'$ ,  $r'$ ,  $c$ , and  $D$  as on page 36. Observe that  $K_n$  is isomorphic to  $X(n-1; L) \rtimes K_1$ , where  $L = \{1, \dots, D\}$ .

First assume that  $c < \frac{d}{2}$ .

If  $m' \equiv 3 \pmod{8}$  and  $n' \equiv 2 \pmod{4}$ , or  $m' \equiv 7 \pmod{8}$  and  $n' \equiv 0 \pmod{4}$ , or  $m' \equiv 1 \pmod{8}$  and  $n' \equiv 2 \pmod{4}$  and  $m' \geq 17$ , or  $m' \equiv 5 \pmod{8}$  and

$n' \equiv 0 \pmod{4}$ , or  $m' = 13$  and  $n' = 14$ , or  $m' = 7$  and  $n' = 10$ , or  $m' = 5$ , or  $m' = 3$ , let  $L_P$  be an edge length set satisfying the conditions of Lemma 3.3.4 so that  $X(n-1; L_P) \oplus \langle \rho^2 \rangle(u_1u_3)$  is  $C_m$ -decomposable. The set  $L_C = L - (L_P \cup \{2\})$  then satisfies the conditions of Lemma 3.3.1 so that  $(X(n-1; L_C) \oplus \langle \rho^2 \rangle(u_0u_2)) \rtimes K_1$  is  $C_m$ -decomposable as well. Hence  $X(n-1; L) \rtimes K_1$  is  $C_m$ -decomposable.

If  $m' \equiv 3 \pmod{8}$  and  $n' \equiv 0 \pmod{4}$  and  $m' \geq 11$ , or  $m' \equiv 7 \pmod{8}$  and  $n' \equiv 2 \pmod{4}$  and  $m' \geq 15$ , or  $m' \equiv 1 \pmod{8}$  and  $n' \equiv 0 \pmod{4}$  and  $m' \geq 17$ , or  $m' \equiv 5 \pmod{8}$  and  $n' \equiv 2 \pmod{4}$  and  $m' \geq 21$ , or  $m' = 13$  and  $n' = 18$ , or  $m' = 9$ , let  $L_P$  be an edge length set satisfying the conditions of Lemma 3.3.5 so that  $X(n-1; L_P) \oplus \langle \rho^2 \rangle(\{u_0u_2, u_0u_3, u_0u_5\})$  is  $C_m$ -decomposable. The set  $L_C = L - (L_P \cup \{2, 3, 5\})$  then satisfies the conditions of Lemma 3.3.2 so that  $(X(n-1; L_C) \oplus \langle \rho^2 \rangle(\{u_1u_3, u_1u_4, u_1u_6\})) \rtimes K_1$  is  $C_m$ -decomposable. Hence  $X(n-1; L) \rtimes K_1$  is  $C_m$ -decomposable.

Now let us look at the case  $c \geq \frac{d}{2}$ .

If  $\frac{d}{2}$  is odd, let  $L_{CP}$  be a set satisfying the conditions of Lemma 3.3.7 and let  $L_C = L - L_{CP}$ . Since  $X(n-1; L_{CP})$  is  $C_m$ -decomposable by Lemma 3.3.7 and  $X(n-1; L_C) \rtimes K_1$  is  $C_m$ -decomposable by Lemma 3.3.8,  $X(n-1; L) \rtimes K_1$  is  $C_m$ -decomposable.

For  $\frac{d}{2}$  even, let the zig-zag  $m'$ -path  $P_{0,0}$  with edge length set  $L_0$  be defined as in Lemma 3.3.9, let  $L_{CP}$  be a set satisfying the conditions of Lemma 3.3.9, and let  $L_C = L - (L_0 \cup L_{CP})$ . Then  $X(n-1; L) \rtimes K_1$  can be partitioned into  $X(n-1; L_{CP}) \oplus \langle \rho^2 \rangle(P_{0,0})$ , which is  $C_m$ -decomposable by Lemma 3.3.9, and  $(X(n-1; L_C) \oplus \langle \rho^2 \rangle(\rho^{-1}(P_{0,0}))) \rtimes K_1$ , which is  $C_m$ -decomposable by Lemma 3.3.10.

This covers all cases. Therefore  $K_n$  is  $C_m$ -decomposable.  $\square$

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