UNIVERSITY OF LJUBLJANA
INSTITUTE OF MATHEMATICS, PHYSICS AND MECHANICS
DEPARTMENT OF MATHEMATICS
JADRANSKA 19, 1000 LJUBLJANA, SLOVENIA

Preprint series, Vol. 39 (2001), 753

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S. A. Abramov M. Petkovšek

ISSN 1318-4865

April 11, 2001

Ljubljana, April 11, 2001

Minimal Decomposition of Indefinite Hypergeometric Sums

S. A. Abramov [*]
Computer Center of
the Russian Academy of Science,
Vavilova 40, Moscow 117967, Russia
abramov@ccas.ru

M. Petkovšek [†]
Department of Mathematics and Mechanics,
Faculty of Mathematics and Physics,
University of Ljubljana,
Jadranska 19, 1000 Ljubljana, Slovenia
marko.petkovsek@uni-lj.si

Abstract

We present an algorithm which, given a hypergeometric term $T(n)$, constructs hypergeometric terms $T_1(n)$ and $T_2(n)$ such that $T(n) = T_1(n+1) - T_1(n) + T_2(n)$, and $T_2(n)$ is minimal in some sense. This solves the decomposition problem for indefinite sums of hypergeometric terms: $T_1(n+1) - T_1(n)$ is the "summable part" and $T_2(n)$ the "non-summable part" of $T(n)$.

1 Introduction

A sequence $T(n)$ is a *hypergeometric term* (or simply a *term*) if the ratio $T(n+1)/T(n)$ is a rational function of n. We call this function the *certificate* of T.

The well-known Gosper's algorithm [4] solves the *problem of indefinite hypergeometric summation*: Given a hypergeometric term $T(n)$, find another hypergeometric term $T_1(n)$ such that

$$T(n) = T_1(n+1) - T_1(n), \tag{1}$$

provided that such a term exists. If it does, and if $T(k)$ and $T_1(k)$ are defined for $k = n_0, n_0+1, \ldots, n$, then we obtain from (1) the summation identity

$$\sum_{k=n_0}^{n} T(k) = T(n) + T_1(n) - T_1(n_0).$$

If no hypergeometric term $T_1(n)$ satisfies (1), we can ask for *two* hypergeometric terms $T_1(n)$ and $T_2(n)$ such that

$$T(n) = T_1(n+1) - T_1(n) + T_2(n) \tag{2}$$

and $T_2(n)$ is minimal in some sense. We call this the *decomposition problem for indefinite hypergeometric sums*. For example, when $T(n)$ is a rational function of n, the algorithms given in [1, 2, 7] yield rational functions $T_1(n)$ and $T_2(n)$ such that (2) holds and the denominator of T_2 is of the least possible degree. These approaches are unified in [5].

Throughout the paper K is a field of characteristic zero, n an indeterminate, and E the shift operator on sequences with elements in K defined by $ET(n) = T(n+1)$. We write $p \perp q$ to indicate that polynomials $p, q \in K[n]$ are relatively prime. The *leading coefficient* of a rational function is the quotient of the leading coefficients of its numerator and denominator. A rational function is *monic* if its leading coefficient is 1.

Following [5] we introduce the notion of shift-equivalence among polynomials.

[*]Partially supported by Natural Sciences and Engineering Research Council of Canada under grant GRD215442-98, and by the French-Russian Lyapunov Institute, Project No 98-03.

[†]Partially supported by MZT RS under grant J2-8549.

Definition 1 *Polynomials $p, q \in K[n]$ are* shift-equivalent *if $p = E^k q$ for some $k \in \mathbb{Z}$. In this case we write $p \overset{\text{sh}}{\sim} q$. A monic rational function is* shift-homogeneous *if all nonconstant irreducible monic factors of its numerator and denominator belong to the same shift-equivalence class.*

By grouping together shift-equivalent irreducible monic factors of its numerator and denominator every rational function can be written in the form

$$R(n) = z\, R_1(n) R_2(n) \cdots R_k(n) \tag{3}$$

where $z \in K$, $k \geq 0$, each R_i is a shift-homogeneous rational function, and any two nonconstant monic irreducible factors p of R_i and q of R_j are pairwise shift-inequivalent whenever $i \neq j$. We call (3) a *shift-homogeneous factorization* of R. It is clear that the shift-homogeneous factorization is unique up to the order of the factors.

Definition 2 *Let $r, s \in K[n]$. If $r \perp E^k s$ for all $k \in \mathbb{Z}$ then the rational function r/s is* shift-reduced*. If $r \perp E^k r$ for all $k \in \mathbb{Z} \setminus \{0\}$ then the polynomial r is* shift-free.

Now we define a normal form for rational functions which reveals the shift structure of its factors. We need this normal form to define a measure of complexity of a hypergeometric term.

Definition 3 *Let $R \in K(n) \setminus \{0\}$ be a nonzero rational function. If $z \in K$ and monic polynomials $r, s, u, v \in K[n]$ satisfy $R = z \cdot \frac{r}{s} \cdot \frac{EF}{F}$ where r/s is shift-reduced, $F = u/v$ and $u \perp v$, then (z, r, s, u, v) is a* rational normal form (RNF) *of R. The rational function zr/s is the* kernel *of this RNF.*

In [3] it is shown that every nonzero rational function has an RNF, and that regarding uniqueness we have the following results:

Lemma 1 *Let $R \in K(n)$ be shift-homogeneous. If $(1, r, s, u, v)$ and $(1, r_1, s_1, u_1, v_1)$ are two RNF's of R then $r = r_1 = 1$ and $\deg s = \deg s_1$, or $s = s_1 = 1$ and $\deg r = \deg r_1$.*

Theorem 1 *Let (z, r, s, u, v) and (z', r', s', u', v') be two RNF's of $R \in K(n) \setminus \{0\}$. Then*

(i) $z = z'$,

(ii) $\deg r = \deg r'$ *and* $\deg s = \deg s'$,

(iii) *there is a one-to-one correspondence f between the multisets of nonconstant irreducible monic factors of r and r' such that $p \overset{\text{sh}}{\sim} f(p)$ for all $p \,|\, r$,*

(iv) *there is a one-to-one correspondence g between the multisets of nonconstant irreducible monic factors of s and s' such that $q \overset{\text{sh}}{\sim} g(q)$ for all $q \,|\, s$.*

Now we can formulate the decomposition problem for indefinite hypergeometric sums in the following way:

Given a hypergeometric term T, find hypergeometric terms T_1, T_2 such that

$$T = (E - 1)T_1 + T_2 \tag{4}$$

and the certificate ET_2/T_2 has an RNF

$$(z, r, s, u, v) \tag{5}$$

with v of minimal degree.

This formulation agrees with the decomposition problem for indefinite sums of rational functions [1, 2, 7] because if $T_2 \in K(n)$ then $r = s = 1$ and v is the denominator of T_2.

Definition 4 *If (z, r_1, s_1, u_1, v_1) and (z, r_2, s_2, u_2, v_2) are two RNF's for the same rational function we will say that the former is* simpler *than the latter if $\deg v_1 < \deg v_2$.*

If the terms T, T_1, T_2 satisfy (4) then they are *similar*, i.e., there exist $R_1, R_2 \in K(n)$ such that $T_1 = R_1 T$ and $T_2 = R_2 T$ (cf. [6, Prop. 5.6.2]).

Definition 5 *A rational function $F \in K(n)$ is* adequate *for a hypergeometric term T if the certificate ET/T has an RNF with F as its kernel.*

Let T, T_1, T_2 satisfy (4). By the similarity of these terms each rational function that is adequate for one of them is adequate for the other two.

We will describe a solution of the decomposition problem in three stages:

(A) We describe an algorithm which, given a term T, constructs terms T_1, T_2 similar to T, and a rational function F adequate for T, T_1, T_2, such that if one restricts attention to RNF's with kernel F then T_1, T_2 solve the decomposition problem.

(B) We prove that for any other rational function F' adequate for T and for any term T_1' similar to T, the RNF (with kernel F') of ET_2'/T_2' where $T_2' = T - (E-1)T_1'$ is not simpler than the RNF of the certificate of T_2 constructed by the algorithm in (A).

(C) Finally we show how one can "improve" T_2 without changing $\deg v$ in such a way that $\deg u$ in (5) is not too large.

2 Part (A): An algorithm

Definition 6 *Let $n_0 \in \mathbb{Z}$ and $D, U \in K(x)$. The triple (D, U, n_0)* regularly describes *a term $T(n)$ if for all integer $n \geq n_0$*

- *$D(x), U(x)$ have neither a pole nor a zero at $x = n$,*
- *$T(n) = U(n) \prod_{k=n_0}^{n-1} D(k)$.*

Let (D, U, n_0) regularly describe a term T, $n_1 > n_0$, and

$$V(x) = U(x)\frac{T(n_1)}{U(n_1)} = U(x) \prod_{k=n_0}^{n_1-1} D(k).$$

Then it is evident that (D, V, n_1) regularly describes T.

Lemma 2 *Let the triples (D, U, n_0) and (D, U_1, n_0) regularly describe (similar) terms T and T_1. Then the certificate of the term $T_2 = T - (E-1)T_1$ is*

$$D\frac{EU_2}{U_2} \tag{6}$$

where

$$U_2 = U - D(EU_1) + U_1. \tag{7}$$

Proof: For all integer $n \geq n_0$ we have

$$\begin{aligned} T_2(n) &= U(n) \prod_{k=n_0}^{n-1} D(k) - (E-1)U_1(n) \prod_{k=n_0}^{n-1} D(k) \\ &= U(n) \prod_{k=n_0}^{n-1} D(k) - (EU_1(n)) \prod_{k=n_0}^{n} D(k) + U_1(n) \prod_{k=n_0}^{n-1} D(k) \\ &= (U(n) - (EU_1(n))D(n) + U_1(n)) \prod_{k=n_0}^{n-1} D(k). \end{aligned}$$

It follows that the values of ET_2/T_2 are equal to the values of (6) for all integer $n \geq n_0$. This proves the claim. □

Lemma 3 *Let $D, U \in K(n)$, $D = d_1/d_2$, $U = u_1/u_2$ where $d_1, d_2, u_1, u_2 \in K[n]$, and D is shift-reduced. Then there is $U_1 \in K(n)$ such that substituting it into the right-hand side of (7) one gets*

$$U_2 = \frac{v_1}{(E^{-1}d_1)^i d_2^j v_2}, \tag{8}$$

where $i, j \in \{0, 1\}$, $v_1, v_2 \in K[n]$, and for any irreducible $p \in K[n]$ dividing v_2 and any $h \in \mathbb{Z}$ the following conditions are satisfied:

$$E^h p \mid v_2 \Rightarrow h = 0, \tag{9}$$

$$E^h p \mid d_1 \Rightarrow h < 0, \tag{10}$$

$$E^h p \mid d_2 \Rightarrow h > 0 \tag{11}$$

(in words, (9) means that v_2 is shift-free).

Proof: Let q be an irreducible in $K[n]$ and $u_2 = u_2' q^k$ where $q \perp u_2'$ and $k > 0$. Then there are $a, b \in K[n]$ such that

$$U = \frac{a}{u_2'} + \frac{b}{q^k}. \tag{12}$$

We distinguish two cases.

a) There is an integer $h \leq 0$ such that $E^h q \mid d_2$. Let $U_1{}' = E^{-1}\left(b/(Dq^k)\right)$. Then $D(EU_1{}') = b/q^k$, so $U - D(EU_1{}') + U_1{}'$ can be written as

$$\frac{c_0}{u_2'} + \frac{c_1}{E^{-1}d_1} + \frac{c_2}{(E^{-1}q)^l}$$

where $l \leq k$, $c_0, c_1, c_2 \in K[n]$.

b) There is an integer $h \geq 0$ such that $E^h q \mid d_1$. Let $U_1{}' = -b/q^k$. Then $U - D(EU_1{}') + U_1{}'$ can be written as

$$\frac{c_0}{u_2'} + \frac{c_1}{d_2} + \frac{c_2}{(Eq)^l}$$

where $l \leq k$, $c_0, c_1, c_2 \in K[n]$.

Since D is shift-reduced, at most one of the cases a), b) can occur. Repeating these steps if necessary (using $U_1{}'', U_1{}''', \ldots$) we obtain a rational function $U - DE(U_1{}' + U_1{}'' + \cdots) + (U_1{}' + U_1{}'' + \cdots)$ whose denominator is divisible by at most one polynomial of the form $E^\gamma q$. If such a γ exists then for $p = E^\gamma q$ we have (9), (10), and (11). Similarly we can proceed with the irreducible factors of u_2 that are different from $E^\sigma q, \sigma \in \mathbb{Z}$.

Suppose that u_2 is not shift-free. If there is an integer $h > 0$ such that $E^h q | u_2$ for some irreducible q and if (12) holds, then we can transform U as it is described in a). Similarly we can use the way described in b) if we have $h < 0$. □

Lemma 4 *Let T be a term with the certificate $D\frac{EU}{U}$, where $D = d_1/d_2$ is a shift-reduced rational function and $U = u_1/u_2 \in K(n)$. Let U_1, U_2 be rational functions that exist by Lemma 3. Then there exists a term T_1, similar to T, such that the term $T_2 = T - (E-1)T_1$ has $D\frac{EU_2}{U_2}$ as its certificate.*

Proof: Let n_0 be an integer such that $D(n), U(n), U_1(n)$ have no pole or zero for integer $n \geq n_0$ and the value $T(n)$ is defined for $n \geq n_0$. Set $\alpha = T(n_0)/U(n_0)$. By (7) we have

$$\alpha U_2 = \alpha U - D(E(\alpha U_1)) + \alpha U_1.$$

The triple $(D, \alpha U, n_0)$ regularly describes T, the triple $(D, \alpha U_1, n_0)$ in turn regularly describes the term $T_1(n) = \alpha U_1(n) \prod_{k=n_0}^{n-1} D(k)$. Then by Lemma 2 the term $T_2 = T - (E-1)T_1$ has the certificate $D\frac{E\alpha U_2}{\alpha U_2} = D\frac{EU_2}{U_2}$. □

Using Lemmas 3 and 4 one obtains the terms T_1 and $T_2 = T - (E-1)T_1$ with certificates in the form $D\frac{EU_1}{U_1}$ and $D\frac{EU_2}{U_2}$, respectively. The certificate of T_2 can be rewritten in a simpler form: suppose that U_2 has the form (8), then we can remove the factors $(E^{-1}d_1)^i, d_2^j,\ i,j \in \{0,1\}$ from the denominator of U_2, since

$$D\frac{EU_2}{U_2} = \frac{d_1\left(\frac{E^{-1}d_1}{d_1}\right)^i}{d_2\left(\frac{Ed_2}{d_2}\right)^j}\frac{E\left(\frac{v_1}{v_2}\right)}{\frac{v_1}{v_2}}.$$

Setting

$$F = \frac{d_1\left(\frac{E^{-1}d_1}{d_1}\right)^i}{d_2\left(\frac{Ed_2}{d_2}\right)^j},\quad V = \frac{v_1}{v_2},$$

we get the certificate of T_2 in the form $F\frac{EV}{V}$. Reformulating properties (9), (10), (11) in terms of F and V, we have the following theorem.

Theorem 2 *Let T be a term. Then it is possible to find a term T_1 similar to T, and a shift-reduced rational function $F = f_1/f_2$ that is adequate for T, T_1, such that the certificate of the term $T_2 = T - (E-1)T_1$, written in the RNF with F as the kernel:*

$$F\frac{EV}{V},\ V = \frac{v_1}{v_2},\ v_1 \perp v_2,$$

has the following properties:

(Pa) *v_2 is shift-free;*

(Pb) *if p is an irreducible from $K[n]$ such that $p \mid v_2$, then*

$$E^h p \mid f_1 \quad \Rightarrow \quad h < 0, \tag{13}$$
$$E^h p \mid f_2 \quad \Rightarrow \quad h > 0. \tag{14}$$

As the proofs of Lemmas 3 and 4 are constructive, we can now describe an algorithm to compute the terms T_1, T_2 mentioned in Theorem 2. In the case a) of the proof of Lemma 3 we considered irreducible q and integer h such that $q|u_2$, $E^h q|d_2$, $h \le 0$. All the q's (say $q_1, \dots, q_\kappa$) that relate to the minimal possible h can be considered together. Using a resultant and gcd techniques, we can find the minimal value of h along with $q' = q_1^{\nu_1} \dots q_\kappa^{\nu_\kappa}$, $q'|u_2$, $\nu_1, \dots, \nu_\kappa > 0$. After this we can compute $\tilde{q} = q_1^{\mu_1} \dots q_\kappa^{\mu_\kappa}$, where $\mu_1, \dots, \mu_\kappa$ are maximal possible values such that $q_1^{\mu_1} \dots q_\kappa^{\mu_\kappa}|u_2$. We can use the following simple algorithm *pump*:

input: $f, g \in K[n],\ f|g;$
output: $\tilde{f}, \tilde{g} \in K[n],\ f|\tilde{f},\ \tilde{f}\tilde{g} = g,\ \tilde{f} \perp \tilde{g};$

  $\tilde{f} := f;\ \tilde{g} := g/\tilde{f};$
  repeat $d = \gcd(\tilde{f}, \tilde{g});$
    $\tilde{f} := \tilde{f}d;\ \tilde{g} := \tilde{g}/d;$
  until $\deg d = 0.$

We can compute $(\tilde{q}, \tilde{u}_2) = pump(q, u_2)$ and then use the partial-fraction decomposition

$$U = \frac{\tilde{a}}{\tilde{u}_2} + \frac{\tilde{b}}{\tilde{q}},$$

where $\tilde{a}, \tilde{b} \in K[n]$, in place of decomposition (12). We can similarly proceed in the case b) of the proof of Lemma 3. Thus we finally arrive at the following algorithm *hgdecomp*:

input: $D = \frac{d_1}{d_2}, U = \frac{u_1}{u_2}$, where $d_1 \perp d_2$ and D is shift-reduced.
output: $U_1, F, V \in K(n)$ such that the term with certificate $D\frac{EU}{U}$ has decomposition $T = T_2 + (E-1)T_1$,

where T_1, T_2 have certificates $D\frac{EU_1}{U_1}$ and $F\frac{EV}{V}$, resp., with F, V satisfying **Pa**, **Pb** of Theorem 2.

$U_1 := 0;\ U_2 := U;$
$N_1 := -1;\ N_2 := 1;\ M := -1;$
$R_1(m) := Res_x(u_2(x+m), d_1(x));$
$R_2(m) := Res_x(u_2(x+m), d_2(x));$
$R(m) := Res_x(u_2(x+m), u_2(x));$
`if` $R_1(m)$ has some nonnegative integer root
  `then` $N_1 := \max\{m : m \in \mathbb{Z}, R_1(m) = 0\}$
`fi;`
`if` $R_2(m)$ has some nonpositive integer root
  `then` $N_2 := \min\{m : m \in \mathbb{Z}, R_2(m) = 0\}$
`fi;`
`if` $R(m)$ has some positive integer root
  `then` $M := \max\{m : m \in \mathbb{Z}, R(m) = 0\}$
`fi;`
$N_1 = \max\{N_1, M\};$
`for` $h = N_1$ `downto` 0 `do`
  $q := \gcd(u_2, E^{-h}d_1);$
  $t := u_2/q;$
  $q := q\gcd(t, E^{-h}t);$
  $(\tilde{q}, \tilde{u}_2) := pump(q, u_2);$
  represent U_2 in the form $U_2 = \frac{\tilde{a}}{\tilde{u}_2} + \frac{\tilde{b}}{\tilde{q}},\ \tilde{a}, \tilde{b} \in K[n];$
  ${U_1}' := -\tilde{b}/\tilde{q};$
  $U_2 := U_2 - D(E{U_1}') + {U_1}';\ U_1 := U_1 + {U_1}'$
`od;`
`for` $h = N_2$ `to` 0 `do`
  $q := \gcd(u_2, E^{-h}d_2);$
  $t := u_2/q;$
  $q := q\gcd(t, E^{-h}t);$
  $(\tilde{q}, \tilde{u}_2) := pump(q, u_2);$
  represent U_2 in the form $U_2 = \frac{\tilde{a}}{\tilde{u}_2} + \frac{\tilde{b}}{\tilde{q}},\ \tilde{a}, \tilde{b} \in K[n];$
  ${U_1}' := E^{-1}(\tilde{b}/(D\tilde{q}));$
  $U_2 := U_2 - D(E{U_1}') + {U_1}';\ U_1 := U_1 + {U_1}'$
`od;`
$v_1 := numerator(U_2);\ v_2 := denominator(U_2);$
`if` $E^{-1}d_1 | v_2$
  `then` $v_2 := v_2/(E^{-1}d_1);\ f_1 := E^{-1}d_1$
  `else` $f_1 := E^{-1}d_1$
`fi;`
`if` $d_2 | v_2$
  `then` $v_2 := v_2/d_2;\ f_2 := Ed_2$
  `else` $f_2 := d_2$
`fi;`
$F := \frac{f_1}{f_2};\ V := \frac{v_1}{v_2}.$

If $n_0 \in \mathbb{Z}$ is such that $D(x), U(x), U_1(x), F(x), V(x)$ have neither a pole nor a zero for $x \geq n_0$, then the proof of Lemma 4 allows one to obtain regular descriptions of terms T and T_1. It is also possible to get a regular description of T_2. Indeed, it is sufficient to find a constant γ such that $T_2(n) = \gamma V(n) \prod_{\alpha=n_0}^{n-1} F(k)$. For $n = n_0$ we have

$$\gamma V(n_0) = T_2(n_0) = T(n_0) - T_1(n_0+1) + T_1(n_0) = \alpha\,(U(n_0) - U_1(n_0+1)D(n_0) + U_1(n_0))$$

where $\alpha = T(n_0)/U(n_0)$. This gives γ.

So, after applying *hgdecomp* we can find regular descriptions of the terms T_1 and T_2. We will prove in the rest of the paper that these terms give a solution of the decomposition problem.

Theorem 3 *Let T, T_1, T_1' be similar terms, $T_2 = T-(E-1)T_1$, $T_2' = T-(E-1)T_1'$, and $F = f_1/f_2$ a shift-reduced rational function adequate for these terms. Let $ET_2/T_2 = F\frac{EV}{V}$ where $F, V \in K(n)$ have properties* **Pa** *and* **Pb** *of Theorem 2, and $ET_2'/T_2' = F\frac{EV'}{V'}$. If $V = v_1/v_2$ and $V' = v_1'/v_2'$ where $v_1, v_2, v_1', v_2' \in K[n]$ and $v_1 \perp v_2$ then $\deg v_2 \le \deg v_2'$.*

Proof: We have

$$T_2' = T_2 - (E-1)(T_1' - T_1).$$

Suppose that the certificate of $T_1' - T_1$ is equal to $F(EW)/W$ where $W = w_1/w_2$ and $w_1 \perp w_2$. Then, by (7),

$$\frac{v_1'}{v_2'} = \frac{v_1}{v_2} - \frac{f_1}{f_2}\frac{Ew_1}{Ew_2} + \frac{w_1}{w_2}. \tag{15}$$

Consider an arbitrary irreducible $p \in K[n]$ such that $p \,|\, v_2$. We set

$$k = \max\{\alpha : p^\alpha \,|\, v_2\}$$

and claim that $E^l p^k \,|\, v_2'$ for some $l \in \mathbb{Z}$. Since the pair F, V has property **Pa**, this claim will imply the statement of the theorem. Suppose that p^k does not divide v_2'. Equation (15) implies that v_2 and hence p^k divides the lcm of v_2', f_2Ew_2, and w_2. By (14) we have $p \perp f_2$, therefore $p^k \,|\, Ew_2$ or $p^k \,|\, w_2$.

Let $p^k \,|\, Ew_2$. Then

$$E^{-1}p^k \,|\, w_2. \tag{16}$$

Set $l = \min\{m : E^m p^k \,|\, w_2\}$. Apparently $E^l p^k$ does not divide Ew_2. It follows from (16) that $l \le -1$; together with (14) this gives $E^l p \perp f_2$. As v_2 is shift-free and $p \,|\, v_2$, it follows that $E^l p^k$ does not divide v_2. Therefore (15) implies

$$E^l p^k \,|\, v_2'. \tag{17}$$

Let $p^k \,|\, w_2$. Then

$$Ep^k \,|\, Ew_2. \tag{18}$$

Set $l = \max\{m : E^m p^k \,|\, Ew_2\}$. Apparently $E^l p^k$ does not divide w_2. It follows from (18) that $l \ge 1$; together with (13) this gives $E^l p \perp f_1$. Therefore (15) implies (17) in this case as well. □

Corollary 1 *Let $F, U, S_1, S_2 \in K(n)$ where F is shift-reduced. Let rational functions*

$$V_1 = U - FES_1 + S_1,\ V_2 = U - FES_2 + S_2$$

be such that the pairs F, V_1 and F, V_2 have properties **Pa** *and* **Pb** *of Theorem 2. Then the degrees of the denominators of V_1 and V_2 are equal.*

3 Part (B): A verification

If both rational functions F_1, F_2 are adequate for a term T then there exists $G \in K(n)$ such that

$$\frac{F_1}{F_2} = \frac{EG}{G}. \tag{19}$$

Indeed, for some $U_1, U_2 \in K(n)$ we have

$$F_1\frac{EU_1}{U_1} = F_2\frac{EU_2}{U_2},$$

and therefore $G = U_1^{-1}U_2$. The case where $G \in K[n]$ is especially interesting for us.

Theorem 4 *Let F_1, F_2 be rational functions that are adequate for a term T. Let (19) hold with $G \in K[n]$. Let V be such that the pair F_1, V has properties* **Pa** *and* **Pb**. *Then the denominator of V is coprime with G and the pair F_2, GV has properties* **Pa** *and* **Pb**.

Proof: First we prove that the denominator of V is coprime with G. If they have a common irreducible factor p then the set $\{\nu : E^\nu p \,|\, G\}$ is a non-empty finite set. Suppose that m, M are, resp., the minimal and the maximal elements of this set. Write

$$W = \frac{G}{EG} = \frac{F_2}{F_1} = \frac{w_1}{w_2}, \quad w_1 \perp w_2.$$

Then $E^{M+1}p \,|\, w_2$ and $E^m p \,|\, w_1$. We have $F_2 = WF_1$. As p divides the denominator of V and the pair F_1, V has properties **Pa** and **Pb**, the numerator of F_1 is not divisible by $E^{M+1}p$ since $M + 1 > 0$. Similarly the denominator of F_1 is not divisible by $E^m p$ since $m \leq 0$. Therefore the numerator of F_2 is divisible by $E^m p$ while the denominator of F_2 is divisible by $E^{M+1}p$. But F_2 is shift-reduced by Definition 3(ii), a contradiction.

Now we prove that the pair F_2, GV has properties **Pa** and **Pb**. We have

$$F_2 = \frac{G}{EG} F_1$$

and the pair F_2, GV has property **Pa** because the denominator of GV divides the denominator of V. Now we shall be concerned with **Pb**. Let p be an irreducible from $K[n]$ that divides the denominator of GV and thereby divides the denominator of V. Let $E^h p$, $h \leq 0$, divide the denominator of F_2. Then $E^h p$ does not divide the denominator of F_1 since the pair F_1, V has properties **Pa** and **Pb**. The equality $(EG)F_2 = GF_1$ implies that $E^h p \,|\, EG$. Set $h_0 = \min\{\nu : E^\nu p \,|\, EG\}$. Then $h_0 \leq h \leq 0$ and $E^{h_0 - 1}p \,|\, G$, but $E^{h_0-1}p$ does not divide EG. The denominator of F_1 is not divisible by $E^{h_0-1}p$ since the pair F_1, V has properties **Pa** and **Pb**. Therefore $E^{h_0-1}p$ divides the numerator of F_2. But as $E^h p$ divides the denominator of F_2, this contradicts the fact that F_2 is shift-reduced.

Similarly it can be shown that $E^h p$, $h \geq 0$, cannot divide the numerator of F_2. □

Lemma 5 *Let $F, F_1, U, U_1 \in K(n)$, $G \in K[n]$ be such that $F/F_1 = EG/G$, $G \in K[n]$ and $F\frac{EU}{U} = F_1\frac{EU_1}{U_1}$. Then there exists $\overline{G} \in K[n]$ such that $\overline{G}U = U_1$ and for any $S \in K(n)$ we have*

$$\overline{G}(U - FES + S) = U_1 - F_1 E(\overline{G}S) + \overline{G}S.$$

Proof: We have

$$\frac{E(U^{-1}U_1)}{U^{-1}U_1} = \frac{EG}{G}.$$

It follows from this that there exists $\alpha \in K$ such that $U^{-1}U_1 = \alpha G$. Set $\overline{G} = \alpha G$. We get

$$\frac{E\overline{G}}{\overline{G}} F_1 = F, \quad U_1 = \overline{G}U.$$

Substituting U_1 for $\overline{G}U$ and $(E\overline{G}/\overline{G})F_1$ for F in $\overline{G}U - (\overline{G}F)ES + \overline{G}S$ gives $U_1 - F_1 E(\overline{G}S) + \overline{G}S$. □

Theorem 5 *Let F_1, F_2 be rational functions that are adequate for a term T. Let $U_1, U_2, R \in K(n)$ be such that*

$$F_1 \frac{EU_1}{U_1} = F_2 \frac{EU_2}{U_2} = R. \tag{20}$$

For $S_1, S_2 \in K(n)$, let

$$V_1 = U_1 - F_1 ES_1 + S_1, \; V_2 = U_2 - F_2 ES_2 + S_2 \tag{21}$$

be such that the pairs F_1, V_1 and F_2, V_2 have properties **Pa** *and* **Pb**. *Then the denominators of V_1 and V_2 have equal degrees.*

Proof: First of all we show that there exists a rational function a/b, $a \perp b$, such that for the rational functions

$$F_0 = \frac{a}{b},\ F_{-1} = \frac{E^{-1}a}{b},\ F_{-2} = \frac{a}{Eb},\ F_{-3} = \frac{E^{-1}a}{Eb} \tag{22}$$

the equalities

$$\frac{F_i}{F_1} = \frac{EG'_i}{G'_i},\ \frac{F_i}{F_2} = \frac{EG''_i}{G''_i},\ G'_i, G''_i \in K[n], \tag{23}$$

hold for $i = -1, -2, -3$. It is sufficient to prove the theorem for shift-homogeneous F_1, F_2 which belong to the same shift-homogeneous class. Then, by Lemma 1, either both F_1 and F_2 are polynomials, or both F_1 and F_2 are reciprocals of polynomials. By Theorem 1(ii) we have

$$F_1 = \prod_{i=1}^{\tau} E^{h_i}p,\ F_2 = \prod_{i=1}^{\tau} E^{l_i}p, \tag{24}$$

in the former case, and

$$F_1 = \frac{1}{\prod_{i=1}^{\tau} E^{h_i}p},\ F_2 = \frac{1}{\prod_{i=1}^{\tau} E^{l_i}p}. \tag{25}$$

in the latter, where $p \in K[n]$ is irreducible. In case of (24), set

$$a = \prod_{i=1}^{\tau} E^{\max\{h_i,l_i\}+1}p,\ b = 1,$$

and in case of (25), set

$$a = 1,\ b = \prod_{i=1}^{\tau} E^{\min\{h_i,l_i\}-1}p.$$

It is easy to see that if $F_0, F_{-1}, F_{-2}, F_{-3}$ are defined as in (22) then the equalities (23) hold for some polynomials G'_i, G''_i.

Considering the RNF of R with the kernel a/b and using algorithm *hgdecomp* we can get $i,\ -3 \le i \le 0$, and $F, U, V, S \in K[n]$ such that

- $F = F_i$,
- $R = F\frac{EU}{U},\ U = \frac{u_1}{u_2},\ u_1 \perp u_2$;
- $V = U - FES + S$,
- the pair F, V has properties **Pa** and **Pb**.

Set

$$G' = G'_i,\ G'' = G''_i$$

for the computed i. By Lemma 5 we have a polynomial $\overline{G}'$ such that

$$\overline{G}'V = \overline{G}'(U - FES + S) = U_1 - F_1E(\overline{G}'S) + \overline{G}'S.$$

By Theorem 4 the pair F_1, $U_1 - F_1E(\overline{G}'S) + \overline{G}'S$ has properties **Pa** and **Pb** and the degree of denominator of $\overline{G}'V$ is equal to the degree of the denominator of V. By Corollary 1 we have that the denominator of V is of the same degree as the denominator of V_1, and similarly for the degrees of the denominators of V and V_2. The claim follows. □

Theorem 6 *Let T, T_1, T'_1 be similar terms. Let the certificates of the terms $T_2 = T - (E-1)T_1, T'_2 = T - (E-1)T'_1$ be written in the form*

$$F\frac{EV}{V},\ F'\frac{EV'}{V'}$$

with shift-reduced F, F'. Let the pair F, V have properties **Pa** *and* **Pb**. *Then the degree of the denominator of V is less than or equal to the degree of the denominator of V'.*

Proof: It is possible to find $U, S, S' \in K(n)$, $n_0 \in \mathbb{Z}$ and $\alpha, \beta \in K$ such that the triples (F, S, n_0), $(F, \alpha V, n_0)$ regularly describe the terms T, T_1, T_2 and the triples (F, S', n_0), $(F, \beta V', n_0)$ regularly describe the terms T'_1, T'_2. The claim follows from Lemma 2 and Theorems 3, 5. □

4 Part (C): Decreasing the degree of the numerator

So, the denominator of V has the minimal possible degree. How to reduce the degree of the numerator of V to a tolerable size? Recall that when one solves the decomposition problem for indefinite sums of rational functions, it is always possible to have the degree of the numerator less than the degree of the denumerator (this is because any rational function is the sum of a polynomial and a proper rational function, and the equation $(E-1)y = u$ has a polynomial solution for any $u \in K[n]$).

In the hypergeometric case the situation is not so simple. Consider a term which is regularly described by a triple (D, V, n_0), $V \in K[n]$, $D = d_1/d_2$, $d_1 \perp d_2$. Let's try first to find a polynomial S such that the polynomial

$$d_2V - d_1ES + d_2S \tag{26}$$

has a "reasonable" degree. Rewrite (26) as

$$P = d_2V - d_1(E-1)S + (d_2 - d_1)S \tag{27}$$

and set

$$M = d_2V, \quad T = -d_1(E-1)S + (d_2 - d_1)S.$$

Evidently $\deg(E-1)S = \deg S - 1$, $\mathrm{lc}\,((E-1)S) = \deg S \cdot \mathrm{lc}\, S$. By a judicious choice of S we can cancel out some leading terms of M if the degree of M is large enough. The number of those terms depends first of all on the relation between $\deg(d_2 - d_1)$ and $\deg d_1$:

1. $\deg(d_2 - d_1) > \deg d_1$. Then $\deg d_2 > \deg d_1$ and $\deg T = \deg d_2 + \deg S$, $\mathrm{lc}\, T = \mathrm{lc}\, d_2 \cdot \mathrm{lc}\, S$. We can transform M to P of degree $< \deg d_2$.

2. $\deg(d_2 - d_1) = \deg d_1$. Then $\deg d_2 \leq \deg d_1$ and $\deg T = \deg d_1 + \deg S$, $\mathrm{lc}\, T = \mathrm{lc}\,(d_2 - d_1) \cdot \mathrm{lc}\, S$. We can transform M to P of degree $< \deg d_1$.

3. $\deg(d_2 - d_1) < \deg d_1$. Then $\deg d_2 = \deg d_1$, $\mathrm{lc}\, d_1 = \mathrm{lc}\, d_2$.

   3a. $\deg(d_2 - d_1) < \deg d_1 - 1$. Then the coefficients of $x^{\deg d_1 - 1}$ in d_1 and d_2 are equal. We have $\deg T = \deg d_1 + \deg S - 1$, $\mathrm{lc}\, T = -\mathrm{lc}\, d_1 \cdot \mathrm{lc}\, S \cdot \deg S$. We can transform M to P of degree $\sigma < \deg d_1$. (More, if $\sigma = \deg(d_2 - d_1)$, then we can by an additional elimination transform P to P' of degree $< \deg(d_2 - d_1)$.)

   3b. $\deg(d_2 - d_1) = \deg d_1 - 1$. Then the coefficients of $x^{\deg d_1 - 1}$ in d_1 and d_2 are not equal. If

   $$-\mathrm{lc}\, d_1 \cdot \deg S + \mathrm{lc}\,(d_2 - d_1) \neq 0$$

   then

   $$\deg T = \deg d_1 + \deg S - 1 = \deg(d_2 - d_1) + \deg S,$$
   $$\mathrm{lc}\, T = \mathrm{lc}\, S \cdot (-\mathrm{lc}\, d_1 \cdot \deg S + \mathrm{lc}\,(d_2 - d_1)).$$

   In such a case if the equation

   $$-\mathrm{lc}\, d_1 \cdot X + \mathrm{lc}\,(d_2 - d_1) = 0$$

   has no integer root on the segment $[0; \deg M - \deg(d_2 - d_1)]$ or, equivalently, on the segment $[0; \deg M - \deg d_1 + 1]$, then we can transform M to P of degree $< \deg d_1 - 1$. If

   $$-\mathrm{lc}\, d_1 \cdot \tau + \mathrm{lc}\,(d_2 - d_1) = 0$$

   for an integer τ from the segment $[0; \deg M - \deg(d_2 - d_1)]$, then we can transform M to P of degree $< \deg(d_2 - d_1) + \tau + 1$, or, equivalently, $< \deg d_1 + \tau$.

Therefore we have the following

Theorem 7 *Let* $M, d_1, d_2 \in K[n]$, $d_2 \neq 0$. *Then it is possible to find* $S \in K[n]$ *such that the degree of* $P = M - d_1ES + d_2S$ *is less than*

$$\lambda = \begin{cases} \deg d_2 & \text{if } \deg(d_2 - d_1) > \deg d_2, \\ \deg d_1 & \text{if } \deg(d_2 - d_1) = \deg d_2 \text{ or } \deg(d_2 - d_1) < \deg d_1 - 1, \\ \deg d_1 + \tau & \text{if } \deg(d_2 - d_1) = \deg d_1 - 1, \end{cases}$$

where in the last case τ *is equal to* $\mathrm{lc}\,(d_2 - d_1)/\mathrm{lc}\, d_1$ *if this is a nonnegative integer and* -1 *otherwise.*

Observe that if $d_1 = d_2 = 1$, then $\lambda = 0$ and $P = 0$. This is in agreement with the fact that the equation $(E-1)S = V$ has a polynomial solution when $V \in K[n]$.

Thus we can find S such that (26) is equal to a polynomial P whose degree is bounded from above as described in Theorem 7. This implies that

$$\begin{aligned} & V(n) \prod_{k=n_0}^{n-1} D(k) - (E-1)S(n) \prod_{k=n_0}^{n-1} D(k) \\ = \; & \frac{P(n)}{d_2(n)} \prod_{k=n_0}^{n-1} D(k) \; = \; \frac{P(n)}{d_2(n_0)} \prod_{k=n_0}^{n-1} \frac{d_1(k)}{d_2(k+1)}. \end{aligned}$$

Now suppose that V is a rational function of the form v_1/v_2. This leads to the expression

$$d_2v_1 - v_2d_1ES + v_2d_2S.$$

We can use the described techniques, considering v_2d_1, v_2d_2 instead of d_1, d_2. We get S, P such that

$$d_2v_1 - v_2d_1ES + v_2d_2S = P,$$

in other words

$$\begin{aligned} & \frac{v_1(n)}{v_2(n)} \prod_{k=n_0}^{n-1} \frac{d_1(k)}{d_2(k)} - (E-1)\left(S(n) \prod_{k=n_0}^{n-1} \frac{d_1(k)}{d_2(k)} \right) \\ = \; & \frac{P(n)}{v_2(n)d_2(n)} \prod_{k=n_0}^{n-1} \frac{d_1(k)}{d_2(k)} \; = \; \frac{P(n)}{v_2(n)d_2(n_0)} \prod_{k=n_0}^{n-1} \frac{d_1(k)}{d_2(k+1)}. \end{aligned}$$

Hence we have proven

Theorem 8 *Let a term* T *be regularly described by a triple* $(d_1/d_2, v_1/v_2, n_0)$, *i.e.,*

$$T(n) = \frac{v_1(n)}{v_2(n)} \prod_{k=n_0}^{n-1} \frac{d_1(k)}{d_2(k)}.$$

Then there exists a term T_1 *of the form*

$$T_1(n) = S(n) \prod_{k=n_0}^{n-1} \frac{d_1(k)}{d_2(k)},$$

$S \in K[n]$, *such that the term* $T_2 = T - (E-1)T_1$ *is of the form*

$$\frac{P(n)}{v_2(n)} \prod_{k=n_0}^{n-1} \frac{d_1(k)}{f(k)},$$

where $f(k)$ *is either* $d_2(k)$ *or* $d_2(k+1)$ *and* P *is a polynomial whose degree is less than*

$$\lambda = \begin{cases} \deg v_2 + \deg d_2 & \text{if } \deg(d_2 - d_1) > \deg d_2, \\ \deg v_2 + \deg d_1 & \text{if } \deg(d_2 - d_1) = \deg d_2 \text{ or } < \deg d_1 - 1, \\ \deg v_2 + \deg d_1 + \tau & \text{if } \deg(d_2 - d_1) = \deg d_1 - 1, \end{cases}$$

where in the last case τ *is equal to* $\mathrm{lc}\,(d_2 - d_1)/\mathrm{lc}\, d_1$ *if this is a nonnegative integer, and* -1 *otherwise.*

Observe that if $d_1 = d_2 = 1$, then by this theorem $\deg P < \deg v_2$. This is in agreement with the properties of the decomposition of indefinite sums of rational functions.

Note the following. If the pair $d_1/d_2, v_1/v_2$ has properties **Pa** and **Pb**, then the rational function P/v_2 is irreducible (i.e., $P \perp v_2$) by Theorem 3. But we can try to reduce P/d_2. Suppose this gives P'/d_2', $d_2 = d_2' d_2''$. Then

$$\frac{P(n)}{v_2(n)d_2(n)} \prod_{k=n_0}^{n-1} \frac{d_1(k)}{d_2(k)}$$

$$= \frac{P'(n)}{v_2(n)d_2'(n)} \prod_{k=n_0}^{n-1} \frac{d_1(k)}{d_2'(k)d_2''(k)} = \frac{P'(n)}{v_2(n)d_2'(n_0)} \prod_{k=n_0}^{n-1} \frac{d_1(k)}{d_2''(k)d_2'(k+1)}.$$

5 Examples

With algorithm *hgdecomp* we can get the following decompositions:

$$\left(\frac{1}{n+1} - \frac{1}{n}\right) \prod_{k=0}^{n-1} \frac{1}{k+2} = (E-1)\left(\frac{n+1}{n} \prod_{k=0}^{n-1} \frac{1}{k+2}\right) + \prod_{k=0}^{n-1} \frac{1}{k+2}; \tag{28}$$

$$\left(\frac{1}{n+1} - \frac{1}{n} - 1\right) \prod_{k=0}^{n-1} \frac{1}{k+2} = (E-1)\left(\frac{n+1}{n} \prod_{k=0}^{n-1} \frac{1}{k+2}\right); \tag{29}$$

$$\left(\frac{1}{n+1} - \frac{2}{n} - 2\right) \prod_{k=0}^{n-1} \frac{1}{k+2} = (E-1)\left(\frac{n+1}{n} \prod_{k=0}^{n-1} \frac{1}{k+2}\right) - \frac{n+1}{n} \prod_{k=0}^{n-1} \frac{1}{k+2}. \tag{30}$$

Using the approach from section 4 we can rewrite the term

$$\frac{n+1}{n} \prod_{k=0}^{n-1} \frac{1}{k+2}$$

in the right-hand sides of (28), (29), (30) as

$$\frac{1}{n} \prod_{k=0}^{n-1} \frac{1}{k+1}.$$

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