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Preprint series, Vol. 39 (2001), 755

ON STABILITY OF CRITICAL POINTS OF RICCATI DIFFERENTIAL EQUATIONS IN NONASSOCIATIVE ALGEBRAS

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ISSN 1318-4865

April 13, 2001

Ljubljana, April 13, 2001

On stability of critical points of Riccati differential equations in nonassociative algebras

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Abstract. In this note we treat the stability of nonzero critical points of differential equation $\dot{x} = x^2$ in a commutative real nonassociative algebra. As our first result we prove that if critical point lies in some Peirce subspace with respect to a nonzero idempotent, it cannot be stable. This improves previously known result due to Kinyon and Sagle. As a second result we show that there exists a 2-dimensional algebra, with nonzero critical point and nontrivial idempotent, such that the critical point is stable, so that the additional assumption in our first result cannot be completely lifted.

1. Introduction

One possibility for approaching polynomial systems of autonomous differential equations in $\mathbb{R}^n$ is via the theory of commutative (nonassociative in general) finite dimensional algebras. It seems that this idea originated with Markus in [Mar]. A standard reference for the state of art in 1990, with many references to older papers, is [Wal1]. Some last decade papers of importance are [GW], [Hop], [KS1], [KS2], [Myu] and [Wal2].

For every autonomous polynomial system of ODE it is possible to construct a homogeneous quadratic system of ODE such that the original solutions can be read off the new solutions. For detail of this construction the reader should consult [Wal1] and [KS2]. The quadratic form in quadratic system $\dot{x} = Q(x)$ can be interpreted as diagonal of a bilinear form, i.e. $Q(x) = B(x, x)$. Defining $x \cdot y = B(x, y)$ we can interpret obtained system of ODE as Riccati equation $\dot{x} = x^2$ in a finite dimensional algebra. As an example we note that system

$$\dot{x} = ax^2 + 2bxy + cy^2$$
$$\dot{y} = dx^2 + 2exy + fy^2$$

can be viewed as Riccati equation corresponding to the algebra whose multiplication table is given by

$\cdot$	i	j
i	ai+dj	bi+ej
j	bi+ej	ci+dj

1991 *Mathematics Subject Classification.* Primary 17; Secondary 17C27, 34A34, 34D20.

Key words and phrases. Riccati differential equation, nonassociative algebra, critical points, ray solutions, projections, nilpotents, Peirce subspaces.

Under this interpretation critical points correspond to nilpotents of order two, i.e. $n^2 = 0$, while ray solutions correspond to idempotents, i.e. $p^2 = p$. For applications of these and other algebraic concepts in theory of ODE we again refer to the above mentioned references.

The topic of stability of critical points of the Riccati equation $\dot{x} = x^2$ was taken up in [KS1] which is also a starting point for the present investigation. Among other things Kinyon and Sagle proved

- if algebra has a nonzero idempotent, then 0 (which clearly is always a critical point) is not stable;
- the critical point 0 can never be asymptotically stable;
- if p is an idempotent and a (nonzero) critical point lies in the Pierce subspace $A_{\frac{1}{2}}(p)$, then it is not stable.

The last result (Corollary 3.8 of [KS1]) is a direct motivation of the present paper and we intend to show that the spectral value $\frac{1}{2}$ can in fact be replaced by an arbitrary real λ while the conclusion still remains true. As an interesting remark we note that it was proved long time ago by Kaplan, York and Rohrl that every finite dimensional real algebra contains at least one nonzero idempotent or nilpotent (see [KY] and [Roh]).

2. Critical points lying in Peirce subspaces

Let A be a finite dimensional real commutative algebra, which is not assumed to be associative. Let $p \in A$ be a nonzero idempotent and $n \in A$ a nilpotent of order 2. As we mentioned in the introduction, n is a critical point of the Riccati equation $\dot{x} = x^2$. If $n = 0$, it was already proved by Kinyon and Sagle that n is not stable, so we assume in the sequel that n is also nonzero.

If λ is a real constant, we can define

$$A_\lambda(p) = \{x \in A : p \cdot x = \lambda x\}$$

which can be viewed as eigenspace of the multiplication operator $L_p : x \longmapsto p \cdot x$. For almost all values of λ we shall have $A_\lambda(p) = \{0\}$. If this is not the case, we can call λ the eigenvalue of p and $A_\lambda(p)$ is the Peirce $\lambda-$ subspace. The reader is perhaps familiar with the theory of Jordan algebras, where one of the crucial results states that only possible eigenvalues are 0, 1 and $\frac{1}{2}$. In general algebras this is not true as any value of λ can occur. The only generally valid remark (trivial) is that $p \in A_1(p)$, so 1 is always an eigenvalue. The purpose of this section is to prove

THEOREM 1. *If the critical point n belongs to the Peirce $\lambda-$ subspace for some λ, then n is not stable.*

With the correct idea the proof appears to be remarkably simple.

PROOF. Let ϵ be a positive constant. Every neighborhood of the critical point n contains a point of the form $n + \epsilon p$ for some $\epsilon > 0$. We shall prove that the solution of the equation $\dot{x} = x^2$ satisfying the initial condition $x(0) = n + \epsilon p$, where ϵ is small enough, goes to infinity. This clearly implies that n cannot be stable.

Since n belongs to $\lambda - subspace$, we have $pn = \lambda n$. We shall seek the solution in the form

$$x(t) = f(t)p + g(t)n,$$

where f, g are real functions. Since n, p are nonzero, one being idempotent another nilpotent, they are linearly independent. The equation $\frac{dx}{dt} = x^2$ then gives

$$\frac{df}{dt}p + \frac{dg}{dt}n = (fp + gn)^2 = f^2p^2 + 2fgpn + g^2n^2 = $$
$$= f^2p + 2\lambda fgn + 0,$$

which implies

$$\frac{df}{dt} = f^2,$$
$$\frac{dg}{dt} = 2\lambda fg,$$

where our initial conditions are

$$f(0) = \epsilon > 0,$$
$$g(0) = 1.$$

It is quite easy to solve this explicitly and obtain a solution

$$x(t) = \frac{\epsilon}{1-\epsilon t}p + \left(\frac{1}{1-\epsilon t}\right)^{2\lambda} n$$

which is well-defined on the interval $[0, \frac{1}{\epsilon})$ for every real λ and even has a finite escaping time property. □

3. Two dimensional counterexample

After the result in previous section one might ask a natural question whether with some additional work it may be possible to prove even much more general result, i.e. that in the presence of idempotents (ray solutions) no critical point can be stable. This however is not true. Now we show that already among two dimensional algebras containing idempotents there are such for which system $\dot{x} = x^2$ has stable critical points.

Let a commutative real algebra A be given with the following multiplication table

·	n	a
n	0	-a
a	-a	-n

It is obvious that A contains a nilpotent n of order two which is critical point of the system $\dot{x} = x^2$. Further we have

Proposition 1. *Algebra A has exactly two idempotents $p = -\frac{1}{2}n + \frac{1}{\sqrt{2}}a$ and $q = -\frac{1}{2}n - \frac{1}{\sqrt{2}}a$. They satisfy $pq = -\frac{1}{2}(p+q)$ and both have 1, -$\frac{1}{2}$ as eigenvalues. Furthermore*

$$A = A_1(p) \oplus A_{-\frac{1}{2}}(p) = A_1(q) \oplus A_{-\frac{1}{2}}(q)$$

where

$$A_1(p) = \mathbb{R}|, \qquad \mathbb{A}_1(q) = \mathbb{R}||, \qquad \mathbb{A}_{-\frac{1}{2}}(p) = \mathbb{R}(p+3q), \qquad A_{-\frac{1}{2}}(q) = \mathbb{R}(q+3p).$$

With respect to those idempotents critical point can be decomposed as

$$n = -\frac{2}{3} \cdot p - \frac{1}{3} \cdot (p + 3q),$$
$$n = -\frac{2}{3} \cdot q - \frac{1}{3} \cdot (q + 3p).$$

PROOF. The equation

$$(\alpha n + \beta a)^2 = \alpha n + \beta a$$

gives, using the above mentioned table, system for two real variables

$$\alpha = -\beta^2,$$
$$-2\alpha\beta = \beta,$$

whose nonzero solutions are

$$\alpha = -\frac{1}{2}, \qquad \beta = -\frac{1}{\sqrt{2}},$$

hence p and q are obtained. Then it can be directly verified that $pq = -\frac{1}{2}(p + q)$. The Pierce decomposition can be obtained via the equation

$$p(\alpha p + \beta q) = \lambda \alpha p + \lambda \beta q$$

which implies

$$\alpha - \frac{\beta}{2} = \lambda \alpha,$$
$$-\frac{\beta}{2} = \lambda \beta$$

whose nonzero solutions are possible when $\lambda = 1$ $(\beta = 0)$ or $\lambda = -\frac{1}{2}$ $(3\alpha = \beta)$. In a similar way we find decomposition with respect to q. The decomposition of n is then obtained from relation $p = -\frac{1}{2}n + \frac{1}{\sqrt{2}}a = -\frac{1}{2}n + \frac{1}{2}(p - q)$. □

We have seen that our algebra can also be given with multiplication table

$\cdot$	p	q
p	p	$-\frac{1}{2}(p+q)$
q	$-\frac{1}{2}(p+q)$	q

The equation $\dot{x} = x^2$, written in this basis becomes

$$\frac{df}{dt} \cdot p + \frac{dg}{dt} \cdot q = f^2 \cdot p + g^2 \cdot q - fg \cdot p - fg \cdot q$$

and therefore we considered the following system (called BS in the sequel) of ODE

$$\frac{df}{dt} = f^2 - fg,$$
$$\frac{dg}{dt} = g^2 - fg.$$

We are interested in behavior of the solutions in the neighborhood of the critical point $n = -p - q$, and therefore in (f, g) near the point (-1,-1). Note that we choose among possible inner products on A the product in which $\{p, q\}$ is an orthonormal base. We shall prove that n is stable by showing that for all initial conditions x_0 which are not too far from n there exists a solution of (BS) which is defined for all positive times and stays within the circle with center at critical point and passing through x_0.

LEMMA 1. *The solutions of (BS) lie on hyperbolas $fg = const$.*

PROOF. Since

$$\frac{d(fg)}{dt} = \frac{df}{dt} \cdot g + f \cdot \frac{dg}{dt} = \\ = (f^2 - fg)g + f(g^2 - fg) = 0,$$

the result follows. □

LEMMA 2. *For all initial conditions $x_0 = (f_0, g_0)$ which lie in the square $(-1.5, -0.5)^2$ the solution $x(t) = (f(t), g(t))$ of (BS) exists for all $t \geq 0$. Every solution has a limit point $x_\infty = \lim_{t\to\infty} x(t)$.*

PROOF. In the neighborhood of (-1,-1) the product fg is positive, so we may write $fg = \gamma^2$. Hence the first equation of (BS) becomes

$$\frac{df}{dt} = f^2 - \gamma^2$$

whose solution is

$$f(t) = \gamma \frac{1 + \delta e^{-2\gamma t}}{-1 + \delta e^{-2\gamma t}}.$$

In a similar way we obtain

$$g(t) = \gamma \frac{-1 + \delta e^{-2\gamma t}}{1 + \delta e^{-2\gamma t}}$$

Solving for initial conditions

$$f_0 = \gamma \frac{1+\delta}{-1+\delta}$$
$$g_0 = \gamma \frac{-1+\delta}{1+\delta}$$

we obtain

$$\gamma = \sqrt{f_0 g_0}$$
$$\delta = \frac{\sqrt{\frac{f_0}{g_0}} - 1}{\sqrt{\frac{f_0}{g_0}} + 1}$$

so the final solutions, with initial condition x_0, is

$$f(t) = \sqrt{f_0 g_0} \frac{1 + \frac{\sqrt{\frac{f_0}{g_0}}-1}{\sqrt{\frac{f_0}{g_0}}+1} e^{-2\sqrt{f_0 g_0}t}}{-1 + \frac{\sqrt{\frac{f_0}{g_0}}-1}{\sqrt{\frac{f_0}{g_0}}+1} e^{-2\sqrt{f_0 g_0}t}},$$

$$g(t) = \sqrt{f_0 g_0} \frac{-1 + \frac{\sqrt{\frac{f_0}{g_0}}-1}{\sqrt{\frac{f_0}{g_0}}+1} e^{-2\sqrt{f_0 g_0}t}}{1 + \frac{\sqrt{\frac{f_0}{g_0}}-1}{\sqrt{\frac{f_0}{g_0}}+1} e^{-2\sqrt{f_0 g_0}t}}.$$

Since δ is obviously smaller than 1, this solution is defined for all positive times. It is also elementary to verify that

$$x(t) = (f(t), g(t)) \longrightarrow (-\sqrt{f_0 g_0}, -\sqrt{f_0 g_0}) = x_\infty$$

as $t \to \infty$. □

Lemma 3. *For all initial conditions $x_0 = (f_0, g_0)$ which lie in the square $(-1.5, -0.5)^2$ the solution $x(t) = (f(t), g(t))$ of (BS) has the property*

$$dist(n, x_\infty) \leq dist(n, x_0).$$

Proof. We must prove that

$$2(1 - \sqrt{f_0 g_0})^2 \leq (1 + f_0)^2 + (1 + g_0)^2$$

for all $f_0, g_0 \in [-1.5, -0.5]$. If we write $f_0 = -u^2$ and $g_0 = -v^2$, we must prove that

$$(1 - u^2)^2 + (1 - v^2)^2 - 2(1 - uv)^2 \geq 0,$$

which can first be expanded into

$$u^4 + v^4 - 2u^2 - 2v^2 - 2u^2 v^2 + 4uv \geq 0.$$

and then factored as

$$\left(u^2 + 2uv - 2 + v^2\right)(u - v)^2 \geq 0.$$

The last factor is always nonnegative, while $\sqrt{0.5} < u, v$ implies $u^2 + v^2 + 2uv > 2$, so the first factor is positive as well. □

Lemma 4. *For all initial conditions $x_0 = (f_0, g_0)$ which lie in the square $(-1.5, -0.5)^2$ the solution $x(t) = (f(t), g(t))$ of (BS) has the property that the distance function*

$$D(t) = dist(n, x(t))$$

has no local extrema. Thus, combined with the above lemma,

$$D(t) \leq D(0) = dist(n, x_0)$$

Proof. We know that the solution lies on hyperbola $fg = f_0 g_0$ between points (f_0, g_0) and $\left(-\sqrt{f_0 g_0}, -\sqrt{f_0 g_0}\right)$. We must therefore investigate the function

$$D(f, g) = (1 + f)^2 + (1 + g)^2 =$$

$$(1 + f)^2 + (1 + \frac{f_0 g_0}{f})^2.$$

After taking derivative we obtain the following condition for extrema

$$(1 + f) - (1 + \frac{f_0 g_0}{f})\frac{f_0 g_0}{f^2} = 0$$

which can be factored as

$$(f - \sqrt{f_0 g_0})(f + \sqrt{f_0 g_0})(f^2 + f + f_0 g_0) = 0.$$

The first factor is nonzero for negative f, the second represents x_∞, while the last is positive when f_0, g_0 lie in $(-1.5, -0.5)$ because $f_0 g_0 > \frac{1}{4}$. □

All these results combined show us that for every $\epsilon > 0$ we have a $\delta = \min\left\{\frac{1}{2}, \varepsilon\right\}$ such that when the initial condition is in $\delta-$ neighborhood of the critical point n, then the solution stays within $\epsilon-$neighborhood for all later times. Therefore we have

Theorem 2. *There exists a real algebra with nonzero idempotent and nonzero nilpotent of order two such that the later is a stable critical point of the quadratic system $\dot{x} = x^2$.*

Remark 1. *By the result of Kinyon and Sagle mentioned in the introduction we know that this system also has a nonstable (namely 0) critical point. From the above proof concerning* x_∞ *it is also obvious that* n *is not asymptotically stable. In general case it seems that the problem of stability of nilpotents is still widely open.*

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