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ANALYTICITY ON CIRCLES FOR RATIONAL AND REAL-ANALYTIC FUNCTIONS OF TWO REAL VARIABLES

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Abstract. Conditions for rational and real-analytic functions of two real variables to be holomorphic are given in terms of holomorphic extendibility from families of circles.

1. Introduction.

This article is devoted to the following problem:

Let Ω be a domain in the complex plane and let $\mathcal{F}$ be a family of simple closed curves in Ω. Let f be a continuous function in Ω. Characterize the families $\mathcal{F}$ such that if for any $\gamma \in \mathcal{F}$ the restriction $f\big|\gamma$ admits an analytic extension to the domain bounded by γ then f is an analytic function in Ω.

Theorems of this sort for rotation-invariant families of closed curves in $\mathbb{C}$ were proved in [AV], [G1]-[G6] and for $\mathbb{C}^n$ in [AS]. The main tool in these works is harmonic analysis on orthogonal and unitary groups.

In this article we study the above mentioned problem for rational, and, more generally, for real-analytic, functions of two real variables in the case when the family $\mathcal{F}$ consists of circles or, more generally, images of circles under rational conformal maps. In this case the problem becomes purely complex analytic and is related to simple geometric objects in $\mathbb{C}^2$. More specifically, analytic (rational) extensions

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from circles with singularities always exist for rational functions in the real plane, and are naturally defined on algebraic curves, quadrics, in $\mathbb{C}^2$. Families of circles correspond to families of these complex quadrics. The condition of extendibility is equivalent to the absence of singularities of the extensions and understanding families of these quadrics with respect to the singularities is the key point of our approach.

One can put the problem under consideration into a broader geometric context. The graph $M = \{(z, f(z)) \in \mathbb{C}^2 : z \in \Omega\}$ of a function $f \in C(\Omega)$ is is a real 2-dimensional manifold in $\mathbb{R}^4 \simeq \mathbb{C}^2$. If a Jordan curve $\gamma \subset \Omega$ bounds a domain D then the analytic extension F_γ of the restriction $f|\gamma$ into D defines an analytic disk $\psi : D \ni z \to (z, F|_\gamma) \in \mathbb{C}^2$ attached to M along its boundary, i.e. $\psi(\partial D) \subset M$. Since the analyticity of f means that M is a complex submanifold in $\mathbb{C}^2$, our problem is a particular case of the following general problem: characterize families of analytic disks attached to M which imply that M is a complex manifold?

Note also that the problem we deal with can be regarded as a variation of Morera theorem. The Morera theorem states that a continuous function is analytic if its integrals along closed contours vanish (in fact, it suffices to take arbitrary circles, and, moreover, circles of only two appropriate radii, see [Z]). Our result is that, under the stronger condition of vanishing of *all* complex moments, any *one*-parametric family of circles, except very special ones, tests analyticity of rational and real-analytic functions of two real variables.

The paper is organized as follows. Sections 1 and 2 are introductory. Section 3 is devoted to the case of polynomials. In Section 4 we give analyticity conditions for rational functions in $\mathbb{R}^2$ in terms of analytic extendibility into a single disk. Sections 5-7 deal with finite and infinite families of disks and in Section 8 we treat the case of continuous one-parametric families of disks. In Section 10 we obtain analogous results for families of rational disks, i. e. images of the unit disk under rational conformal mappings. Finally, in Section 11 we study a more general case of real-analytic functions and present one-parametric families of circles that test analyticity of such functions.

2. Preliminaries.

Given a domain $\Omega \subset \mathbb{C}$ and a Jordan curve $\gamma \subset \Omega$ bounding a subdomain $D \subset\subset \Omega$, we say that f *extends analytically* from γ (into D) if there exists a function f^*, continuous in the closure $\overline{D}$ and analytic in D, such that $f^*|_\gamma = f|_\gamma$.

In the present paper we consider rational functions f on $\mathbb{R}^2 \equiv \mathbb{C}$, i.e. quotients of two complex valued polynomials of two real variables $(x,y) \in \mathbb{R}^2$. Identifying $\mathbb{R}^2$ with $\mathbb{C}$ by writing $z = x + iy$ we can write

$$f(z) = \frac{P(z,\overline{z})}{Q(z,\overline{z})}, \tag{2.1}$$

where $P(z,w)$ and $Q(z,w)$ are complex valued polynomials of two complex variables z and w.

With no loss of generality we will assume everywhere that the polynomials P and Q are coprime, that is, that they do not have common polynomial factors. We shall write $Z_P = \{(z,w)\colon\ P(z,w) = 0\}$, $Z_Q = \{(z,w)\colon\ Q(z,w) = 0\}$. These are complex algebraic varieties in $\mathbb{C}^2$.

We will use the decompositions

$$\begin{aligned} P(z,w) &= p_0(z) + p_1(z)w + \cdots + p_n(z)w^n, \\ Q(z,w) &= q_0(z) + q_1(z)w + \cdots + q_m(z)w^m, \end{aligned} \tag{2.2}$$

where p_i and q_i are holomorphic polynomials and $p_n \not\equiv 0$, $q_m \not\equiv 0$.

Finally, we will use the notations:

$\Delta(a,\rho) = \{z \in \mathbb{C} : |z-a| < \rho\}, \quad \Delta^*(a,\rho) = \Delta(a,\rho) \setminus \{a\},$

$\overline{\Delta}(a,\rho) = \{z \in \mathbb{C} : |z-a| \le \rho\}, \quad \Delta = \Delta(0,1), \quad \Delta_\rho = \Delta(0,\rho),$

$\deg P$ denotes the degree of a polynomial P.

We shall use a consequence of the argument principle that we state in a form suitable for our purposes. By a holomorphic curve M in $\mathbb{C}^2$ we understand the image of a disk $D = \Delta(a,r) \subset \mathbb{C}$ under a holomorphic mapping $\varphi\colon D \mapsto \mathbb{C}^2$.

Lemma 2.1. *Let M_0 be a holomorphic curve in $\mathbb{C}^2$ and $u_0 = (z_0, w_0) \in M_0$. Let Q be a holomorphic function in a neighborhood $U \subset \mathbb{C}^2$ of the point u_0, Q not identically zero on M, and $Q(u_0) = 0$. Then given a neighborhood $V \subset U \subset \mathbb{C}^2$ of the point u_0, the function Q vanishes at a point of $V \cap M$ for any holomorphic curve M which is sufficiently close to M_0.*

Proof. We can assume that the parametrizations φ_0, φ of holomorphic curves M_0 and M correspondingly, are defined on the same disk $D = \Delta(a,r) \subset \mathbb{C}$, and that $u_0 = \varphi_0(a)$.

Define $H_0(\zeta) = Q(\varphi_0(\zeta))$, $H(\zeta) = Q(\varphi(\zeta))$. Then $H_0(a)$=0 and since Q does not vanish identically on $M_0 = \varphi(D)$ the point a is an isolated zero fof H_0. Hence there exists $0 < \varepsilon < r$ such that $\min_{|\zeta - a| = \varepsilon} |H_0(\zeta)| = m > 0$.

Choose $\delta > 0$ so that $|\varphi_0(\zeta) - \varphi(\zeta)| < \delta$, $\zeta \in D$ implies that $|H_0(\zeta) - H(\zeta)| < m$ for $|\zeta - a| = \varepsilon$. Once can provide $\varphi(D_\varepsilon) \subset V$, where $D_\varepsilon = \Delta(a, \varepsilon)$, by taking ε and δ small enough.

Now the Rouché theorem implies that if φ is so close to φ_0 that the above inequalities hold then the holomorphic function H has the same number of zeros in the disk D_ε as the function H_0, and therefore H has at least one zero $\zeta_0 \in D_\varepsilon$. Clearly $\varphi(\zeta_0) \in Q^{-1}(0) \cap M \cap V$. □

3. The case of polynomials.

The case of polynomials is particularly simple.

Proposition 3.1. *Let*

$$f(z) = p_0(z) + p_1(z)\overline{z} + \cdots + p_n(z)\overline{z}^n = P(z, \overline{z}) \tag{3.1}$$

where $p_0,\cdots,p_n$ are polynomials. Let $a\in\mathbb{C}$, $\rho>0$. Assume that $p_n(a)\neq 0$ and that f extends analytically from the circle $\partial\Delta(a,\rho)$. Then f is analytic.

Proof. On $\partial\Delta(a,\rho)$ we have $\overline{z}=\overline{a}+\rho^2/(z-a)$ so by our assumption $P(z,\overline{a}+\rho^2/(z-a))$ has no pole in $\Delta(a,\rho)$. Since $p_n(a)\neq 0$ it follows that $n=0$, that is, f is holomorphic. □

Corollary 3.2. *Let $f(x,y)$ be a polynomial on $\mathbb{R}^2$. If f extends analytically from N, $N\geq \deg f$, circles with different centers $a_1,\cdots,a_N$, then f is holomorphic.*

Proof. If f is not holomorphic then $n>0$ and $\deg p_n\leq \deg f-n<N$, Theorem 3.1 implies that $p_n(a_i)=0$, $i=1\cdots N$ and hence $p_n=0$, which leads to contradiction. □

Consider the example

$$f(z)=(r_1^2-z\overline{z})\cdots(r_k^2-z\overline{z})\overline{z},$$

where $0<r_1<r_2<\cdots<r_k$. The circles from which f extends analytically are precisely $\partial\Delta(0,r_i)$, $1\leq i\leq k$. This shows that the assumption $p_n(a)\neq 0$ cannot be dropped in general. Another example is $f(z)=z\overline{z}$ where the circles from which f extends analytically are $\partial\Delta(0,r)$, $r>0$. It is easy to describe the families of circles from which a function (3.1) extends analytically without being analytic:

Theorem 3.3. *Let $f(x,y)$ be a nonholomorphic polynomial on $\mathbb{R}^2$. There are disjoint finite sets $A,B\subset\mathbb{C}$, $\#(A\cup B)<\deg f$, possibly empty, and for each $a\in A$ there is a finite set R_a, $\#R_a<\deg f$, of positive real numbers such that*

(a) f extends analytically from $\partial\Delta(a,r)$ for each $a\in A$ and for each $r\in R_a$,

(b) f extends analytically from $\partial\Delta(b,r)$ for each $b\in B$ and for each $r>0$,

(c) there are no other circles except the ones described in (a), (b), from which f extends analytically.

Conversely, let $A, B \subset \mathbb{C}$ be disjoint finite sets, possibly empty, and for each $a \in A$, let R_a be a finite set of positive real numbers. Then there is a nonholomorphic polynomial $f(x, y)$ satisfying (a), (b), and (c).

Proof. Associate, as in (2.1), with $f(z) = f(x, y)$, $z = x + iy$, the polynomial $P(z, \overline{z}) = f((z + \overline{z})/2, (z - \overline{z})/2i)$.

Since f is not holomorphic, P is not a polynomial of the variable z only. Write $P(z, \overline{z}) = p_0(z) + p_1(z)\overline{z} + \cdots + p_N(z)\overline{z}^N$. By our assumption, $N \geq 1$ and $p_N \not\equiv 0$. Let S be the set of centers of circles from which f extends analytically. By Proposition 3.1 $S \subset \{z \in \mathbb{C}\colon\ p_N(z) = 0\}$ so S is a finite set, possibly empty.

Let $a \in S$. We want to show that the set R_a of all radii r such that f extends analytically from $\partial\Delta(a, r)$ is either finite or equal to $\mathbb{R}_+$. To see this, suppose that R_a is infinite. Let $n \in \mathbb{N}$. Then

$$\int_0^{2\pi} e^{in\theta} f(a + re^{i\theta}) d\theta = 0 \quad (r \in R_a) \tag{3.2}$$

by the analytic extendibility. However, the integral on the left, $\int_0^{2\pi} e^{in\theta} P(a + re^{i\theta}, \overline{a} + re^{-i\theta}) d\theta$, is a polynomial in r which vanishes on the infinite set R_a so it vanishes identically. It follows that

$$\int_0^{2\pi} e^{in\theta} f(a + re^{i\theta}) d\theta = 0 \quad (n \in \mathbb{N}, r > 0)$$

which implies that $R_a = \mathbb{R}_+$. This completes the proof of the first part.

To prove the second part, let $A = \{a_1, \cdots, a_\alpha\}$, $B = \{b_1, \cdots, b_\beta\}$ be arbitrary disjoint finite sets and for each i, $1 \leq i \leq \alpha$, let $R_i = \{r_{i1}, \cdots, r_{ik_i}\}$.

If A and B are both empty put $f(z) = \overline{z}$. If not, let $k = k_1 + \cdots + k_m$ and set

$$f(z) = \overline{z}\Big[\prod_{i=1}^{\alpha} \prod_{j=1}^{k_i} [r_{ij}^2 - (z - a_i)(\overline{z} - \overline{a}_i)]\Big] \prod_{l=1}^{\beta} (z - b_l)^{k+1}. \tag{3.3}$$

If $z \in \partial\Delta(a, \rho)$ then the substitution $\overline{z} = \overline{a} + \rho^2/(z - a)$ in (3.3) and our choice of k imply that f extends analytically from each circle $\partial\Delta(b_l, r)$, $r > 0$. Note that

f extends from each circle $\partial\Delta(a_i, r_{ij})$, $1 \le i \le \alpha$, $1 \le j \le k_i$, since it vanishes identically on each such circle.

If we write $f(z) = P(z, \overline{z})$ as a polynomial in $\overline{z}$ with the coefficients depending on z then the leading coefficient $p_N(z) = \prod_{i=1}^{\alpha}(z-a_i)^{k_i}\prod_{l=1}^{\beta}(z-b_l)$ has zeros only at the points a_i and b_l and Proposition 3.1 implies that there are no circles with other centers than $a_1, \cdots, a_\alpha, b_1, \cdots, b_\beta$ from which f extends analytically.

Consider the circle $\partial\Delta(a_1, \rho)$ where $\rho \neq r_{1j}, 1 \le j \le k_1$ and let $z \in \partial\Delta(a_1, \rho)$. If $z \in \partial\Delta(a_1, \rho)$ then $\overline{z} = \overline{a}_1 + \rho^2/(z - a_1)$ and (3.3) yields :

$$f(z) = c\big((\overline{a_1} + \rho^2/(z-a_1)\big)\prod_{i=2}^{\alpha}\prod_{j=1}^{k_i}[r_{ij}^2 - (z-z_i)(\overline{a_1} - \overline{a_i} + \rho^2/(z-a_1)]\prod_{l=1}^{\beta}(z-b_l),$$

where c is a positive constant. Since $a_1 \neq a_i$ $(2 \le i \le m)$ each factor in the bracket has a pole at a_1 which means that f does not extend analytically from $\partial\Delta(a_1, \rho)$. The same reasoning applies for a_i in place of a_1. This proves (c). $\square$

4. Analytic extension into a single disk.

We now pass to rational functions of real variables x, y. We begin with the condition for analyticity of a rational function f on $\mathbb{R}^2$ in terms of analytic extendibility from a single circle.

Observe that given a and $\rho > 0$ any rational function $f(z) = P(z, \overline{z})/Q(z, \overline{z})$ has the unique meromorphic extension f^* into the disk $\Delta(a, \rho)$, defined by the inversion around the boundary circle:

$$f^*(z) = \frac{P(z, \overline{a} + \rho^2/(z-a))}{Q(z, \overline{a} + \rho^2/(z-a))}, \tag{4.1}$$

so to say that f extends analytically from $\partial\Delta(a, \rho)$ is the same as to say that f^* has no singularities (poles) in $\Delta(a, \rho)$.

Given $a \in \mathbb{C}$, $\rho > 0$, we introduce the manifold in $\mathbb{C}^2$:

$$\Lambda_{a,\rho} = \{(z, w) \in \mathbb{C}^2 : (z-a)(w-\overline{a}) = \rho^2,\ 0 < |z-a| < \rho\}. \tag{4.2}$$

Note that $\Lambda_{a,\rho}$ is a closed complex submanifold of $\mathbb{C}^2 \setminus \{(z,\overline{z}):\ z \in \mathbb{C}\}$ which is attached to the real 2-plane $\{(z,\overline{z}):\ z \in \mathbb{C}\}$ in $\mathbb{C}^2$ along the circle $\{(z,\overline{z}):\ z \in \partial\Delta(a,\rho)\}$. It is one of the two components of the complement of the circle $\{(z,\overline{z}):\ z \in \partial\Delta(a,\rho)\}$ in the complex quadric (complex hyperboloid) $M_\rho = \{(z,w) : (z-a)(w-\overline{a}) = \rho^2\}$ in $\mathbb{C}^2$. We shall call $\Lambda_{a,\rho}$ *the manifold associated with the circle* $\partial\Delta(a,\rho)$, and we shall call the circle $\partial\Delta(a,\rho)$ *the circle associated with the manifold* $\Lambda_{a,\rho}$.

The analytic extension of a rational function $f(x,y)$ from the circle $\partial\Delta(a,\rho)$ into the disk $\Delta(a,\rho)$ is given by the restriction of of the rational function P/Q of two complex variables z,w to the complex manifold $\Lambda_{a,\rho}$, as it is seen from (4.1).

The following theorem is a key one in all our considerations of rational functions in the sequel.

Theorem 4.1. *Let f be a rational function (2.1), having analytic extension from a circle $\partial\Delta(a,\rho)$. Suppose that*

(a) $Q(z,\overline{z}) \neq 0$ *for* $z \in \overline{\Delta}(a,\rho)$,

(b) $p_n(a) \neq 0$, $q_m(a) \neq 0$, *where* p_n, q_m *are as in* (2.2),

(c) *the polynomials $P(z,w)$ and $Q(z,w)$ have no common zero on $\Lambda_{a,\rho}$.*

Then f is an analytic rational function of z with poles outside of the closed disk $\overline{\Delta}(a,\rho)$.

The proof is based on the analysis of the intersections of the null-variety Z_Q with the manifolds $\Lambda_{a,t}$ when t decreases from ρ to 0.

Proof. Let us first describe the idea of the proof in more geometric terms (see Fig.1). Observe first that our assumptions imply that Q does not vanish at any point of $\Lambda_{a,\rho}$. Indeed, the analytic extendibility of f from $\partial\Delta(a,\rho)$ implies that f_* has no singularity in $\Delta(a,\rho)$. Thus, if Q vanishes at some point of $\Lambda_{a,\rho}$, P must vanish at this point to cancel the pole produced by the zero of $Q(z,\overline{a}+\rho^2/(z-a))$ which contradicts (c).

For a large $R > 0$ (to be chosen below) consider the smooth foliation $M_t,\ 0 \leq t \leq \rho$, where

$$M_t = \Lambda_{a,t} \cap \{(z,w)\colon\ |w-\overline{a}| < R\}\ \ (0 < t \leq \rho)$$

and

$$M_0 = \{(a,w)\colon\ 0 < |w-\overline{a}| < R\}.$$

The boundary of each leaf $M_t,\ 0 \leq t \leq \rho$, has two components, one of which is $\Lambda_{a,t} \cap \{(z,w)\colon\ |w-\overline{a}| = R\}$, and the other is $\{(z,\overline{z})\colon\ |z-a| = t\}$. On the second boundary component of $M_t,\ 0 \leq t \leq \rho$, the polynomial Q is different from zero because of (a).

We prove that $Q(z,\overline{z})$ is a holomorphic polynomial which is equivalent to $Q(z,w)$ being a polynomial of z only. Then the holomorphicity of the polynomial $P(z,\overline{z}) = f(z)Q(z,\overline{z})$ follows, for instance from Proposition 3.1 or directly from the condition (b).

Suppose that $Q(z,w)$ does not depend only on z, that is, that $m \geq 1$. Since $q_m(a) \neq 0$ it follows that there are $R < \infty$ and $\delta > 0$ such that Z_Q misses $\{(z,w)\colon\ |z-a| \leq \delta,\ |w-\overline{a}| \geq R\}$. Passing to a larger R if necessary we may assume that for each $t,\ 0 \leq t \leq \rho$, Z_Q misses $\Lambda_{a,t} \cap \{(z,w)\colon\ |w-\overline{a}| = R\}$, the first boundary component of M_t. By Lemma 2.1 it follows that either Z_Q meets M_t for each $t,\ 0 \leq t \leq \rho$, or Z_Q misses M_t for each $t,\ 0 \leq t \leq \rho$.

However, since $m \geq 1$ and $q_m(a) \neq 0$ it follows that $w \mapsto Q(a,w)$ is a nonconstant polynomial so there is some w_0 such that $Q(a,w_0) = 0$, that is, Z_Q meets M_0. By the preceding discussion it follows that Z_Q meets M_t for each $t,\ 0 \leq t \leq \rho$. In particular, Z_Q meets $\Lambda_{a,\rho}$, contradicting our assumptions. This proves that $m = 0$, that is, that $Q(z,w)$ depends only on z.

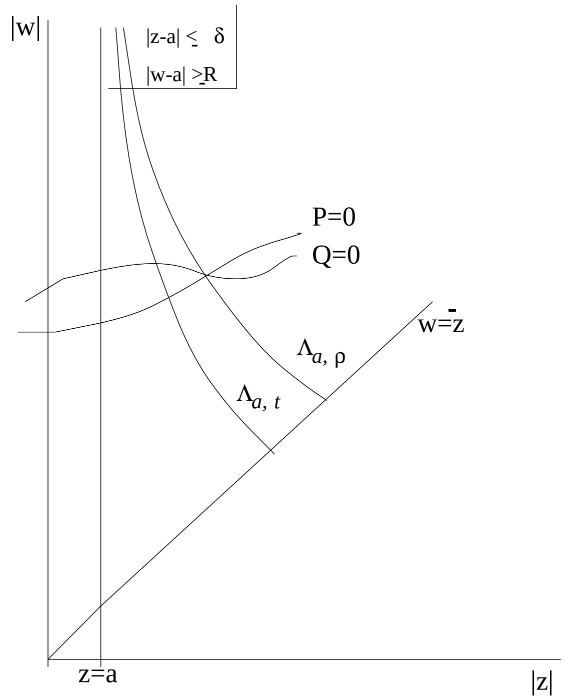

Fig.1

Now we turn to the precise analytic proof. Without loss of generality, we assume that $a = 0$.

For any $t \in (0, \rho]$ denote by f_t^* the meromorphic (rational) extension of $f\big|_{\partial\Delta_t}$ into the disk Δ_t :

$$f_t^*(z) = \frac{P(z, t^2/z)}{Q(z, t^2/z)} = z^{m-n}\frac{P_t^*(z)}{Q_t^*(z)},$$

where we have denoted

$$P_t^*(z) = z^n P(z, t^2/z) = \sum_{k=0}^{n} p_k(z) t^{2k} z^{n-k},$$

$$Q_t^*(z) = z^m Q(z, t^2/z) = \sum_{k=0}^{m} q_k(z) t^{2k} z^{m-k},$$

and where $p_k(z)$, $q_k(z)$ are the polynomial coefficients in (2.2).

If $z \in \partial\Delta_t$, $z\overline{z} = t^2$, then $Q_t^*(z) = z^m Q(z, \overline{z}) \neq 0$, due to (a), and the logarithmic residue

$$N_Q(t) = \frac{1}{2\pi i}\int_{|z|=t} \frac{(Q_t^*)'(z)}{Q_t^*(z)} dz = \#\{Q_t^*(z) = 0,\ z \in \Delta_t\}$$

is defined.

We will start with the leaf M_ρ rather than with the singular leaf M_0 as in the geometric explanation above. Since, by our assumption, f_ρ^* has no pole in Δ_ρ, condition (c) implies that $Q(z,\rho^2/z)$ does not vanish at any point z in the punctured disk Δ_ρ^* and therefore Q_ρ^* has the same property. Since $Q_\rho^*(0) = q_m(0)\rho^{2m} \neq 0$ it follows that Q_ρ^* has no zero in Δ_ρ. Hence $N_Q(\rho) = 0$.

The function $t \mapsto N_Q(t)$ is continuous and integer-valued, and therefore constant, so $N_Q(t) = N_Q(\rho) = 0$ for all $t \in (0,\rho]$, which means that for each $t \in (0,\rho]$, the function Q_t^* has no zero in the closed disk $\overline{\Delta_t}$.

Now assume that $Q(z,\overline{z})$ is not holomorphic, that is, $m \geq 1$ in the decomposition (2.2). Since $q_m(0) \neq 0$, the polynomial $w \mapsto Q(0,w)$ is of degree precisely m and thus has $m \geq 1$ roots. None of them equals 0 due to (a). Let $w_0 \neq 0$ be one of the roots.

Introduce the rational function

$$R(t,w) = Q(t^2/w, w),$$

which is holomorphic in $\mathbb{C} \times (\mathbb{C} \setminus \{0\})$ and satisfies $R(0,w_0) = 0$. Let $\varepsilon > 0$ be such that $R(0,w) \neq 0$ for $0 < |w - w_0| \leq \varepsilon$. The argument principle yields that the equation $R(t,w) = 0$ has a solution $w_t \in \Delta(w_0,\varepsilon)$ when t is close to 0. Indeed, take $\delta \in (0,\rho)$ so small that $R(t,w) \neq 0$ for $(t,w) \in (0,\delta) \times \partial\Delta(w_0,\varepsilon)$. Then the logarithmic residue

$$N_R(t) = \frac{1}{2\pi i} \int_{|w-w_0|=\varepsilon} \frac{R'_w(t,w)}{R(t,w)} dw$$

is defined for $t \in [0,\delta)$, is continuous in t and is an integer and therefore is constant.

Then for $t \in (0,\delta)$ we have

$$N_R(t) = N_R(0) = \#\{w \in \Delta(w_0,\varepsilon) : R(0,w) = 0\} > 0$$

which says that there is $w_t \in \Delta(w_0,\varepsilon)$ such that $R(t,w_t) = 0$.

Since $w_0 \neq 0$, ε and δ can be taken so small that $|w_t| > t$ as soon as $t \in (0, \delta)$. Denote $z_t = t^2/w_t$. Then $Q(z_t, w_t) = R(t, w_t) = 0$ and $Q_t^*(z_t) = z_t^m Q(z_t, w_t) = 0$. However, $|z_t| = t^2/|w_t| < t$ and we arrive at a contradiction with Q_t^* having no zero in Δ_t.

Thus, $m = 0$ and Q is a holomorphic polynomial. But then the polynomial $P = fQ$ extends analytically into the disk $\Delta(a, \rho)$ and Proposition 3.1 tells that P is holomorphic as well. This completes the proof.

Examples.

1. Let $f(z) = z^2/\overline{z}$ $(z \neq 0)$, $f(0) = 0$. This is everywhere continuous rational function on $\mathbb{R}^2$, possessing analytic extension from $|z| = 1$ into $|z| < 1$ by means of the function $f^*(z) = z^3$. Condition (a) fails while (b) and (c) hold. Clearly the function f is not analytic.

2. The functions $f(z) = z^n \overline{z}^m + 2$, $f(z) = 1/(z^n \overline{z}^m + 2)$, $n \geq m$, are simple examples of nonholomorphic rational functions, having analytic extension from the unit circle, and satisfying (a), (c) but not (b).

3. Let $f(z) = (z - \alpha)/(\overline{z} - 1/\alpha)$, $0 < |\alpha| < 1$, so that $P(z, w) = z - \alpha$, $Q(z, w) = w - (1/\alpha)$. The conditions (a) and (b) are fulfiled for $a = 0$ and $\rho = 1$, while (c) fails to hold as $P(\alpha, 1/\alpha) = Q(\alpha, 1/\alpha) = 0$ and $(\alpha, 1/\alpha) \in \Lambda_{0,1}$. If $|z| = 1$, then $f(z) = -\alpha z$ and so f extends analytically into $|z| < 1$, yet f is not analytic.

5. Analytic extensions into finite families of disks.

We start with finite family of concentric disks.

Theorem 5.1. *Let f be a rational function* (2.1) *on* $\mathbb{R}^2$ *and* $a \in \mathbb{C}$, $0 < \rho_1 < \cdots < \rho_N$.

Assume that

(a) $Q(z, \overline{z}) \neq 0$ *for* $z \in \overline{\Delta}(a, \rho_N)$

(b) $p_n(a) \neq 0$, $q_m(a) \neq 0$.

If for each $i = 1, \ldots, N$, *the function* $f\big|\partial\Delta(a, \rho_i)$ *extends analytically into the disk* $\Delta(a, \rho_i)$, *and if* $N > \deg P \cdot \deg Q$, *then* f *is an analytic rational function with poles outside of* $\overline{\Delta}_{\rho_N}$.

Proof. Since P and Q are assumed coprime, the intersection $Z_P \cap Z_Q$ consists of at most $\deg P \cdot \deg Q$ points by the Bezout theorem [GH]. By the condition for N there exists $j = 1, \ldots, N$ such that $\rho_j^2 \neq (z_\nu - a)(w_\nu - \overline{a})$ for any $(z_\nu, w_\nu) \in Z_P \cap Z_Q$. Hence $P(z, w)$ and $Q(z, w)$ have no common zero on $\Lambda_{a,\rho_j} = \{(z - a)(w - \overline{a}) = \rho_j^2,\ 0 < |z - a| < \rho_j\}$ so all the conditions of Theorem 3.1 are satisfied with $\rho = \rho_j$. □

Example. The following example shows that the condition $N > \deg P \cdot \deg Q$ is sharp. Choose a large $R > 0$. Let $w_0 > 0$ and let $0 < z_0 < R^2/w_0$. Then $(z_0, w_0) \in \Lambda_{0,\rho}$ for some ρ, $0 < \rho < R$.

Let $L_0 = \{(z, w)\colon\ w - w_0 = 0\}$ and let E_0 be a slight perturbation of L_0 of the form $\alpha z + \beta w + \gamma = 0$ with α, β, γ real, $\alpha \neq 0$, which passes through (z_0, w_0) and does not coincide with L_0.

Let $\mu, \nu \in \mathbb{N}$. Since $\Lambda_{0,r} \cap \mathbb{R}^2$ is a foliation whose leaves, for r near ρ, are transverse to both $L_0 \cap \mathbb{R}^2$ and $E_0 \cap \mathbb{R}^2$, it is easy to find μ distinct lines

$$L_i = \{(z, w)\colon\ w - w_i = 0\}, 1 \leq i \leq \nu,$$

with w_i real, which are parallel and close to L_0, and ν distinct lines

$$E_i = \{(z, w)\colon\ \alpha z + \beta w + \gamma_j = 0\},\quad 1 \leq j \leq \mu,$$

with γ_j real, which are parallel and close to E_0, such that $L_i \cap E_j$ and $L_{i'} \cap E_{j'}$ lie in different leaves $\Lambda_{0,r}$ whenever $(i, j) \neq (i', j')$. Let $\Lambda_{0,r_{ij}}$ contain the interesection point $(z_{ij}, w_{ij}) \in L_i \cap E_j$, i.e. $r_{ij}^2 = z_{ij} w_{ij}$.

Put

$$f(z) = \frac{\prod_{j=1}^{\mu} (\alpha z + \beta \overline{z} + \gamma_j)}{\prod_{i=1}^{\nu} (\overline{z} - w_i)}. \tag{5.1}$$

Chosing $w_i > R$, $1 \le i \le \nu$, where R is large enough, we provide that the denominator in (5.1) does not vanish on any closed disk $\overline{\Delta(0, r_{ij})}$. Note also that the leading coefficients in the decompositions of P and Q in powers of $\overline{z}$ do not vanish at 0, that is, $p_\nu(0) \ne 0$, $q_\mu(0) \ne 0$ so both conditions (a) and (b) are fullfiled.

It is easy to verify, by substituting $\overline{z} = r_{ij}^2/z$ into (5.1), that f extends analytically to $\partial\Delta(0, r_{ij})$ for each i, j, $1 \le i \le \mu$, $1 \le j \le \nu$. However, f is not analytic. In this example, the number of circles from which f extends analytically is $\nu\mu = \deg P \cdot \deg Q$.

The following theorem treats the case of different centers and equal radii.

Theorem 5.2. *Let $\rho > 0$ and assume that a_i, $1 \le i \le N$, are complex numbers such that $a_i \ne a_j$ if $i \ne j$. Write $\Omega = \bigcup_{i=1}^{N} \Delta(a_i, \rho)$.*

Let f be a rational function (2.1) in $\mathbb{R}^2$ such that for each $i, 1 \le i \le N$, the function $f\big|\partial\Delta_{\rho_i}$ extends analytically to $\Delta(a_i, \rho)$. Suppose that

(a) $Q(z, \overline{z}) \ne 0$ *for* $z \in \overline{\Omega}$

(b) $N > \deg p_n + \deg q_m + \deg P \cdot \deg Q$,

where p_n, q_m are the leading coefficients in (2.2). Then f is an analytic rational function with poles outside of $\overline{\Omega}$.

Proof. Observe first that if $w - \overline{z} = Re^{i\omega}$ with $R > 0$ then

$$a = z - e^{-i\omega}(R + (R^2 + 4\rho^2)^{1/2})/2$$

is the unique a such that $(z, w) \in \Lambda_{a,\rho}$.

Let $S = \{z \in \mathbb{C} : p_n(z)q_m(z) = 0\}$ be the union of sets of zeros of p_n and q_m and $Z = Z_P \cap Z_Q \cap (\overline{\Omega} \times \mathbb{C})$.

Let $(z, w) \in Z$. Then $w \ne \overline{z}$ since $Q(z, \overline{z}) \ne 0$ for $z \in \overline{\Omega}$ and therefore, as we have observed above, $(z, w) \in \Lambda_{a_i,\rho}$ for at most one i. Since $N > \deg p_n + \deg q_m + \deg P \cdot$

$\deg Q$ and $\#S \leq \deg p_n + \deg q_m$, it follows that at least $\deg P \cdot \deg Q + 1$ points a_k are not in S. Among them there is at least one, a_{k_0}, such that $\Lambda_{a_{k_0},\rho} \cap Z = \emptyset$, since due to the Bezout theorem $\#Z \leq \deg P \cdot \deg Q$. Now one uses Theorem 3.1 with $a = a_{k_0}$. $\square$

We do not know whether the estimate (b) for the number of disks is sharp.

Remark. *The condition (b) in Theorem 5.2 can be replaced by* $N \geq \deg P + \deg Q + \deg P \cdot \deg Q$.

Indeed, if $\deg p_n = \deg P$ and $\deg q_m = \deg Q$ then both P and Q depend only on z and so f is analytic. Otherwise (b) holds and f is analytic as well by Theorem 5.2.

The following corollaries tell that any infinite family of circles with either equal centers or equal radii tests holomorphicity of rational functions:

Corollary 5.3. *Let* $\rho > 0$ *and let* $A \subset \mathbb{C}$ *be an infinite set. Write* $\Omega = \bigcup_{a\in A} \Delta(a,\rho)$. *Then every rational function (2.1) such that* $Q(z,\overline{z}) \neq 0$ $(z \in \overline{\Omega})$ *and such that for every* $a \in A$ *the function* $f|\partial\Delta_{a,\rho}$ *extends analytically into* $\Delta(a,\rho)$, *is a rational analytic function with poles outside of* $\overline{\Omega}$.

Corollary 5.4. *Let* $a \in \mathbb{C}$ *and let* S *be an infinite set of positive numbers. Write* $\Omega = \bigcup_{\rho\in S} \Delta(a,\rho)$. *Then every rational function (2.1) such that* $Q(z,\overline{z}) \neq 0$ $(z \in \overline{\Omega})$ *and such that for every* $\rho \in S$ *the function* $f|\partial\Delta_{a,\rho}$ *extends analytically into* $\Delta(a,\rho)$, *is a rational analytic function with poles outside of* $\overline{\Omega}$.

6. Family of manifolds $\Lambda_{a,\rho}$ containing a given point.

Recall that $\Lambda_{a,\rho} = \{(z,w)\colon\ (z-a)(w-\overline{a}) = \rho^2,\ |z-a| < \rho\}$.

Lemma 6.1. *Let* $z, w \in \mathbb{C},\ w \neq \overline{z}$. *Then* $(z,w) \in \Lambda_{a,\rho}$ *if and only if there is a* $t > 0$ *such that* $a = z + t(z - \overline{w})$ *and* $\rho = \sqrt{t(t+1)}|z - \overline{w}|$.

Proof. The inclusion $(z,w)\in\Lambda_{a,\rho}$ means that $(z-a)(w-\overline{a})=\rho^2$ and $|z-a|<\rho$. The equality holds if and only if

$$a=z+t(z-\overline{w}),\quad \text{where}\ \ t=(\rho^2-|z-a|^2)/|z-\overline{w}|^2$$

and the inequality $|z-a|<\rho$ is equivalent to $t>0$. The formula for ρ follows by substituting the expression for a in the equation $(z-a)(w-\overline{a})=\rho^2$. □

Lemma 6.2. *Let (z_1,w_1), (z_2,w_2) be two distinct points in $\mathbb{C}^2$ such that $w_1\neq\overline{z_1}$, $w_2\neq\overline{z_2}$. There is at most one manifold $\Lambda_{a,\rho}$ which contains both (z_1,w_1) and (z_2,w_2).*

Proof. Suppose that $(z_1,w_1),(z_2,w_2)\in\Lambda_{a,\rho}\cap\Lambda_{b,r}$. Recall that $|z_1-a|<\rho$ and $|z_2-a|<\rho$. By complex conjugation we obtain another pair $(\overline{w}_1,\overline{z}_1)$, $(\overline{w}_2,\overline{z}_2)$ of points in $\mathbb{C}^2$ that belong to the same quadrics $(z-a)(w-\overline{a})=\rho^2$ and $(z-b)(w-\overline{b})=r^2$.

Since these two quadrics, assuming they are diferent, have at most two common points, it follows that $(a,\rho)\neq(b,r)$ would imply that there are not more than two distinct points among the four ones above. By the assumption, $(z_1,w_1)\neq(z_2,w_2)$ and $(z_1,w_1)\neq(\overline{w}_1,\overline{z}_1)$, $(z_2,w_2)\neq\overline{w_2},\overline{z_2})$, so necessarily $(z_1,w_1)=(\overline{w}_2,\overline{z}_2)$. This is not possible since it would imply that $\rho^2=|(z_1-a)(w_1-\overline{a})|=|(z_1-a)(\overline{z}_2-\overline{a})|<\rho^2$. Hence $a=b,\ \rho=r$. □

By Lemma 6.1 the family of manifolds $\Lambda_{a,\rho}$ (4.2) in C^2 containing a given point (z,w) determines the family of associated circles in the plane:

$$\text{(6.1.)}\qquad \mathcal{R}_{z,w}=\{\partial\Delta(a,\rho)\colon\ a=z+t(z-\overline{w}),\ \rho=\sqrt{t(t+1)}|z-\overline{w}|,\ t>0\}$$

The circles $\{(\zeta,\overline{\zeta})\colon\zeta\in\mathcal{C}\}$, $\mathcal{C}\in\mathcal{R}_{z,w}$, are the boundaries of the complex submanifolds $\Lambda_{a,\rho}$. They are intersections of the plane $\{(\zeta,\overline{\zeta})\colon\zeta\in\mathbb{C}\}$ with the family of quadrics $(z-a)(w-\overline{a})=\rho^2$ which pass through the point (z,w).

We will call the family $\mathcal{R}_{z,w}$ *the ray of circles associated with the point* (z,w). Each circle of such a ray surrounds the point z which we call *the vertex of the ray* $\mathcal{R}_{z,w}$. Let L be the line passing through the midpoint of the segment joining the points z and $\overline{w}$, and perpendicular to the segment. Let Π be the halfplane bounded by L and containing the point z. It is easy to see that Π is the union of all circles belonging to the ray $\mathcal{R}_{z,w}$ of circles associated with (z,w). Note also that (6.1) with $t<-1$ gives $\mathcal{R}_{\overline{w},\overline{z}}$.

7. Analytic extensions into infinite sequences of disks.

We have seen above that an infinite family of circles of fixed radius or of fixed center is sufficient to test holomorphy of any rational function. However, it is not true that every infinite family of circles has this property. We illustrate this by an example.

Example. Let

$$f(z)=\frac{z}{\overline{z}+1},$$

$$\rho(t)=\sqrt{t(t+1)},\ t\in I=(0,\infty).$$

Since $t-\rho(t)>-1$ we have $-1\notin\overline{\Delta}(t,\rho(t))$ so $Q(z,\overline{z})=\overline{z}+1$ does not vanish on $\overline{\Delta}(t,\rho(t)),\ t\in I$.

The function f extends analytically from any circle $\Delta(t,\rho(t))$, $t\in I$. Indeed, the extension is given by

$$f_t^*(z)=\frac{z}{t+(t(t+1)/(z-t))+1}=\frac{z(z-t)}{z(t+1)}=\frac{z-t}{t+1}.$$

However, f is not analytic.

Observe that for every $t\in I$ the relation $(z-t)(w-t)=\rho^2(t)$ holds with $z=0$, $w=-1$. Since $0\in\Delta(t,\rho(t))$ for each $t\in I$ it follows that all the varieties $\Lambda_{t,\rho(t)}$, $t\in I$ pass through the point $(0,-1)$ which is the common zero of the polynomials $P(z,w)=z$ and $Q(z,w)=w+1$.

The following theorem shows that this is the only case when nonanaliticity can occur. More specifically, it states that if a rational function $f = P/Q$ extends analytically from each circle belonging to an infinite family of circles whose cluster point is a circle $\partial\Delta(a,\rho)$ then, provided that $p_n(a) \neq 0$, $q_m(a) \neq 0$, this family is very special near $\partial\Delta(a,\rho)$ - all circles in the family sufficiently close to the circle $\partial\Delta(a,\rho)$ belong to a ray $\mathcal{R}_{z_0,w_0}$ defined in (6.1), where (z_0,w_0) is a common zero of P and Q, and, moreover, the function f extends analytically from all the circles in $\mathcal{R}_{z_0,w_0}$ as long as the denominator Q does not vanish on the closed disks bounded by these circles.

Theorem 7.1. *Let f be the rational function (2.1). Let $a_0 \in \mathbb{C}$ and let $\rho_0 > 0$. Suppose that*

(a) $Q(z,\overline{z}) \neq 0$ for $z \in \overline{\Delta}(a_0,\rho_0)$

(b) $p_n(a_0) \neq 0$, $q_m(a_0) \neq 0$

(c) there are $(a,\rho) \in \mathbb{C}\times\mathbb{R}_+$, $(a,\rho) \neq (a_0,\rho_0)$, arbitrarily close to (a_0,ρ_0) and such that f extends analytically from $\partial\Delta(a,\rho)$.

Suppose that f is not analytic. Then

1) the polynomial Q has precisely one zero (z_0,w_0) on the manifold Λ_{a_0,ρ_0} which is also a zero of the polynomial P,

2) if (a,ρ) is sufficiently close to (a_0,ρ_0) then f extends analytically from $\partial\Delta(a,\rho)$ if and only if the circle $\partial\Delta(a,\rho)$ belongs to the ray $\mathcal{R}_{z_0,w_0}$ of circles associated with (z_0,w_0), that is, if and only if there is a $t > 0$ such that $a = z_0 + t(z_0 - \overline{w_0})$ and $\rho = \rho(a) = |z_0 - \overline{w_0}|\sqrt{t(t+1)}$,

3) if $L = \{z_0 + t(z_0 - \overline{w_0}):\ t > 0\}$ and $T \subset L$ is an open segment containing a_0 such that $Q(z,\overline{z}) \neq 0$ $(z \in \overline{\Delta}(a,\rho(a))$ for each $a \in T$, then f extends analytically from $\partial\Delta(a,\rho(a))$ for each $a \in T$.

Proof. Denote by $\mathcal{A}$ the set of all (a,ρ) such that f extends analytically from $\partial\Delta(a,\rho)$.

By continuity, (c) implies that $(a_0, \rho_0) \in \mathcal{A}$. If $(a, \rho) \in \mathcal{A}$ then the function f^* from (4.1) has no singularity in $\Delta(a, \rho)$. In particular, if $(z, w) \in \Lambda_{a,\rho} \cap Z_Q$ then $(z, w) \in Z_P$.

Since f is not holomorphic, Theorem 4.1 implies that there is a point $(z_0, w_0) \in \Lambda_{a_0,\rho_0} \cap Z_P \cap Z_Q$. By (a), Q does not vanish identically on Λ_{a_0,ρ_0} so one can apply Lemma 2.1 or use the argument principle directly to show that there are points in $\Lambda_{a,\rho} \cap Z_Q$ arbitrarily close to (z_0, w_0) provided that (a, ρ) is sufficiently close to (a_0, ρ_0).

Since P and Q have no common factors, their common zeros are isolated. In particular, (z_0, w_0) is an isolated point of $Z_P \cap Z_Q$. Thus, if $(a, \rho) \in \mathcal{A}$ is sufficiently close to (a_0, ρ_0) then the points of $\Lambda_{a,\rho} \cap Z_Q$ close to (z_0, w_0) must necessarily coincide with (z_0, w_0), that is, $(z_0, w_0) \in \Lambda_{a,\rho}$, so the circle $\partial\Delta(a, \rho)$ belongs to the ray $\mathcal{R}_{z_0,w_0}$ (6.1) of circles associated with (z_0, w_0).

Now suppose that $\Lambda_{a_0,\rho_0} \cap Z_Q$ contains another point $(z_1, w_1) \neq (z_0, w_0)$. As above, we have $(z_1, w_1) \in \Lambda_{a,\rho}$ whenever $(a, \rho) \in \mathcal{A}$ is sufficiently close to (a_0, ρ_0). By (c), this contradicts Lemma 6.2. Consequently, $\Lambda_{a_0,\rho_0} \cap Z_Q$ consists only of point (z_0, w_0). By this, the assertions 1) and 2) are proved.

Write $a(t) = z_0 + t(z_0 - \overline{w_0})$ $(t > 0)$ and let $\rho(t) = \sqrt{t(t+1)}|z_0 - \overline{w_0}|$. According to 2), there is a $t_0 > 0$ such that $a_0 = a(t_0)$ and $\rho_0 = \rho(t_0)$, and there is a sequence t_n converging to t_0, $t_n \neq t_0$ $(n \in \mathbb{N})$, such that for each $n \in \mathbb{N}$, the function

$$\zeta \mapsto f(a(t_n) + \zeta\rho(t_n))$$

extends analytically from $\partial\Delta$, that is, for each $m \in \mathbb{N}$,

$$\int_0^{2\pi} e^{im\theta} f(a(t_n) + \rho(t_n)e^{i\theta})d\theta = 0 \quad (n \in \mathbb{N}) \tag{7.1}$$

By our assumption the function $(z, w) \mapsto F(z, w) = P(z, w)/Q(z, w)$ is analytic in a neighbourhood of $\{(z, \overline{z}): z \in \overline{\Delta}(a_0, \rho_0)\}$ which implies that there is a neighbourhood W of $\{(a_0, \overline{a_0})\} \times \{(z, \overline{z}):\ z \in \overline{\Delta}_{\rho_0}\}$ in $\mathbb{C}^4$ such that for each $m \in \mathbb{N}$ the function

$$(z_1, z_2, w_1, w_2) \mapsto \int_0^{2\pi} e^{im\theta} F(z_1 + w_1 e^{i\theta}, z_2 + w_2 e^{-i\theta}) d\theta$$

is analytic on W. It follows that there is an open segment $I \subset \mathbb{R}_+$ containing t_0 such that for each $m \in \mathbb{N}$ the function

$$t \mapsto \int_0^{2\pi} e^{im\theta} F(a(t) + \rho(t)e^{i\theta}, \overline{a(t)} + \rho(t)e^{-i\theta}) d\theta = \int_o^{2\pi} e^{im\theta} f(a(t) + \rho(t)e^{i\theta}) d\theta$$

is real-analytic on I. By (7.1) it follows that this function vanishes identically on I. As this holds for every $m \in \mathbb{N}$ it follows that for each $t \in I$ the function $\zeta \mapsto f(a(t) + \rho(t)\zeta)$ extends analytically from $b\Delta$.

Finally, the preceding discussion shows that if $A = \{a \in L\colon\ Q(z, \overline{z}) \neq 0\ , z \in \overline{\Delta}(a, \rho(a))\}$ then for each $m \in \mathbb{N}$ the function

$$a \mapsto \int_0^{2\pi} e^{im\theta} f(a + \rho(a)e^{i\theta}) d\theta$$

is real-analytic on A. As it vanishes identically on an open neighbourhood of a_0 in L it vanishes identically on the component of A which contains a_0. $\square$

Remark. In the example in the beginning of this section $I = L = (0, \infty)$ is the maximal open segment in L satisfying the assumptions in the theorem, and the family of circles is $\mathcal{R}_{0,-1}$.

In fact, we shall see below that any ray in $\mathbb{C}$ can be a maximal segment of centers of circles from which a nonholomorphic rational function (2.1), such that $Q(z, \overline{z}) \neq 0$ $(z \in \mathbb{C})$, extends analytically.

8. Analytic extensions into familes of discs depending continuously on a parameter.

In Section 7 we have seen that there are a nonanalytic function f of the form (2.1), a segment K and a positive continuous function ρ on K such that

$$Q(z, \overline{z}) \neq 0 \quad (z \in \cup_{a \in K} \overline{\Delta}(a, \rho(a))), \tag{8.1}$$

(8.2) $\qquad f$ extends analytically from $\partial\Delta(a,\rho(a))$ for each $a\in K$.

The following theorem tells that closed segments are the only continua with this property.

Theorem 8.1. *Let $K\subset\mathbb{C}$ be a continuum (i.e. a compact connected set consisting of more than one point). Suppose that there are a positive continuous function ρ on K and a nonanalytic rational function (2.1) such that (8.1) and (8.2) hold. Then there are $z,w\in\mathbb{C}$, $w\neq\overline{z}$, and a closed segment I such that $K=\{z+t(z-\overline{w}): t\in I\}$ and $\rho(z+t(z-\overline{w}))=\sqrt{t(t+1)}|z-w|$ $(t\in I)$. In particular, K is a closed segment.*

Proof. Part 1. We first show that away from zeros of p_n and q_m, K is locally contained in a segment.

Let $S=\{z\in\mathbb{C}\colon\ p_n(z)=0\}\cup\{z\in\mathbb{C}\colon\ q_m(z)=0\}$. The set S is finite. Let $\zeta\in K\setminus S$. Since K has no isolated points, ζ is a limit of points from K different from ζ. By (8.1), $Q(z,\overline{z})\neq 0$ $(z\in\overline{\Delta}(\zeta,\rho(\zeta)))$, so Theorem 7.1 applies and we get the following

Proposition 8.2. *If $\zeta\in K\setminus S$ then there are $(z(\zeta),w(\zeta))\in Z_P\cap Z_Q$, $w(\zeta)\neq\overline{z(\zeta)}$, and a neighbourhood $U(\zeta)$ of ζ in $\mathbb{C}$ such that $U(\zeta)\cap K\subset L(\zeta)=\{z(\zeta)+t(z(\zeta)-\overline{w(\zeta)})\colon\ t>0\}$ and $\rho\equiv r_\zeta$ on $U(\zeta)\cap K$ where $r_\zeta(z(\zeta)+t(z(\zeta)-\overline{w(\zeta)}))=\sqrt{t(t+1)}|z(\zeta)-\overline{w(\zeta)}|$ $(t>0)$.*

Part 2. If $E\subset K\setminus S$ is an open segment we show that there is a neighbourhood of E in $\mathbb{C}$ containing no other points of K and we examine ρ on E.

Assume that $E\subset K$ is an open segment which contains no point of S. Applying Proposition 8.2 at each point of E we conclude that ρ is real-analytic on E. Fixing a point ζ_0 in E and applying Proposition 8.2 again we conclude that there are $(z_0,w_0)\in Z_P\cap Z_Q$, $w_0\neq\overline{z_0}$, and a neighbourhood $U(\zeta_0)$ of ζ_0 in $\mathbb{C}$ such that $U(\zeta_0)\cap K\subset L=\{z_0+t(z_0-\overline{w_0})\colon\ t>0\}$ and $\rho\equiv r$ on $U(\zeta_0)\cap K$ where $r(z_0+t(z_0-\overline{w_0}))=\sqrt{t(t+1)}|z_0-\overline{w_0}|$ $(t\geq 0)$. Since both r and ρ are real-analytic on $E \cap L$ and coincide on an open segment containing ζ_0 it follows that $\rho \equiv r$ on $E \cap L$ and so $\rho \equiv r$ on $\overline{E} \cap L$ by the continuity. However, the endpoint of L cannot be contained in $\overline{E}$ since $r = 0$ at the endpoint of L and ρ is positive on $\overline{E} \subset K$. This proves the first statement in the following proposition. The second statement follows from Proposition 8.2.

Proposition 8.3. *Suppose that $E \subset K$ is an open segment which contains no point of S. There is $(z_0, w_0) \in Z_P \cap Z_Q$, $w_0 \neq \overline{z_0}$, such that $\overline{E} \subset L = \{z_0 + t(z_0 - \overline{w_0})\colon\ t > 0\}$ and $\rho \equiv r$ on $\overline{E}$ where $r(z_0 + t(z_0 - \overline{w_0})) = \sqrt{t(t+1)}|z_0 - \overline{w_0}|$ $(t > 0)$. Moreover, there is an open neighbourhood U of E in $\mathbb{C}$ such that $U \cap K = U \cap E$, that is, U contains no other points of K than the points of E.*

Part 3. Fix a point $z \in K \setminus S$. We show that K contains a segment γ whose one endpoint is z and show that the maximal closed segment $K_0 \subset K$ which contains γ, coincides with K provided that K_0 contains no point of S.

By Proposition 8.2 there are $(z_0, w_0) \in Z_P \cap Z_Q$, $w_0 \neq \overline{z_0}$, and an open disc V centered at z such that $V \cap K \subset L = \{z_0 + t(z_0 - \overline{w_0})\colon\ t > 0\}$ and $\rho \equiv r$ on $V \cap K$ where $r(z_0 + t(z_0 - \overline{w_0})) = \sqrt{t(t+1)}|z_0 - \overline{w_0}|$ $(t > 0)$. Shrinking V if necessary we may assume that $V \cap L$ is an open segment containing $V \cap K$. Since K is connected and contains other points than z it follows that K contains a segment γ in L with z as one endpoint. Let K_0 be the maximal closed segment in K which contains γ. Our goal is to show that $K = K_0$.

Suppose for a moment that K_0 contains no point of S. By a reasoning similar to the one above we see that $K_0 \subset L$ and $\rho \equiv r$ on K_0. Applying Proposition 8.2 at each point of K_0 and using a compactness argument we see that there is an $\eta > 0$ such that $(K_0 + \eta\Delta) \cap K \subset L$. Passing to a smaller $\eta > 0$ we may assume that $(K_0 + \eta\Delta) \cap L$ is an open segment. Since K_0 is a maximal closed segment contained in K it follows that K_0 is a component of K so $K = K_0$ by the connectedness of K. This completes the proof in the case when K_0 contains no point of S.

Part 4. We now consider the case when K_0 contains points of S and show that if K contains a point b which does not belong to K_0 then there is a segment $\tilde{K} \subset K$ which meets K_0 only at one point, which necessarily belongs to S. We do this by showing that if there is no such segment $\tilde{K}$ then K_0 can be separated from b by a Jordan curve that does not meet K, which is not possible since K is connected..

Suppose that K_0 contains points of S. We begin by the remark that $Z_P \cap Z_Q$ is a finite set so in Proposition 8.2 the set of all possible choices for $(z(\zeta), w(\zeta))$ is finite. Thus, there are only finitely many choices for $L(\zeta)$. It follows that $K \setminus S$ is contained in the union of a finite family $\mathcal{L}$ of rays of the form $L' = \{z' + (z' - \overline{w'}):\ t > 0\}$.

Let $K_1, K_2, \cdots, K_\nu$ be the components of $\mathrm{Int} K_0 \setminus S$. By Proposition 8.3 for each $i,\ 1 \leq i \leq \nu$, there are $(z_i, w_i) \in Z_P \cap Z_Q,\ w_i \neq \overline{z_i}$, and open segments $I_i \subset \mathbb{R}_+$ such that $K_i = \{z_i + t(z_i - \overline{w_i}):\ t \in I_i\}$ and $\rho \equiv r_i$ on K_i where $r_i(z_i + t(z_i - \overline{w_i})) = \sqrt{t(t+1)}|z_i - \overline{w_i}|$ $(1 \leq i \leq \nu)$. By Proposition 8.3 for each $i,\ 1 \leq i \leq \nu$, there is a neighbourhood $U_i \subset \mathbb{C}$ of K_i such that $U_i \cap K = K_i$, that is, that U_i contains no other points of K than those of K_i. Let $K_0 \setminus \cup_{j=1}^{\nu} K_j = \{a_1, a_2, \cdots, a_{\nu+1}\}$.

Suppose that there is a point $b \in K \setminus K_0$. We show that this implies that there are a $k,\ 1 \leq k \leq \nu + 1$, and a closed segment $\tilde{K} \subset K$ such that $\tilde{K} \cap K_0 = \{a_k\}$. To see this, choose pairwise disjoint open discs $D_i,\ 1 \leq i \leq \nu + 1$, centered at a_i, such that $D_i \setminus \{a_i\}$ does not contain b and contains no point of S, $1 \leq i \leq \nu + 1$.

For each $i,\ 1 \leq i \leq \nu + 1$, denote by $\mathcal{L}_i \subset \mathcal{L}$ the family of all those rays that pass through a_i or that have their endpoints at a_i. Clearly L above belongs to all $\mathcal{L}_i$. Since each family $\mathcal{L}_i$ is finite one can shrink D_i if necessary so that $K \cap \{D_i \setminus \{a_i\}\} \subset \cup_{\Lambda \in \mathcal{L}_i} \Lambda,\ 1 \leq i \leq \nu + 1$.

Suppose that there is no $\tilde{K}$ with the properties above. Note that for each $i,\ 1 \leq i \leq \nu + 1$, $D_i \cap K$ is contained in a finite union of rays emanating from a_i. We have assumed that beside segment(s) contained in K_0, $D_i \cap K$ contains no other segments with one endpoint a_i. This implies that there is a Jordan curve $\mathcal{C}_i \subset D_i$ surrounding

a_i and meeting K only at points which belong to K_0. Let Ω_i be the domain bounded by $\mathcal{C}_i$, $1 \le i \le \nu+1$. By the discussion above there is a neighbourhood U of $\cup_{j=1}^{\nu} K_j$ in $\mathbb{C}$ such that $U \cap K = \cup_{j=1}^{\nu} K_j$. Let $\Omega_0 = K_0 + \gamma\Delta$ with $\gamma > 0$ very small. With a little care in choosing $\mathcal{C}_i$, $1 \le i \le \nu$, $\Omega = \Omega_0 \cup \Omega_1 \cup \cdots \cup \Omega_\nu$ will be a Jordan domain whose boundary will not meet K and $\mathbb{C} \setminus \overline{\Omega}$ will contain a point $b \in K$, a contradiction since K is connected. Thus, there are a closed segment $\tilde{K}$ and k, $1 \le k \le \nu+1$, such that $K_0 \cap \tilde{K} = \{a_i\}$.

Part 5. We now complete the proof by showing that there is no segment $\tilde{K} \subset K$ which meets K_0 only at one point, and that $(z_i, w_i) = (z_0, w_0)$ $(1 \le i \le \nu+1)$. We need the following

Lemma 8.4. *Suppose that* (z_1, w_1), (z_2, w_2) *are points in* $\mathbb{C}^2$ *such that* $w_1 \neq \overline{z_1}$, $w_2 \neq \overline{z_2}$. *Let* I_1, I_2 *be closed segments in* $[0, \infty)$ *such that the segments* $T_i = \{z_i + t(z_i - \overline{w_i}) \colon t \in I_i\}$, $i = 1, 2$, *meet only at a point* a_0 *which is a common endpoint of* T_1 *and* T_2. *Let* $r_i(z_i + t(z_i - \overline{w_i})) = \sqrt{t(t+1)}|z_i - \overline{w_i}|$ $(t \in I_i)$, $i = 1, 2$, *and suppose that* $r_1(a_0) = r_2(a_0)$. *Define* r *on* $T_1 \cup T_2$ *by putting* $r \equiv r_i$ *on* T_i, $i = 1, 2$. *Assume that there is a nonanalytic function* f *of the form (2.1) such that (8.1) and (8.2) hold with* $T_1 \cup T_2$ *in place of* K *and* r *in place of* ρ. *Then* $(z_1, w_1) = (z_2, w_2)$.

Proof. Assume first that $r(a_0) \neq 0$. By Lemma 6.1 the assumptions imply that $(z_1, w_1) \in \Lambda_{a,\rho(a)}$ $(a \in T_1)$ and $(z_2, w_2) \in \Lambda_{a,\rho(a)}$ $(a \in T_2)$. In particular, $\Lambda_{a_0,\rho(a_0)}$ contains (z_1, w_1) and (z_2, w_2). Since f is not analytic (8.2) implies by Theorem 7.1 that $(z_i, w_i) \in Z_P \cap Z_Q$, $i = 1, 2$. By (8.1) Q does not vanish identically on $\Lambda_{a_0,\rho(a_0)}$ so by Lemma 2.1 $Z_Q \cap \Lambda_{a,\rho(a)}$ contains points that are arbitrarily close to (z_1, w_1) and points that are arbitrarily close to (z_2, w_2) provided that $a \in T_1 \cup T_2$ is sufficiently close to a_0. By (8.2) these points must belong to Z_P. Since $Z_P \cap Z_Q$ is finite it follows that $\Lambda_{a,\rho(a)}$ contains both (z_1, w_1) and (z_2, w_2) provided that $a \in T_1 \cup T_2$ is sufficiently close to a_0. By Lemma 6.2 it follows that $(z_1, w_1) = (z_2, w_2)$.

Assume now that $r(a_0) = 0$. Then $z_1 = z_2 = a_0$, $w_1 \neq \overline{a_0}$, $w_2 \neq \overline{a_0}$. As above, $(a_0, w_1) \in Z_P \cap Z_Q$, $(a_0, w_2) \in Z_P \cap Z_Q$. By (8.1) Q does not vanish identically on the line $z = a_0$ so by Lemma 2.1 it follows that $Z_Q \cap \Lambda_{a,\rho(a)}$ contains points that are arbitrarily close to (a_0, w_1) and points that are arbitrarily close to (a_0, w_2) provided that $a \in T_1 \cup T_2 \setminus \{a_0\}$ is sufficiently close to a_0. As above this implies that $\Lambda_{a,\rho(a)}$ contains (a_0, w_1) and (a_0, w_2) provided that $a \in T_1 \cup T_2 \setminus \{a_0\}$ is sufficiently close to a_0. By Lemma (6.2) it follows that $w_1 = w_2$. This completes the proof of Lemma 8.4. □

The completion of the proof of Theorem 8.1 By Lemma 8.4 the only possibility for a segment $\tilde{K}$ as above to exist is that it is attached to an endpoint of K_0. Lemma 8.4 then implies that $\tilde{K}$ must lie on the same line as K_0, a contradiction since K_0 is a maximal segment contained in K. Lemma 8.4 also implies that $(z_i, w_i) = (z_j, w_j)$ and $r_i = r_j$ for all i, j. □

Theorem 8.5. *Let f be a function of the form (2.1) and let ρ be a nonnegative continuous function on a line S which does not vanish identically on an open subset of S. Suppose that $Q(z, \overline{z}) \neq 0$ $(z \in \overline{\Delta}(a, \rho(a)),\ a \in S)$ and that f extends analytically from $\partial\Delta(a, \rho(a))$ for each $a \in S$. Then f is an analytic rational function with poles outside of $\cup_{a \in S}\overline{\Delta}(a, \rho(a))$.*

Proof. Suppose that there is a nonanalytic function f of the form (2.1) with the properties above. Let E be a component of $\{z \in S\colon\ \rho(z) > 0\}$. E is an open segment, finite or infinite. By Proposition 8.3 there are (z, w), $w \neq \overline{z}$, such that $E \subset L = \{z + t(z - \overline{w})\colon\ t > 0\}$ and $\rho \equiv r$ on $\overline{E}$ where $r(z + t(z - \overline{w})) = \sqrt{t(t+1)}|z - \overline{w}|$ $(t > 0)$. L is a ray in S and r vanishes only at the endpoint of L. Since $\rho \equiv r$ on $\overline{E}$ it follows that E is not a finite segment since in this case, as ρ vanishes at the endpoint(s) of E, r would have to vanish at two points of $\overline{L}$, a contradiction. The same reasoning shows that the endpoint of E must coincide with the endpoint of L. Thus, $E = L$.

It follows that $\{z \in S\colon\ \rho(z) \neq 0\}$ has two components and ρ vanishes precisely at one point. This is impossible by Lemma 8.4. $\square$

9. Remarks on rational functions which are real-analytic in the entire plane.

In this section we make a few remarks about functions of the form (2.1) which satisfy

$$Q(z,\overline{z}) \neq 0 \quad (z \in \mathbb{C}). \tag{9.1}$$

Suppose that a function f of the form (2.1) satisfies (9.1) and is not holomorphic. If there are a continuum K and a positive continuous function ρ on K such that f extends analytically from $\partial\Delta(a,\rho(a))$ for each $a \in K$ then by Theorems 8.1 and 7.1 the family of circles $\{\partial\Delta(a,\rho(a))\colon\ a \in K\}$ is contained in a ray of circles $\mathcal{R}_{z_0,w_0}$ associated with a point $(z_0,w_0) \in Z_P \cap Z_Q,\ w_0 \neq \overline{z_0}$, that was defined in Section 6 as

$$\mathcal{R}_{z_0,w_0} = \{\partial\Delta(z_0 + t(z_0 - \overline{w_0}), \sqrt{t(t+1)}|z_0 - \overline{w_0}|)\colon\ t > 0\},$$

Moreover, these theorems say that f extends analytically from each circle belonging to $\mathcal{R}_{z_0,w_0}$.

Lemma 8.2 implies that if f extends analytically also from all circles belonging to another ray $\mathcal{R}_{z_1,w_1}$ where $(z_1,w_1) \neq (z_0,w_0)$ then the rays $\mathcal{R}_{z_0,w_0}$ and $\mathcal{R}_{z_1,w_1}$ do not contain a common circle, and even the vertices z_0 and z_1 of $\mathcal{R}_{z_0,w_0}$ and $\mathcal{R}_{z_1,w_1}$ are different.

The following example (with n=1) shows that for every ray of circles there is a nonholomorphic function f of the form (2.1) which satisfies (9.1) and extends analytically from each circle of this ray.

Example (finite union of exceptional rays of centers emanating from one point).

Let $p \in \mathbb{C}$ and let $z_j \in \mathbb{C} \setminus \{p\}$ $(1 \le j \le n)$. Let

$$f(z) = \frac{(z - z_1)(z - z_2) \cdots (z - z_n)}{(z - p)(\overline{z} - \overline{p}) + 1}. \tag{9.2}$$

We shall show that for each i, $1 \le i \le n$, and for each $t > 0$, the function f extends analytically from $\partial\Delta(a_i(t), \rho_i(t))$ where

$$a_i(t) = z_i + t\big(1 + |z_i - p|^{-2}\big)(z_i - p) \tag{9.3}$$

$$\rho_i(t) = \big(|z_i - p| + |z_i - p|^{-1}\big)\sqrt{t(t+1)} \tag{9.4}$$

Notice that $\{a_i(t)\colon\ t > 0,\ 1 \le i \le n\}$ is a finite union of rays. Each of these rays is contained in a ray emanating from p. Denote

$$Q(z, w) = (z - p)(w - \overline{p}) + 1.$$

Let $1 \le i \le n$ and define $w_i = \overline{p} - (z_i - p)^{-1}$, that is, w_i is chosen so that $Q(z_i, w_i) = 0$.

Recall that $(z_i - a)(w_i - \overline{a}) = \rho^2 > 0$ where $|z_i - a| < \rho$ if and only if there is a $t > 0$ such that $a = z_i + t(z_i - \overline{w_i})$ and $\rho = \sqrt{t(t+1)}|z_i - \overline{w_i}|$. Fix a and $\rho > 0$ such that $(z_i - a)(w_i - \overline{a}) = \rho^2$ and $|z_i - a| < \rho$. Clearly ζ is a zero of $Q_{a,\rho}(\zeta) = Q(\zeta, \overline{a} + \rho^2/(\zeta - a))$ if and only if for some η

$$Q(\zeta, \eta) = 0, \quad (\zeta - a)(\eta - \overline{a}) = \rho^2. \tag{9.5}$$

Observe that $\zeta = z_i$, $\eta = w_i$ is a solution of (9.5) and by conjugating the equations (9.5) we see that $\zeta = \overline{w_i}, \eta = \overline{z_i}$ is the other solution (by degree, (9.5) has only two solutions). Thus, z_i and $\overline{w_i}$ are the zeros of $Q_{a,\rho}$.

Since $|z_i - a| < \rho$ the second equality in (9.5) implies that $|\overline{w_i} - a| > \rho$, so z_i is the only zero of $Q_{a,\rho}$ in $\Delta(a, \rho)$ which implies that the function $z \mapsto (z - z_i)/Q(z, \overline{z})$ extends analytically from $\partial\Delta(a, \rho)$.

To see that (9.3) and (9.4) hold observe that $z_i - \overline{w_i} = (z_i - p) + (\overline{z_i} - \overline{p})^{-1}$. Note that the circles mentioned above are not the only ones from which f from (9.2) extends analytically. For instance, f extends analytically from the circles $\partial\Delta(p, r)$, $r > 0$.

If $n = 2,\ p = 0,\ z_1 = \alpha,\ z_2 = -\alpha$ where $\alpha > 0$ then the nonanalytic function f from (9.2) extends analytically from circles with centers on $(-\infty, -\alpha] \cup [\alpha, \infty)$ and continuously varying radii. Theorem 8.3 tells that there is no such example with $\alpha = 0$.

We have seen above that if a nonanalytic function f of the form (2.1), which satisfies (9.1), extends analytically from all circles belonging to two different rays of circles then the closures of these rays are disjoint. This does not mean that the rays of the centers of these circles do not meet. We illustrate this by the following

Example (two intersecting exceptional rays of centers). Let $\alpha \in \mathbb{R}$ and put

$$f(z) = \frac{(z - e^{i\alpha})(z - 2)}{z\overline{z} + i}. \tag{9.6}$$

Here $P(z, w) = (z - e^{i\alpha})(z - 2)$ and $Q(z, w) = zw + i$. We show first that $Z_Q \cap \Lambda_{a,\rho}$ consists of one point. Let $a \in \mathbb{C},\ \rho > 0$ and let $(z_1, w_1),\ (z_2, w_2)$ be the solutions of the system

$$zw + i = 0, \quad (z - a)(w - \overline{a}) = \rho^2. \tag{9.7}$$

We get $(z - a)(i - \overline{a}z) = \rho^2 z$ which we rewrite as

$$\overline{a}(z - a)^2 + (\rho^2 + a\overline{a} - i)(z - a) + \rho^2 a = 0.$$

This shows that the roots $z_1,\ z_2$ satisfy $|z_1 - a||z_2 - a| = \rho^2$. It is not possible that $|z_1 - a| = \rho$; indeed, this would imply that $(z_1 - a)(\overline{z_1} - \overline{a}) = \rho^2$ which, since $z_1,\ w_1$ satisfy the second equation in (9.7), would imply that $w_1 = \overline{z_1}$, contradicting the first equation in (9.7). Thus, one of the roots, say z_1, satisfies $|z_1 - a| < \rho$ and the other root satisfies $|z_2 - a| > \rho$, which implies that $Z_Q \cap \Lambda_{a,\rho} = \{(z_1, -i/z_1)\}$.

It follows that whenever $\Lambda_{a,\rho}$ passes through either $(2,-i/2)$ or $(e^{i\alpha},-ie^{-i\alpha})$, the common zeros of P and Q, the function f extends analytically from $\partial\Delta(a,\rho)$. By Lemma 6.1 the rays of centers of these circles are $\{e^{i\alpha}[1+t(1-i)]\colon\ t>0\}$ and $\{2+t(2-i/2)\colon\ t>0\}$. It is easy to see that one can choose α so that these two rays intersect.

Remark. Lev Aizenberg noticed that Theorem 8.1 indirectly implies that any ray $\mathcal{R}_{z_0,w_0}$ of circles

$$|z-(z_0+tb)|=|b|\sqrt{t(t+1)},\ b\in\mathbb{C},\ t>0,\ (b=z_0-\overline{w}_0)$$

is mapped into a ray of circles by any fractional-linear transformation of the complex plane which sends to infinity a point that does not belong to any of the closed discs bounded by $\mathcal{C}\in\mathcal{R}_{z_0,w_0}$. Indeed, these transformations preserve rational functions and the image of a circle of analytic extendibility under such a transformation is again a circle of analytic extendibility provided that the transformation preserves the orientation. In particular, the property that the circles in these families have their centers on a straight line is preserved under fractional-linear transformations. This interesting property can also be checked directly by a simple computation.

We conclude this section by an open question:

Question. Let $\mathcal{R}_{z_1,w_1},\mathcal{R}_{z_2,w_2},\cdots,\mathcal{R}_{z_k,w_k}$ be rays of circles with distinct vertices. Suppose that no two rays contain a common circle. Does there exist a nonanalytic function f of the form (2.1) which satisfies (9.1) and extends analytically from each circle in $\mathcal{R}_{z_1,w_1}\cup\mathcal{R}_{z_2,w_2}\cup\cdots\cup\mathcal{R}_{z_k,w_k}$?

10. Analytic extension into rational disks.

10.1. The Schwarz function.

Let D be a bounded simply-connected domain in $\mathbb{C}$. We will call D a *rational disk* if the Riemann mapping $\omega:\Delta\to D$ is a rational function (with poles outside of the closed disk $\overline{\Delta}$). In potential theory, such domains are called *quadrature domains* (see e.g. [Gu] or [S]).

Recall that the function S_D, analytic in a neighborhood of the boundary ∂D, is called *Schwarz function* if $\overline{z} = S_D(z)$ for $z \in \partial D$. It is known ([D], Ch.14, p.158) that $S_D(z)$ is a meromorphic function if and only if D is a rational disk. In this case

$$S_D(z) = \overline{\omega\left(1/\overline{\omega^{-1}(z)}\right)}, \quad z \in \overline{D}$$

and S_D has poles at the points $d_i = \omega(1/\overline{b}_i)$, where $b_i \in \overline{\mathbb{C}} \setminus \overline{\Delta}$ are poles of ω. S_D has a single simple pole if and only if D is a disk.

If f is a rational function in $\mathbb{R}^2$, $f(z) = P(z,\overline{z})/Q(z,\overline{z})$, then the function

$$f_D^*(z) = \frac{P(z, S_D(z))}{Q(z, S_D(z))}$$

is a rational function of the complex variable z providing analytic extension from the boundary of the domain D, that is $f_D^*(z) = f(z)$ for $z \in \partial D$.

In the present section we generalize Theorem 4.1 to rational disks.

10.2. The case of a single rational disk.

Let D be a rational disk, $\omega : \Delta \to D$ be the (rational) conformal mapping, $d_0, d_1, \ldots, d_\ell \in D$ be the poles of the Schwarz function $S_D(z)$. Denote Λ_D the graph of the Schwarz function over the domain D with removed singular points:

$$\Lambda_D = \{(z, S_D(z)) \in \mathbb{C}^2 : z \in D \setminus \{d_0, d_1, \ldots, d_\ell\}\}.$$

The following theorem generalizes Theorem 4.1.

Theorem 10.1. *Let f be a rational function* (2.1) *in $\mathbb{R}^2$ and D a rational disk. Suppose that*

(a) $Q(z,\overline{z}) = 0$, $z \in \overline{D}$

(b) $p_n(d_i) \neq 0$, $q_m(d_i) \neq 0$, $i = 0, 1, \ldots, \ell$, *where, as before, $p_n(z)$ and $q_n(z)$ are the leading coefficients in* (2.2)

(c) *the polynomials* $P(z,w)$ *and* $Q(z,w)$ *have no common zero on* Λ_D.

If $f\big|\partial D$ *extends analytically into* D *then* f *is an analytic rational function with poles outside of* $\overline{D}$.

Proof. We follow, essentially, the idea of the proof of Theorem 4.1. The main difference with the case of the disk is that for general rational disk the Schwarz function has more than one pole in D.

We can assume, after applying a fractional-linear conformal automorphism of the unit disk, that the Riemann mapping $\omega : \Delta \to D$ has a pole at ∞, say, $b_0 = \infty$. Then $d_0 = \omega(0)$.

Define $g(\xi) = f(\omega(\xi))$. Then g is also a rational function in $\mathbb{R}^2$ and $g\big|\partial\Delta$ extends analytically into Δ. The extension is given by

$$g^*(\xi) = \frac{P(\omega(\xi), S_D(\omega(\xi))}{Q(\omega(\xi), S_D(\omega(\xi))} = \frac{P\left(\omega(\xi), \overline{\omega(1/\overline{\xi})}\right)}{Q\left(\omega(\xi), \overline{\omega(1/\overline{\xi})}\right)}, \quad \xi \in \Delta. \tag{10.1}$$

Consider the family of rational subdisks:

$$D_t = \omega(\Delta_t), t \in [0,1]$$

and let S_{D_t} be the Schwarz function of the domain vD_t :

$$S_{D_t}(\omega(\xi)) = \overline{\omega(t^2/\overline{\xi})}. \tag{10.2}$$

When $t \to 0$ then the disks Δ_t shrink to 0 and the domains D_t shrink to the point d_0.

Denote $c_i = 1/\overline{b_i},\ i = 0,1,\dots,\ell$ the poles of $S_D(\omega(\xi)) = \overline{\omega(1/\overline{\xi})}$, so that $c_0 = 0$ and $d_i = \omega(c_i),\ i = 0,1,\dots,\ell$. Clearly, the function $S_{D_t}(\omega(\xi))$ has poles at the points $\xi = t^2 c_i$, all of which are in Δ_t. Now represent the rational function $S_D(\omega(\xi))$ in the form

$$S_D(\omega(\xi)) = \frac{\psi(\xi)}{\prod\limits_{i=0}^{\ell}(\xi - c_i)^{k_i}}, \tag{10.3}$$

where ψ is a polynomial and $\psi(c_i) \neq 0$, $i = 0, 1, \ldots, \ell$. Since $S_D(\omega(\infty)) = \overline{\omega(0)} = \overline{d}_0$ is finite, it follows that $\deg \psi \leq k_0 + \cdots + k_n$.

It follows from (10.2) and (10.3) that for $S_{D_t}(\omega(\xi))$ we have

$$S_{D_t}(\omega(\xi)) = S_D(\omega(\xi/t^2)) = \frac{\psi(\xi/t^2)t^{2(k_0+\cdots+k_n)}}{\prod\limits_{i=0}^{\ell}(\xi - t^2 c_i)^{k_i}}. \tag{10.4}$$

The substitution $z = \omega(\xi)$, $\omega = S_{D_t}(\omega(\xi))$ into the decomposition (2.2) of the polynomials $P(z, w)$ and $Q(z, w)$ yields:

$$P(\omega(\xi), S_{D_t}(\omega(\xi))) = \sum_{k=0}^{n} p_k(\omega(\xi))[S_{D_t}(\omega(\xi))]^k, \quad \xi \in D_t. \tag{10.5}$$

Introduce the product

$$\rho_t(\xi) = \prod_{i=0}^{\ell}(\xi - t^2 c_i)^{k_i},$$

and denote

$$P_t^*(\xi) = \rho_t^n(\xi)P(\omega(\xi), S_{D_t}(\omega(\xi)), \quad Q_t^*(\xi) = \rho_t^n(\xi)Q(\omega(\xi), S_{D_t}(\omega(\xi))), \tag{10.6}$$

The formulas (10.4) and (10.5) imply

$$\begin{aligned} \lim_{\xi \to t^2 c_j} P_t^*(\xi) &= \frac{t^{2nk_j}\psi^n(c_j)}{\prod_{\substack{i=0\\ i\neq j}}(c_j - c_i)^{nk_i}} p_n(\omega(t^2(c_j)), \\ \lim_{\xi \to t^2 c_j} Q_t^*(\xi) &= \frac{t^{2mk_j}\psi^m(c_j)}{\prod_{\substack{i=0\\ i\neq j}}(c_j - c_i)^{mk_i}} q_m(\omega(t^2(c_j)). \end{aligned} \tag{10.7}$$

Lemma 10.2. *$Q_t^*(\xi) \neq 0$ when $\xi \in \Delta_t$, $0 < t \leq 1$*

Proof. Observe that if $\xi \in \partial\Delta_t$, then $\omega(\xi) \in \partial\Delta_t$ and $S_{D_t}(\omega(\xi)) = \overline{\omega(\xi)}$, hence

$$Q_t^*(\xi) = \rho_t^n(\xi)Q(\omega(\xi), \overline{\omega(\xi)}) \neq 0$$

because $\rho_t(\xi) \neq 0$ for $\xi \in \partial\Delta_t$ and condition (a).

Therefore the logarithmic residue is defined:

$$N_Q(t) = \int_{|\xi|=t} \frac{(Q_t^*)'(\xi)}{Q_t^*(\xi)} d\xi = \#\{\xi \in \Delta_t : Q_t^*(\xi) = 0\}.$$

By the main assumption about analytic extendibility,

$g^*(\xi) = P(\omega(\xi), S_D(\omega(\xi)))/Q(\omega(\xi), S_D(\omega(\xi)))$ has no pole in Δ.

When $t = 1$, then by (10.1) and (10.6)

$$g^*(\xi) = \prod_{i=1}^{\ell} (\xi - c_j)^{(m-n)k_i} \frac{P_1^*(\xi)}{Q_1^*(\xi)},$$

and therefore any zero $\xi \in \Delta \setminus \{c_0, c_1, \dots, c_\ell\}$ of Q_1^* must be also a zero of P_1^* because g^* is analytic in Δ. Since $\rho_1(\xi) \neq 0$ when $\xi \neq c_0, c_1, \dots, c_\ell$, one has from (10.6):

$$P(z, S_D(z)) = Q(z, S_D(z)) = 0, \quad \text{where} \quad z = \omega(\xi) \notin D_t \setminus \{d_0, \dots, d_\ell\}.$$

This is impossible, according to condition (c), because $(z, S_D(z)) \in \Lambda_D$.

Hence Q_1^* does not vanish on $\Delta \setminus \{c_0, c_1, \dots, c_\ell\}$. Additionally, by (10.7), for any $j = 0, 1, \dots, \ell$:

$$Q_1^*(c_j) = \frac{\psi^m(c_j)}{\prod\limits_{\substack{i=0 \\ i \neq j}} (c_j - c_i)^{mk_i}} q_m(d_j) \neq 0$$

because $\psi(c_j) \neq 0$ and $q_m(d_j) \neq 0$ by assumption (b).

Thus $Q_1^*(\xi) \neq 0$ for all $\xi \in \Delta$ and hence $N_Q(1) = 0$. The function $N_Q(t)$ is interger-valued and continuous for $t \in (0, 1]$; hence $N_Q(t) = 0$, $t \in [0, 1]$ and $Q_t^*(\xi) \neq 0$ for $\xi \in \Delta_t$. The lemma is proved. $\square$

We proceed with the proof of the theorem. Assume that $Q(z, w)$ is not a polynomial in z only, that is, $m \geq 1$ in the decomposition (2.2).

The polynomial $Q(d_0, w) = \sum\limits_{k=0}^{m} q_k(d_0) w^k$ is of degree m as $q_m(d_0) \neq 0$ and has m roots counting multiplicities.

Let w_0 be one of these roots. Observe that $w_0 \neq \overline{d_0}$ because $d_0 \in D$ and by condition (a). Let $\eta_0 \in \overline{\omega^{-1}(\overline{w_0})}$. Then $\eta_0 \neq 0$ as $\overline{w_0} \neq \overline{\omega(0)} = d_0$. Also $\eta_0 \neq \overline{b_i}$, $i = 1, \dots, \ell$ because $\omega(b_i) = \infty \neq w_0$.

Introduce the function

$$R(t,\eta) = Q(\omega(t^2/\eta), \overline{\omega(\overline{\eta})}),\ t \in (0,1],$$

which, for each t, is rational in $\eta \in \mathbb{C}$ and has singular points $t^2\overline{c}_i$ and $\overline{b}_i$, $i = 0,1,\dots,\ell$. Recall that b_i are poles of ω, $b_0 = \infty$, and $c_i = 1/\overline{b_i}$. Put $t = 0$. Then

$$R(0,\eta_0) = Q(\omega(0), \overline{\omega(\overline{\eta}_0)}) = Q(d_0, w_0) = 0.$$

Choose $\varepsilon > 0$ so small that the disk $\overline{\Delta}(\eta_0,\varepsilon)$ does not contain 0 and $\overline{b}_1,\dots,\overline{b}_\ell$, and also contains no zero of $R(0,\eta)$ different from η_0.

Then the logarithmic residue

$$N_R(0) = \int_{|\eta-\eta_0|=\varepsilon} \frac{R'_\eta(0,\eta)}{R(0,\eta)} d\eta$$

is defined and equals the multiplicity of the root w_0; hence $N_R(0) > 0$.

Now choose $\delta > 0$ so smal that when $t \in [0,\delta]$ and $|\eta - \eta_0| = \varepsilon$, then $\frac{t^2}{\eta} \neq b_1, b_2, \dots, b_\ell$ (recall that $|b_i| > 1$), and also $R(t,\eta) \neq 0$ (which is possible to provide, as $R(t,\eta)$ is uniformly continuous on $[0,\delta] \times \partial\Delta(\eta_0,\varepsilon)$).

Then one can define for $t \in [0,\delta]$ the logarithmic residue

$$N_R(t) = \int_{|\eta-\eta_0|=\varepsilon} \frac{R'_\eta(t,\eta)}{R(t,\eta)} d\eta = \#\{\eta \in \Delta(\eta_0,\varepsilon) : R(t,\eta) = 0\}.$$

This function is continuous and an integer and hence constant, therefore $N_R(t) > 0$, $t \in [0,\delta]$ and the equation $R(t,\xi) = 0$ has at least one solution $\eta = \eta_t \in \Delta(\eta_0,\varepsilon)$.

Note that $0 \notin \Delta(\eta_0,\varepsilon)$ so $\eta_t \neq 0$ and δ can be chosen so small that $|\eta_t| > t$ for $t \in [0,\delta]$.

Let $\xi_t = t^2/\eta_t$. Then $\xi_t \in \Delta_t$ and, by the construction of η_t and (10.3):

$$Q(\omega(\xi_t), S_{D_t}(\omega(\xi_t)) = Q(\omega(\xi_t), \omega(t^2/\overline{\xi}_t)) = Q(\omega(t^2/\eta_t)), \overline{\omega(\overline{\eta}_t)}) = R(t,\eta_t) = 0.$$

However, then

$$Q_t^*(\xi_t) = \rho_t^m(\xi_t) Q(\omega(\xi_t), S_{D_t}(\omega(\xi_t))) = 0,$$

and we arrive at a contradiction to Lemma 10.2 because $\xi_t \in \Delta_t$.

Therefore, the assumption that $Q(z,w)$ depends on w is not true, $Q(z,\overline{z})$ is a holomorphic polynomial and $m = 0$. It remains to note that, according to (10.8),

$$P_1^*(c_0) = \frac{\psi^n(c_0)}{\prod_{i=1}^{\ell}(c_0 - c_i)^{nk_i}} p_n(d_0) \neq 0$$

and

$$Q_1^*(c_0) = \frac{\psi^m(c_0)}{\prod_{i=1}^{\ell}(c_0 - c_i)^{mk_i}} q_m(d_0) \neq 0;$$

and since the analytic extension g^* in (10.1) has no singularlity at $\xi = c_0$, it follows from (10.9) that $m - n \geq 0$. Therefore $n = 0$ and so $P(z,\overline{z})$ is also a holomorphic polynomial. □

10.3. Families of rational disks.

For a rational disk D denote by $singD$ the set of poles (in D) of the Schwarz function S_D.

Consider a continuous family D_t, $t \in (t_0, t_1)$, of rational disks, which means that the Riemann mappings $\omega_t : \Delta \to D_t$ can be chosen to depend continuously on the parameter t.

The following theorem is a version of Theorem 8.1 for contimuous families of rational disks.

Theorem 10.3. *Let D_t, $t \in I = (t_0, t_1)$ be a continuous family of rational disks such that $singD_t \cap singD_t' = \emptyset$ for $t \neq t'$. Let $\Omega = \bigcup_{t\in I} D_t$.*

Let f be a rational function (2.1) and suppose that

(1) $Q(z,\overline{z}) \neq 0$, $z \in \overline{\Omega}$;

(2) *no relation of the form*

$$w_k = S_{D_t}(z_k),\ t \in I_k,$$

holds with some $z_k \in \bigcap_{t\in I_k} D_t$ and $w_k \in \mathbb{C}$, $w_k \neq \overline{z}_k$.

If $f|\partial D_t$ extends analytically into D_t for all $t \in I$, then f is analytic in Ω.

Proof. The proof uses Theorem 10.1 and follows the idea of the proof of Theorem 8.1.

First of all, if S is the set of all values of the parameter t such that the singular set $singD_t$ contains at least one zero of the leading coefficients p_n and q_m then S is finite, due to the assumption about the sets $singD_t$, and therefore for each $t \in I \setminus S$ Theorem 10.1 is applicable for the domain D_t.

Suppose that f extends analytically from ∂D_t for each $t \in I$ and assume that f is not analytic. By Theorem 10.1 each associated manifold Λ_{D_t}, $t \in I \setminus S$ contains at least one common zero of the polynomials P and Q. Let ν be the number of common zeros of P and Q and for each $(z_k, w_k) \in Z_P \cap Z_Q$, $k = 1, \cdots \nu$, denote

$$\Sigma_k = \{t \in I \setminus S : (z_k, w_k) \in \Lambda_{D_t}\}.$$

It is obvious that the set Σ_k is closed in $I \setminus S$. It is also open for the following reason. If $t_0 \in \Sigma_k$ then by Lemma 2.1 the variety Z_Q meets Λ_{D_t} at a point that is close to (z_k, w_k), provided that t is suffciently close to t_0. Then P must vanish at the intersection point by the analytic extendibility condition, and since the common zeros of P and Q are isolated, we conclude that $(z_k, w_k) \in \Lambda_{D_t}$, i.e. $t \in \Sigma_k$.

Thus Σ_k are open-closed subsets of $I \setminus S$ and therefore they are unions of adjacent intervals in $I \setminus S$. In other words, the set of points $(z_k, w_k) \in Z_P \cap Z_Q \cap \Lambda_{D_t}$ is the same for all t in each connected component of $I \setminus S$.

Take two connected components - adjacent intervals I_1 and I_2 with a common endpoint $s \in S$. Suppose that $I_1 \subset \Sigma_k$. This means that $z_k \in D_t$ and $w_k = S_{D_t}(z_k)$ for all $t \in I_1$. The point z_k does not belong to the singular set $singD_s$ because then $S_{D_s}(z_k) = \infty$ which contradicts the previous identity when $t \to s$.

By (1), Q does not vanish identically on Δ_{D_s}. Lemma 2.1 implies that Z_Q meets Δ_{D_t} at a point near (z_k, w_k) when $t \in I_2$ is sufficiently close to s. This intersection point must be a zero of P because of the analytic extendibility condition, and therefore this point conicides with (z_k, w_k) because the common zeros of P and Q are isolated.

Thus, Σ_k contains both intervals I_1 and I_2 and since this is true for any pair of adjacent intervals in $I \setminus S$ we arrive to $\Sigma_k = I$. This means that the relation $w_k = S_{D_t}(z_k),\ t \in I$, holds which contradicts the assumption (2). Therefore, f is analytic. $\square$

11. A test for analyticity of real-analytic functions.

In the previous sections we considered rational functions whose denominators do not vanish on the union of the disks from the test families. Such functions are real-analytic on the union of these disks. In this section we study real-analytic, not necessary rational, functions and give a sufficient condition for holomorphicity in terms of analytic extendibility from families of circles centered on a smooth curve with radius smoothly depending on the center. The proof uses ideas different from those in the previous sections.

Theorem 11.1. *Let $a : (0,1) \mapsto \mathbb{C}$ be a C^1-function with the derivative $a'(t) \neq 0,\ t \in (0,1)$, and let $\rho : (0,1) \mapsto \mathbb{R}_+$ be a positive C^1-function. Denote $\Delta(t) = \Delta(a(t), \rho(t))$ and let f be a real-analytic function in a neighborhood of $\overline{\Omega}$ where $\Omega = \bigcup_{t\in(0,1)} \overline{\Delta}(t)$. Suppose that $|\rho'(t)| \leq |a'(t)|$ for all $t \in (0,1)$. If f extends analytically from each circle $\partial\Delta(t),\ t \in (0,1)$, then f is holomorphic in a neighbourhood of $\overline{\Omega}$.*

The proof will follow from several lemmas.

Lemma 11.2. *If a function f is real analytic in the disk $\Delta(a,r)$ then for any continuous function p in the disk, the integral*

$$J_p(\rho) = \int_{|\zeta|=1} f(a + \rho\zeta)p(\zeta)d\zeta$$

is a real-analytic function of $\rho \in (0, r)$.

Proof. Write $f(\zeta) = F(\zeta, \overline{\zeta})$ where F is a function holomorphic in a neighbourhood of $\{(\zeta, \overline{\zeta})\colon\ \zeta \in \Delta(a, \rho)\}$ in $\mathbb{C}^2$. Then the function

$$(z, w) \mapsto \int_{|\zeta|=1} F(a + z\zeta, \overline{a} + w\overline{\zeta}) p(\zeta) d\zeta$$

is well defined and holomorphic in a neighbourhood of $\{(\zeta, \overline{\zeta})\colon\ \zeta \in \Delta(0, \rho)\}$ and so the function

$$\rho \mapsto \int_{|\zeta|=1} F(a + \rho\zeta, \overline{a} + \rho\overline{\zeta}) p(\zeta) d\zeta$$

is real-analytic on $(0, 1)$. □

The following lemma relates derivatives of the function $(t, r, \theta) \mapsto f(a(t) + re^{i\theta})$ We denote $\partial_r = \frac{\partial}{\partial r}$, $\partial_\theta = \frac{\partial}{\partial \theta}$, $\partial_t = \frac{\partial}{\partial t}$.

Lemma 11.3. *Define* $\Psi(t, r, \theta) = f(a(t) + re^{i\theta})$. *Then for every* $k \in \mathbb{N}$

$$[a'(t)e^{-i\theta} + \overline{a}'(t)e^{i\theta} + 2\rho'(t)]\ (\partial_\rho^k \Psi)(t, \rho(t), \theta) = 2\frac{\partial}{\partial t}\left[(\partial_\rho^{k-1}\Psi)(t, \rho(t), \theta)\right]$$

$$+\ i[a'(t)e^{-i\theta} - \overline{a}'(t)e^{i\theta}]\ \left[\partial_r^{k-1}\left(\frac{1}{r}\partial_\theta \Psi\right)\right](t, \rho(t), \theta).$$

Proof. We have

$$\frac{\partial}{\partial t}\left[(\partial_r^{k-1}\Psi)(t, \rho(t), \theta)\right] = (\partial_t \partial_r^{k-1}\Psi)(t, \rho(t), \theta) + (\partial_r^k \Psi)(t, \rho(t), \theta)\rho'(t). \tag{11.1}$$

We eliminate the derivative with respect to t in the first term on the right. To this end, observe that

$$\partial_t \Psi(t, r, \theta) = a'(t)\partial_z f(a(t) + re^{i\theta}) + \overline{a}'(t)\partial_{\overline{z}} f(a(t) + re^{i\theta}), \tag{11.2}$$

and express the operators ∂_z, $\partial_{\overline{z}}$ in polar coordinates r, θ :

$$2\partial_z = e^{-i\theta}(\partial_r - \frac{i}{r}\partial_\theta), \quad 2\partial_{\overline{z}} = e^{i\theta}(\partial_r + \frac{i}{r}\partial_\theta).$$

Using this in (11.2) and noticing that $\partial_r f(a(t)+re^{i\theta}) = \partial_r\Psi(t,r,\theta)$, $\partial_\theta f(a(t)+re^{i\theta}) = \partial_\theta\Psi(t,r,\theta)$ we get

$$2(\partial_t\Psi)(t,r,\theta) = a'(t)e^{-i\theta}[(\partial_r\Psi)(t,r,\theta) - \frac{i}{r}(\partial_\theta\Psi)(t,r,\theta)]$$
$$+ \overline{a}'(t)e^{i\theta}[(\partial_r\Psi)(t,r,\theta) + \frac{i}{r}(\partial_\theta\Psi)(t,r,\theta)].$$

Using this in (11.1) gives the formula we want to prove. □

Lemma 11.4. *Let f be as in Theorem 11.1. Then f extends analytically from any circle $\partial\Delta(a(t),r)$, $0<r<\rho(t)$, $t\in(0,1)$.*

Proof. We have to show that for each $t\in(0,1)$

$$\int_{|\zeta|=1} f(a(t)+r\zeta)p(\zeta)d\zeta = 0 \quad (0<r<\rho(t))$$

for every holomorphic polynomial p, that is, that $J_{t,p}(r)=0$, $0<r<\rho(t)$, where

$$J_{t,p}(r) = \int_{|\zeta|=1} f(a(t)+r\zeta)p(\zeta)d\zeta.$$

For each $t\in(0,1)$ the function f is real-analytic in a disk $\Delta(a(t),R(t))$ for some $R(t)>\rho(t)$, so Lemma 11.2 implies that the function $J_{t,p}$ is real-analytic on $(0,R(t))$. Therefore, to prove that $J_{t,p}(r)=0$, $0<r<\rho(t)$, $t\in(0,1)$ it suffices to prove that for each $t\in(0,1)$ all the derivatives $J_{t,p}^{(k)}(\rho(t))$, $k=0,1,2,\cdots$, vanish. The latter happens if and only if for each k and for each $t\in(0,1)$ the radial derivative $e^{i\theta}\mapsto\partial_r^k f(a(t)+re^{i\theta}) = \partial_r^k\Psi(t,r,\theta)$ evaluated at $r=\rho(t)$, extends analytically from the unit circle.

We will prove this by induction in k. For $k=0$ this holds by the assumption. Assume that this holds for all derivatives up to order $k-1$.

Multiplying both sides of the equality in Lemma 11.3 by $\zeta=e^{i\theta}$ we obtain:

$$\partial_r^k\Psi(t,\rho(t),\theta) = \frac{G_t(\zeta)}{H_t(\zeta)}, \tag{11.4}$$

where

$$H_t(\zeta) = \overline{a}'(t)\zeta^2 + 2\rho'(t)\zeta + a'(t),$$

and

$$G_t(\zeta) = \quad 2\zeta \frac{\partial}{\partial t}\big[(\partial_r^{k-1}\Psi)(t,\rho(t),\theta) \ + \ i[a'(t) - \overline{a}'(t)\zeta^2]\big[\partial_r^{k-1}\big(\frac{1}{r}\partial_\theta\Psi\big)\big](t,\rho(t),\theta).$$

The term $\partial_r^{k-1}\big(\frac{1}{r}\partial_\theta\Psi\big)$ is a linear combination of terms of the form $r^{-s-1}\partial_\theta\partial_r^s\Psi$, $0 \leq s \leq k-1$.

We have assumed that for each $t \in (0,1)$ and for each j, $0 \leq j \leq k-1$, the function $e^{i\theta} \mapsto (\partial_r^j\Psi)(t,\rho(t),\theta)$ extends analytically from the unit circle. Since ∂_θ preserves this extendibility property it follows that for each s the function

$$e^{i\theta} \mapsto \frac{1}{\rho(t)^{s+1}}(\partial_\theta\partial_r^s\Psi)(t,\rho(t),\theta)$$

extends analytically from the unit circle. Moreover, since for each $t \in (0,1)$ the function $e^{i\theta} \mapsto (\partial_r^{k-1}\Psi)(t,\rho(t),\theta)$ extends analytically from the unit circle it follows that the function

$$e^{i\theta} \mapsto \frac{\partial}{\partial t}\big[(\partial_r^{k-1}\Psi)(t,\rho(t),\theta)\big]$$

extends analytically from the unit circle for each $t \in (0,1)$. Thus, G_t extends analytically from the unit circle.

The condition $|\rho'(t)| \leq |a'(t)|$ implies that both roots of the quadratic polynomial H_t lie on the unit circle $|\zeta| = 1$ and therefore the denominator in (11.4) does not vanish in the open unit disk. Although H_t has zeros on $|\zeta| = 1$ they are removable singularities of the ratio G_t/H_t since G_t is real analytic in a neghborhood of the unit disk and G_t/H_t is smooth on $|\zeta| = 1$.

It follows that $e^{i\theta} \mapsto (\partial_r^k\Psi)(t,\rho(t),\theta)$ extends analytically from the unit circle for each $t \in (0,1)$. $\square$

Proof of Theorem 11.1. By Lemma 11.4 the function f extends analytically from each circle $\partial\Delta(a(t),r)$, $0 < r \leq \rho(t)$, $t \in (0,1)$. Pick $t_0 \in (0,1)$ and let $0 < \rho_0 < \rho(t_0)$. Choose $\varepsilon > 0$ so that $\rho_0 < \rho(t)$ for all $t \in I = (t_0 - \varepsilon, t_0 + \varepsilon)$.

We have

$$\int_{|z|=r} f(a(t)+z)dz = 0 \quad (0 < r \le \rho_0,\ t \in I). \tag{11.5}$$

By Green's formula the equality can be rewritten as

$$\int_{|z|\le r} g((a(t)+z)dzd\overline{z} = 0 \quad (0 < r \le \rho_0,\ t \in I) \tag{11.6}$$

where $g = \partial_{\overline{z}} f$.

Differentiating the integral with respect to t we get

$$\int_{|z|\le r} \left[\partial_z g(a(t)+z)a'(t) + \partial_{\overline{z}} g(a(t)+z)\overline{a}'(t)\right] dzd\overline{z} = 0 \quad (0 < r \le \rho_0,\ t \in I).$$

By Green's formula $\int_{|z|\le\rho} \partial_{\overline{z}} g(a(t)+z)dzd\overline{z} = \int_{|z|=r} g(a(t)+z)dz$ and $\int_{|z|\le\rho} \partial_z g(a(t)+z)dzd\overline{z} = \int_{|z|=r} g(a(t)+z)d\overline{z}$, which implies that

$$\int_0^{2\pi} g(a(t)+re^{i\theta})[a'(t)re^{-i\theta} - \overline{a}'(t)re^{i\theta}]d\theta = 0 \quad (0 < r \le \rho_0,\ t \in I).$$

Integrating both sides with respect to rdr we get

$$\begin{aligned} \int_{|z|\le r} [a'(t)\overline{z} - \overline{a}'(t)z]\ g(a(t)+z)dzd\overline{z} &= a'(t)\int_{|z|\le r} \overline{z}\ g(a(t)+z)dzd\overline{z} \\ &- \overline{a}'(t)\int_{|z|\le r} z\ g(a(t)+z)dzd\overline{z} = 0 \quad (0 < r \le \rho_0,\ t \in I). \end{aligned} \tag{11.7}$$

Since (11.5) holds with f replaced by zf it follows that (11.6) holds with g replaced by zg and hence the second integral is zero. Since $a'(t) \ne 0$, (11.7) implies that (11.6) holds with g replaced by $\overline{z}g$. Continuing this way we conclude that (11.6) holds with $\overline{z}^m g$ in place of g for each $m \in \mathbb{N}$. Repeating the process with $z^n f$ in place of f we conclude that

$$\int_{|z|\le\rho_0} z^n\overline{z}^m g(a(t_0)+z)dzd\overline{z} = 0 \quad (m \in \mathbb{N}\cup\{0\},\ n \in \mathbb{N}\cup\{0\})$$

which means that g is orthogonal on the disk $\Delta(a(t_0),\rho_0)$ to any polynomial in $z,\ \overline{z}$. Hence $g = 0$ on this disk and therefore f is holomorphic there. Since ρ_0 can be chosen arbitrarily close to $\rho(t_0)$ it follows that the function f is holomorphic on $\Delta(t_0)$. $\square$

Remark. In Theorem 11.1 it suffices to require the inequality $|\rho'(t) \leq |a'(t)|$ for t in an open subset of $(0,1)$ so that failure of the condition means that $|\rho'(t)| > |a'(t)|$ everywhere except on a discrete set where the equality holds.

The example from Section 7 shows that the condition about the derivatives of $\rho(t)$ and $a(t)$ can not be dropped. In this example $a(t) = 1, \quad \rho(t) = \sqrt{t(t+1)}$ and $|\rho'(t)| = (2t+1)/(2\sqrt{t(t+1)}) > |a'(t)| = 1$.

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