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RELATIVE EMBEDDABILITY INTO LIPSCOMB'S 0-DIMENSIONAL UNIVERSAL SPACE

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Abstract

Let $\Sigma(\tau)$ be the generalized Sierpiński curve constructed [7, 8], which is naturally identified with the Lipscomb's space $\mathcal{J}(\tau)$ [3, 4]. Then $L_0(\tau) \subseteq \Sigma(\tau)$, the set of irrational points of $\Sigma(\tau)$, is universal for 0-dimensional metric spaces of weight $\leq \tau$. We prove that any embedding of a compact subspace of a 0-dimensional metric spaces of weight $\leq \tau$ into $L_0(\tau)$ can be extended to the embedding of the whole space.

Keywords: covering dimension, generalized Sierpiński curve, universal space, Lipscomb's universal space, embedding

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1 Introduction and definitions

Let $\tau \geq \aleph_0$ be a cardinal number and let Λ be a set of cardinality τ. In his papers [3, 4] S. L. Lipscomb has defined the space $\mathcal{J}(\tau)$ as a factor-space of Baire's universal 0-dimensional space and used it in his construction of a universal n-dimensional metrizable space of weight τ, (which is a subspace of $\mathcal{J}(\tau)^{n+1}$). In [7, 8] it has been proved that $\mathcal{J}(\tau)$ is naturally homeomorphic to a generalized Sierpiński curve $\Sigma(\tau)$.

The subspace $L_0(\tau) \subseteq \Sigma(\tau)$ of all so called irrational points is a universal space for 0-dimensional metrizable spaces of weight $\leq \tau$.

We wish to present an embedding theorem of the relative type, i.e. when an embedding is given in advance on a certain subspace, and then it is extended to an embedding of the whole space. In our case, the embedding into $L_0(\tau)$ is given on a compact subspace X_0 of a 0-dimensional metric space X of weight $\leq \tau$, and we show that it can be extended to an embedding of X into $L_0(\tau)$.

We shall use the notation of [1, 3] (with a few slight modifications).

$|X|$ denotes the cardinal number of the set X.

For the sake of completeness we include here the descriptions of Lipscomb's space $\mathcal{J}(\tau)$, of the generalized Sierpiński curve $\Sigma(\tau)$, and of the homeomorphism between them.

Baire's universal 0-dimensional space of weight τ is the set $\Lambda^{\mathbf{N}}$ ($\mathbf{N} = \{1, 2, 3, \ldots\}$) of all sequences of elements of Λ equipped with the product topology, while Λ is equipped with the discrete topology. *Lipscomb's space* $\mathcal{J}(\tau)$ is defined as the quotient space $\mathcal{J}(\tau) = \Lambda^{\mathbf{N}}/\!\sim$, where the equivalence relation $\sim$ is defined as follows:

for $\lambda = (\lambda_1, \ldots, \lambda_m, \ldots), \mu = (\mu_1, \ldots, \mu_m, \ldots)$

$\lambda \sim \mu \iff \lambda = \mu \, \text{or} \, \exists j \in \mathbf{N} \, \text{such that}:$

i) $\forall k, k < j \implies \lambda_k = \mu_k,$

ii) $\forall s \in \mathbf{N}, \lambda_j = \mu_{j+s},$

iii) $\forall s \in \mathbf{N}, \lambda_{j+s} = \mu_j.$

In the case $\mu \neq \lambda$ such a j is uniquely determined and is called the *tail index* of λ and μ. We also say that the two sequences are *interwoven*.

The equivalence class of $\lambda = (\lambda_1, \ldots, \lambda_m, \ldots)$ will be denoted by $[\lambda]$ or $[\lambda_1, \ldots, \lambda_m, \ldots]$. An equivalence class may be a singleton — in which case it is called an *irrational point* of $\mathcal{J}(\tau)$ — or a dyad — in which case it is called a *rational point* of $\mathcal{J}(\tau)$. $\mathcal{J}(\tau)$ is a one-dimensional metrizable space of weight τ [3].

The classic *triangular Sierpiński curve* may be described as a subset of $\mathbf{R}^3$ as follows:

Let $e^1 = (1, 0, 0)$, $e^2 = (0, 1, 0)$, $e^3 = (0, 0, 1)$. Let $\varphi_1, \varphi_2, \varphi_3 : \mathbf{R}^3 \longrightarrow \mathbf{R}^3$ be the homotheties with the coefficients $1/2$ and the centers e^1, e^2, e^3, respectively. If the convex hull of these three points (i.e. the standard 2-simplex) is denoted by Σ it is obvious that the set obtained from Σ by n removals of the middle triangles may be described as

$$\Sigma_n = \bigcup_{(\lambda_1, \ldots, \lambda_n) \in \Lambda^n} \varphi_{\lambda_1} \circ \cdots \circ \varphi_{\lambda_n} \Sigma, \tag{1}$$

where $\Lambda = \{1, 2, 3\}$. After that, the classic triangular Sierpiński curve is obtained as the intersection of all this unions Σ_n.

The *generalized Sierpiński curve* $\Sigma(\tau)$ is defined analogously using the Hilbert space $\ell_2(\tau) = \{(x_\lambda) \in \mathbf{R}^\Lambda \,|\, \sum_{\lambda \in \Lambda} x_\lambda^2 < \infty\}$ as the ambient space instead of $\mathbf{R}^3$. Using e^λ, $\lambda \in \Lambda$, defined by $\forall \mu \in \Lambda$, $e^\lambda_\mu = \delta_{\lambda,\mu}$ (Kronecker's symbol) we describe the "homotheties" with the centers e^λ and the coefficients $1/2$, i.e. the functions

$\varphi_\lambda : \ell_2(\tau) \longrightarrow \ell_2(\tau)$ defined by

$$(\varphi_\lambda(x))_\mu = \begin{cases} (x_\lambda + 1)/2, & \mu = \lambda \\ x_\mu/2, & \mu \neq \lambda. \end{cases}$$

Let $\sigma = \{(x_\lambda) \in \ell_2 | \sum_{\lambda\in\Lambda} x_\lambda = 1 \ \& \ \forall\lambda, 0 \le x_\lambda \le 1\}$. Then $\Sigma = \mathrm{Cl}\,\sigma = \mathrm{Cl}\,\sigma = \{(x_\lambda) \in \ell_2 | \sum_{\lambda\in\Lambda} x_\lambda \le 1 \ \& \ \forall\lambda, 0 \le x_\lambda \le 1\}$ is the closed convex hull of the set $\{e^\lambda | \lambda \in \Lambda\}$ and it may be called the *standard τ-simplex* by an analogy to the standard n-simplex. Now the generalized Sierpiński curve $\Sigma(\tau)$ may be described in the same way as previously the classic curve: subspaces Σ_n of ℓ_2 are defined by (1) (only Σ now has a different meaning, and Λ is of cardinality τ), and then $\Sigma(\tau)$ is defined as

$$\Sigma(\tau) = \bigcap_{n\in\mathbf{N}} \Sigma_n.$$

The points $\varphi_{\lambda_1}\circ\cdots\circ\varphi_{\lambda_n}(e^\lambda)$, $n \ge 1$, are called the *rational points* of $\Sigma(\tau)$ (more precisely for a fixed n they are called the *nth level vertices*), and all other points (including all e^λs) are *irrational points* of $\Sigma(\tau)$.

That $\chi : \mathcal{J}(\tau) \longrightarrow \Sigma(\tau)$, defined by

$$\chi([\lambda_1, \ldots, \lambda_n, \ldots]) = \bigcap_{n\in\mathbf{N}} \varphi_{\lambda_1}\circ\cdots\circ\varphi_{\lambda_n}\Sigma \tag{2}$$

is a homeomorphism mapping rational points to rational points and irrational points to irrational points has been proved in [7], and we identify the spaces $\mathcal{J}(\tau)$ and $\Sigma(\tau)$ via χ freely, choosing the description that is more convenient for use in the context.

Every point of $\Sigma(\tau)$ is thus described by a unique equivalence class of indices $[\lambda] = [(\lambda_1, \ldots, \lambda_n, \ldots)]$, where the λ_ns are the indices of the homotheties from (2). Any rational point is represented by two interwoven sequences, while any irrational point is represented by a unique sequence.

S.L. Lipscomb [4] proved that the n-dimensional subspace $L_n(\tau) \subseteq \mathcal{J}(\tau)^{n+1}$, consisting of all points having at least one irrational coordinate, is a universal space for metrizable spaces of weight $\le \tau$ and covering dimension $\le n$. U. Milutinović [7, 9] has used $\Sigma(\tau)$ and indexing of special type of certain sequences of decompositions of a given metrizable space X of weight $\le \tau$ and covering dimension $\le n$, in order to obtain an embedding of X into $L_n(\tau) \subseteq \Sigma(\tau)$ (and thus giving a new proof of the universality of $L_n(\tau)$). I. Ivanšić and U. Milutinović have used the same approach (but more complicated decompositions and indexing) in [6] in to order to prove that the classic Sierpiński curve may be used in the construction of a universal space in the separable case. In [9] U. Milutinović proved a result on approximation of maps by embeddings using another modification of decompositions and indexing. In this paper we rely on the same approach, but because of the specific type of control needed here, we construct a new modification of decompositions and indexing.

Let us introduce some additional definitions and notation (some of them has already been used).

Let $\mathcal{U}$ be a family of subsets of X, $x \in X$. The *local order* of $\mathcal{U}$ at x is defined as $\operatorname{lord}_x \mathcal{U} = \inf\{k : x \text{ has a neighborhood intersecting } k \text{ elements of } \mathcal{U}\} \in \{0, 1, 2, \ldots, \infty\}$. The *local order* of $\mathcal{U}$ is defined as $\operatorname{lord} \mathcal{U} = \sup\{\operatorname{lord}_x \mathcal{U} : x \in X\}$.

$\operatorname{Bd} \mathcal{U} = \bigcup_{U \in \mathcal{U}} \operatorname{Bd} U$, where $\operatorname{Bd} U$ denotes the boundary of the set U; $\operatorname{Cl} \mathcal{U} = \bigcup_{U \in \mathcal{U}} \operatorname{Cl} U$, where $\operatorname{Cl} U$ denotes the closure of the set U.

A *decomposition* of the space X is a pairwise disjoint locally finite family of open nonempty subsets of X whose closures cover X.

As in all mentioned papers of Ivanšić and Milutinović, the main tool enabling us to construct the needed decompositions was the following Lipscomb's lemma, which we use in the present paper as well, and therefore quote it for the sake of completeness (the notation is changed, to fit ours):

Lemma 1.1 *([4, Lemma 4, p.152])*

Let $n \in \{0, 1, 2, \ldots\}$. *Let* X *be a metric space such that* $\dim X = n$, $wX = \tau \geq \aleph_0$.

Let $X = X_1 \cup X_2 \cup \cdots \cup X_{n+1}$, *where* $X_1, \ldots, X_{n+1}$ *are pairwise disjoint* 0*-dimensional subsets of* X.

Let $\mathcal{T}$ *be an arbitrary open covering of* X. *For each* j, $1 \leq j \leq n+1$, *let* $\mathcal{V}_j$ *be a decomposition of* X *such that* $|\mathcal{V}_j| \leq \tau$ *and* $\operatorname{lord} \mathcal{V}_j \leq 2$. *Let* $\mathcal{F}_j$, $|\mathcal{F}_j| \leq \tau$, *be a discrete closed family such that*

$$\operatorname{Bd} \mathcal{V}_j = \bigcup \mathcal{F}_j,$$

and let for each $k \in \{1, \ldots, n+1\}$ *and distinct* $j_1, \ldots, j_k \in \{1, \ldots, n+1\}$

$$\dim(\operatorname{Bd} \mathcal{V}_{j_1} \cap \cdots \cap \operatorname{Bd} \mathcal{V}_{j_k}) \leq n - k$$

hold.

Let $\mathcal{O}_j = \{O_F : F \in \mathcal{F}_j\}$ *be an open family such that* $F \subseteq O_F$ *for each* $F \in \mathcal{F}_j$.

Then for each j, $1 \leq j \leq n+1$, *there are discrete families* $\mathcal{W}_j^S$, $\mathcal{W}_j^B$, *and* $\mathcal{W}_j^Q$ *of cardinality* $\leq \tau$, *which are disjoint in pairs, such that*

$$\mathcal{W}_j = \mathcal{W}_j^S \cup \mathcal{W}_j^B \cup \mathcal{W}_j^Q$$

is a decomposition of X *satisfying (for each* j, $1 \leq j \leq n+1$*):*

(a) $\operatorname{lord} \mathcal{W}_j \leq 2$;

(b) $\{\operatorname{Cl} W : W \in \mathcal{W}_j^S\}$ *refines* $\mathcal{T}$; $\bigcup_{j=1}^{n+1} \mathcal{W}_j^S$ *covers* X;

(c) if $x \in \operatorname{Bd} \mathcal{W}_j$ *then there are distinct elements* W_1, W_2 *in* $\mathcal{W}_j$ *such that* $x \in \operatorname{Bd} W_1 \cap \operatorname{Bd} W_2$;

(d) $\mathcal{W}_j$ *covers* X_j *(hence* $\operatorname{Bd} \mathcal{W}_j$ *misses* X_j*);*

(e) $\operatorname{Bd} \mathcal{W}_j \cap \operatorname{Bd} \mathcal{V}_j = \emptyset$;

(f) $\mathcal{W}_j^S \cup \mathcal{W}_j^Q$ *refines* $\mathcal{V}_j$;

(g) $\mathcal{W}_j^S \cup \mathcal{W}_j^B$ *is a discrete family;*

(h) $\mathcal{W}_j^B = \{W_F : F \in \mathcal{F}_j\}$ *(the indexing is faithful, i.e. injective) and* $F \subseteq W_F \subseteq \operatorname{Cl} W_F \subseteq O_F$ *for each* $F \in \mathcal{F}_j$.

In fact, we shall need only

Lemma 1.2 *Let X be a 0-dimensional metric space. Then for any cover $\mathcal{V}$ of X, consisting of pairwise disjoint clopen sets and for any open cover $\mathcal{T}$ of X, there exists a common refinement $\mathcal{W}$ of $\mathcal{V}$ and $\mathcal{T}$, consisting of pairwise disjoint clopen sets.*

Proof. This lemma is a simple corollary of Lemma 1.1. Using it for $\tau = w(X)$, $n = 0$, $\mathcal{V}_1 = \mathcal{V}$, $\mathcal{F}_1 = \emptyset$, $\mathcal{O}_1 = \emptyset$, we obtain discrete open families $\mathcal{W}_1^S$, $\mathcal{W}_1^B$, $\mathcal{W}_1^Q$ of cardinality $\leq \tau$, which are disjoint in pairs, such that $\mathcal{W}_1 = \mathcal{W}_1^S \cup \mathcal{W}_1^B \cup \mathcal{W}_1^Q$ is a decomposition of X satisfying (a) – (h).

Since by (b) X is covered by $\mathcal{W}_1^S$, it follows that $\mathcal{W}_1^B = \mathcal{W}_1^Q = \emptyset$ (the only other possibilities $\mathcal{W}_1^B = \{\emptyset\}$ or $\mathcal{W}_1^Q = \{\emptyset\}$ are eliminated by the definition of decompositions, since they contain only nonempty sets).

Also, from the discreteness of $\mathcal{W}_1^S = \mathcal{W}_1$ it follows that its elements are pairwise disjoint clopen sets ($W = X \setminus \cup(\mathcal{W}_1 \setminus \{W\})$, for any $W \in \mathcal{W}_1$).

By (b) $\mathcal{W}_1$ refines $\mathcal{T}$, and by (f) $\mathcal{W}_1$ refines $\mathcal{V}$. Therefore, $\mathcal{W} = \mathcal{W}_1$ is the required refinement. ∎

2 The relative embeddability theorem

Theorem 2.1 *Let X be a 0-dimensional metric space of weight $|\Lambda| = \tau \geq \aleph_0$ and let X_0 be a compact subspace of X. Then any embedding $f_0 : X_0 \longrightarrow L_0(\tau)$ can be extended to an embedding $f : X \longrightarrow L_0(\tau)$.*

Proof. For any fixed $n \in \mathbf{N}$ there are only finitely many nonempty subsets of X_0 of the type

$$f_0^{-1}(\varphi_{\lambda_1} \circ \cdots \circ \varphi_{\lambda_n} \Sigma) =$$

$$f_0^{-1}(\varphi_{\lambda_1} \circ \cdots \circ \varphi_{\lambda_n} \Sigma \setminus n\text{th level vertices}) =$$

$$f_0^{-1}(\varphi_{\lambda_1} \circ \cdots \circ \varphi_{\lambda_n} \Sigma \setminus \bigcup_{(\mu_1,\ldots,\mu_n)\neq(\lambda_1,\ldots,\lambda_n)} \varphi_{\mu_1} \circ \cdots \circ \varphi_{\mu_n} \Sigma) =$$

$$f_0^{-1}(L_0(\tau) \setminus \bigcup_{(\mu_1,\ldots,\mu_n)\neq(\lambda_1,\ldots,\lambda_n)} \varphi_{\mu_1} \circ \cdots \circ \varphi_{\mu_n} \Sigma),$$

where the first equality holds true since $f_0(X_0)$ is disjoint with the set of all rational points of $\Sigma(\tau)$, and consequently with the nth level vertices, too. The second equality follows from the fact that $\varphi_{\lambda_1} \circ \cdots \circ \varphi_{\lambda_n} \Sigma \cap \varphi_{\mu_1} \circ \cdots \circ \varphi_{\mu_n} \Sigma$ is either empty, or an nth level vertex (see Lemma 4 of [7]). Note that

$$f_0^{-1}(\varphi_{\lambda_1} \circ \cdots \circ \varphi_{\lambda_n} \Sigma) = f_0^{-1}(L_0(\tau) \setminus \bigcup_{(\mu_1,\ldots,\mu_n)\neq(\lambda_1,\ldots,\lambda_n)} \varphi_{\mu_1} \circ \cdots \circ \varphi_{\mu_n} \Sigma) \quad (3)$$

shows that $f_0^{-1}(\varphi_{\lambda_1} \circ \cdots \circ \varphi_{\lambda_n} \Sigma)$ are closed and open subsets of X_0, since the family $\{\varphi_{\lambda_1} \circ \cdots \circ \varphi_{\lambda_n} \Sigma : (\lambda_1, \ldots, \lambda_n) \in \Lambda^n\}$ is locally finite (see the text following Corollary 15 of [7]). Also, from (3) it follows that nonempty preimages

$\{f_0^{-1}(\varphi_{\lambda_1}\circ\cdots\circ\varphi_{\lambda_n}\Sigma)\colon(\lambda_1,\ldots,\lambda_n)\in\Lambda^n\}$ form an open pairwise disjoint cover of X_0, hence the cover is finite, since X_0 is compact.

Let

$$\Lambda_*^n=\{(\lambda_1,\ldots,\lambda_n)\in\Lambda^n\colon f_0^{-1}(\varphi_{\lambda_1}\circ\cdots\circ\varphi_{\lambda_n}\Sigma)\neq\emptyset\}.$$

For any $(\lambda_1,\ldots,\lambda_n)\in\Lambda_*^n$ we choose an open subset $U_{(\lambda_1,\ldots,\lambda_n)}$ of X, such that

$$U_{(\lambda_1,\ldots,\lambda_n)}\cap X_0=f_0^{-1}(\varphi_{\lambda_1}\circ\cdots\circ\varphi_{\lambda_n}\Sigma). \tag{4}$$

Let

$$T_{(\lambda_1,\ldots,\lambda_n)}=U_{(\lambda_1,\ldots,\lambda_n)}\cap \mathrm{B}(f_0^{-1}(\varphi_{\lambda_1}\circ\cdots\circ\varphi_{\lambda_n}\Sigma),\frac{1}{n}). \tag{5}$$

Let $\mathcal{T}_n$ be the family consisting of all sets $T_{(\lambda_1,\ldots,\lambda_n)}$, $(\lambda_1,\ldots,\lambda_n)\in\Lambda_*^n$, and all open balls of the diameter $<\frac{1}{n}$, which are disjoint with X_0.

Next we inductively construct a sequence $\mathcal{V}_0$, $\mathcal{V}_1$, ... of clopen discrete covers (hence decompositions) of X, putting $\mathcal{V}_0=\{X\}$, and then applying Lemma 1.2 to $\mathcal{V}_n$ in the place of $\mathcal{V}$, with $\mathcal{T}=\mathcal{T}_n$, and then taking so obtained $\mathcal{W}$ for $\mathcal{V}_{n+1}$.

We wish to index the families $\mathcal{V}_1$, $\mathcal{V}_2$, ... in a way similar to [6] (or to [7, 8], where it was done in a slightly different way) by the elements of Λ^n, $n\in\mathbf{N}$, denoting their elements $V_{(\lambda_1,\ldots,\lambda_n)}$, with the goal that the required embedding f can be defined by the formula

$$f(x)=[\lambda_1,\lambda_2,\ldots]\overset{def}{\iff}x\in\bigcap_{n\in\mathbf{N}}V_{(\lambda_1,\ldots,\lambda_n)}.$$

In order to secure the validity of $f/X_0=f_0$, it must hold true that

$$x\in f_0^{-1}(\varphi_{\lambda_1}\circ\cdots\circ\varphi_{\lambda_n}\Sigma)\implies x\in V_{(\lambda_1,\ldots,\lambda_n)},$$

i.e.

$$f_0^{-1}(\varphi_{\lambda_1}\circ\cdots\circ\varphi_{\lambda_n}\Sigma)\subseteq V_{(\lambda_1,\ldots,\lambda_n)}.$$

That means that

$$\{f_0^{-1}(\varphi_{\lambda_1}\circ\cdots\circ\varphi_{\lambda_n}\Sigma)\colon(\lambda_1,\ldots,\lambda_n)\in\Lambda_*^n\}$$

must refine $\mathcal{V}_n$. Since this is not necessarily so, we must modify families $\mathcal{V}_n$, and we do it by replacing families $\mathcal{V}_n$ by $\mathcal{V}_n'$s, which are defined as follows:

$$\begin{gathered}\mathcal{V}_n'=\{\cup\{V\in\mathcal{V}_n\colon V\cap f_0^{-1}(\varphi_{\lambda_1}\circ\cdots\circ\varphi_{\lambda_n}\Sigma)\neq\emptyset\}\colon(\lambda_1,\ldots,\lambda_n)\in\Lambda_*^n\}\cup\\ (\mathcal{V}_n\setminus\{V\in\mathcal{V}_n\colon\exists(\lambda_1,\ldots,\lambda_n)\in\Lambda_*^n,V\cap f_0^{-1}(\varphi_{\lambda_1}\circ\cdots\circ\varphi_{\lambda_n}\Sigma)\neq\emptyset\})=\\ \{\cup\{V\in\mathcal{V}_n\colon V\cap f_0^{-1}(\varphi_{\lambda_1}\circ\cdots\circ\varphi_{\lambda_n}\Sigma)\neq\emptyset\}\colon(\lambda_1,\ldots,\lambda_n)\in\Lambda_*^n\}\cup\\ \{V\in\mathcal{V}_n\colon V\cap X_0=\emptyset\}.\end{gathered}$$

We denote

$$\mathcal{V}_n'^{(1)}=\{\cup\{V\in\mathcal{V}_n\colon V\cap f_0^{-1}(\varphi_{\lambda_1}\circ\cdots\circ\varphi_{\lambda_n}\Sigma)\neq\emptyset\}\colon(\lambda_1,\ldots,\lambda_n)\in\Lambda_*^n\}$$

and

$$\mathcal{V}_n'^{(2)} = \{V \in \mathcal{V}_n : V \cap X_0 = \emptyset\}.$$

We index the sets of the first type (i.e. those belonging to $\mathcal{V}_n'^{(1)}$) by the index, which is naturally offered, putting

$$\bigcup_{V\cap f_0^{-1}(\varphi_{\lambda_1}\circ\cdots\circ\varphi_{\lambda_n}\Sigma)\neq\emptyset} V = V_{(\lambda_1,\ldots,\lambda_n)}. \tag{6}$$

This way both the family $\mathcal{V}_n'$ and the indexing of $\mathcal{V}_n'^{(1)}$ are well defined, and $\mathcal{V}_n'$ obtained this way is still a discrete family of clopen sets.

The only property that may not be obvious is that for any $V \in \mathcal{V}_n$, the conditions

$$V \cap f_0^{-1}(\varphi_{\lambda_1} \circ \cdots \circ \varphi_{\lambda_n}\Sigma) \neq \emptyset,\ V \cap f_0^{-1}(\varphi_{\mu_1} \circ \cdots \circ \varphi_{\mu_n}\Sigma) \neq \emptyset \tag{7}$$

imply

$$(\lambda_1, \ldots, \lambda_n) = (\mu_1, \ldots, \mu_n). \tag{8}$$

For a given V satisfying (7) we prove this property by choosing $(\nu_1, \ldots, \nu_n) \in \Lambda_*^n$ such that $V \subseteq T_{(\nu_1,\ldots,\nu_n)}$. Such an n-tuple exists, since $\mathcal{V}_n$ refines $\mathcal{T}_n$.

Then, by (5)

$$V \subseteq U_{(\nu_1,\ldots,\nu_n)} \cap \mathrm{B}(f_0^{-1}(\varphi_{\nu_1} \circ \cdots \circ \varphi_{\nu_n}\Sigma), \frac{1}{n}),$$

hence, by (4)

$$V \cap X_0 \subseteq X_0 \cap U_{(\nu_1,\ldots,\nu_n)} \cap \mathrm{B}(f_0^{-1}(\varphi_{\nu_1} \circ \cdots \circ \varphi_{\nu_n}\Sigma), \frac{1}{n}) =$$

$$f_0^{-1}(\varphi_{\nu_1} \circ \cdots \circ \varphi_{\nu_n}\Sigma) \cap \mathrm{B}(f_0^{-1}(\varphi_{\nu_1} \circ \cdots \circ \varphi_{\nu_n}\Sigma), \frac{1}{n}) = f_0^{-1}(\varphi_{\nu_1} \circ \cdots \circ \varphi_{\nu_n}\Sigma). \tag{9}$$

Taking into account (7) we choose $x \in V \cap f_0^{-1}(\varphi_{\lambda_1} \circ \cdots \circ \varphi_{\lambda_n}\Sigma)$. Then it follows that $x \in V \cap X_0$, hence by (9) $x \in f_0^{-1}(\varphi_{\nu_1} \circ \cdots \circ \varphi_{\nu_n}\Sigma)$. Therefore

$$f_0(x) \in f_0^{-1}(\varphi_{\lambda_1} \circ \cdots \circ \varphi_{\lambda_n}\Sigma) \cap f_0^{-1}(\varphi_{\nu_1} \circ \cdots \circ \varphi_{\nu_n}\Sigma) \cap L_0(\tau).$$

Since for $(\lambda_1, \ldots, \lambda_n) \neq (\nu_1, \ldots, \nu_n)$ this intersection is empty (using the same result from [7] as before), it follows that $(\lambda_1, \ldots, \lambda_n) = (\nu_1, \ldots, \nu_n)$.

The same reasoning gives $(\mu_1, \ldots, \mu_n) = (\nu_1, \ldots, \nu_n)$, hence (8) is proved.

Note that proving (7) $\Longrightarrow$ (8) we have simultaneously proved

$$V_{(\lambda_1,\ldots,\lambda_n)} \cap X_0 \subseteq f_0^{-1}(\varphi_{\lambda_1} \circ \cdots \circ \varphi_{\lambda_n}\Sigma). \tag{10}$$

Also, we have proved

$$V_{(\lambda_1,\ldots,\lambda_n)} \subseteq \mathrm{B}(f_0^{-1}(\varphi_{\lambda_1} \circ \cdots \circ \varphi_{\lambda_n}\Sigma), \frac{1}{n}). \tag{11}$$

Now we proceed with indexing of the elements of $\mathcal{V}'_n$ of the second type, i.e. of the elements of $\mathcal{V}_n'^{(2)}$.

Let $V \in \mathcal{V}_n'^{(2)}$, $n \in \mathbf{N}$, and $V \subseteq V_{(\lambda_1,\ldots,\lambda_{n-1})} \in \mathcal{V}'_{n-1}$ (for $n = 1$ this condition is vacuously fulfilled). Since there are only finitely many elements in $\mathcal{V}_n'^{(1)}$ (because of the compactness of X_0), the set

$$\Lambda' = \Lambda \setminus \{\lambda \in \Lambda : V_{(\lambda_1,\ldots,\lambda_{n-1},\lambda)} \in \mathcal{V}_n'^{(1)}\}$$

has the cardinality τ. (For $n = 1$ it simply means $\Lambda' = \Lambda \setminus \{\lambda \in \Lambda : V_\lambda \in \mathcal{V}_1'^{(1)}\}$). Since there are at most τ sets V of this type, we may index all of them by different $(\lambda_1, \ldots, \lambda_{n-1}, \lambda)$ choosing $\lambda \in \Lambda'$. Also, because of infiniteness of τ, this may be done in such a way that $\lambda \neq \lambda_{n-1}$. (The last condition will serve in proving irrationality of $f(x)$ for any $x \in X$.)

As announced above, we now define $f : X \longrightarrow L_0(\tau)$ by the formula

$$f(x) = [\lambda_1, \lambda_2, \ldots] \overset{def}{\iff} x \in \bigcap_{n \in \mathbf{N}} V_{(\lambda_1,\ldots,\lambda_n)}. \tag{12}$$

The function f is well-defined by (12) as a function into $\Sigma(\tau)$, since for any fixed $x \in X$, and for any $n \in \mathbf{N}$, there is exactly one $V \in \mathcal{V}'_n$ such that $x \in V$, hence there is uniquely determined $(\lambda_1, \ldots, \lambda_n) \in \Lambda^n$, such that $x \in V_{(\lambda_1,\ldots,\lambda_n)}$. The indices for different n are chosen coherently, i.e.

$$V_{\lambda_1} \supseteq V_{\lambda_1,\lambda_2} \supseteq V_{\lambda_1,\lambda_2,\lambda_3} \supseteq \cdots,$$

so there is a unique $\lambda = (\lambda_1, \lambda_2, \ldots) \in \Lambda^n$, such that for any $n \in \mathbf{N}$, $x \in V_{(\lambda_1,\ldots,\lambda_n)}$, hence $f(x) = [\lambda_1, \lambda_2, \ldots]$ is well-defined.

Our goal $f/X_0 = f_0$ is achieved, since for any $x \in X_0$, $x \in V_{(\lambda_1,\ldots,\lambda_n)}$ implies $x \in f_0^{-1}(\varphi_{\lambda_1} \circ \cdots \circ \varphi_{\lambda_n} \Sigma)$ for any $n \in \mathbf{N}$, by (10). This means $f_0(x) \in \varphi_{\lambda_1} \circ \cdots \circ \varphi_{\lambda_n} \Sigma$, hence

$$f_0(x) \in \bigcap_{n \in \mathbf{N}} \varphi_{\lambda_1} \circ \cdots \circ \varphi_{\lambda_n} \Sigma = \{[\lambda_1, \lambda_2, \ldots]\} = \{f(x)\}.$$

Therefore $f(x) = f_0(x)$.

Now, we proceed by proving that $f(x)$ is an irrational point of $\Sigma(\tau)$ for any $x \in X$.

For $x \in X_0$, $f(x) = f_0(x) \in L_0(\tau)$, which is precisely the set of irrational points of $\Sigma(\tau)$.

If $x \in X \setminus X_0$, we choose an $n_0 \in \mathbf{N}$, such that $\frac{1}{n_0} < \mathrm{d}(x, X_0)$. Then, for any $n \geq n_0$, the set $V_{(\lambda_1,\ldots,\lambda_n)}$ containing x is disjoint with X_0, according to (5) and the fact that $\mathcal{V}_n$ refines $\mathcal{T}_n$. Hence $\lambda_n \neq \lambda_{n-1}$ for any $n \geq n_0$, by the conditions we have imposed on the indexing. Therefore $[\lambda_1, \lambda_2, \ldots]$ is an irrational point.

This proves that we may consider f as a function $f : X \longrightarrow L_0(\tau)$.

Let us prove the continuity of f at any given point $x \in X$. For any $\varepsilon > 0$ we choose an $n \in \mathbf{N}$, such that $\operatorname{diam} \Sigma / 2^n < \varepsilon$. Choose the uniquely determined

$(\lambda_1, \ldots, \lambda_n) \in \Lambda^n$, such that $f(x) \in \varphi_{\lambda_1} \circ \cdots \circ \varphi_{\lambda_n} \Sigma$. (The uniqueness follows from the irrationality of $f(x)$.) This means that $x \in V_{(\lambda_1,\ldots,\lambda_n)}$, and we prove that $V_{(\lambda_1,\ldots,\lambda_n)}$ is a neighborhood of x mapped by f into the ε-ball centered at $f(x)$: for any $y \in V_{(\lambda_1,\ldots,\lambda_n)}$, by (12) $f(y) \in \varphi_{\lambda_1} \circ \cdots \circ \varphi_{\lambda_n} \Sigma$, therefore $\mathrm{d}(f(x), f(y)) \leq \operatorname{diam} \varphi_{\lambda_1} \circ \cdots \circ \varphi_{\lambda_n} \Sigma = \operatorname{diam} \Sigma / 2^n < \varepsilon$. (The equality in the last formula has been proved in [8]; it easily follows from the fact that φ_λs are homotheties with the coefficient $1/2$).

We shall now prove that f is an embedding by proving that for any closed subset F of X, and for any $x \in X \setminus F$, $f(x) \notin \mathrm{Cl}_{L_0(\tau)} f(F)$ (this means that $\{f\}$ separates points and closed sets; see [2, p.82]).

Let F be a closed subset of X and $x \in X \setminus F$.

Case 1 Let $x \notin X_0$. Choose an $n \in \mathbf{N}$ such that $\frac{1}{n} < \min\{\mathrm{d}(x, X_0), \mathrm{d}(x, F)\}$. As in the proof of irrationality of $f(x)$ we see that $x \in V_{(\lambda_1,\ldots,\lambda_n)}$ implies that $V_{(\lambda_1,\ldots,\lambda_n)}$ is contained in a ball of diameter $< \frac{1}{n}$. Therefore $V_{(\lambda_1,\ldots,\lambda_n)} \cap X_0 = \emptyset$ and $V_{(\lambda_1,\ldots,\lambda_n)} \cap F = \emptyset$.

It follows that $f(F) \subseteq \Sigma \setminus \varphi_{\lambda_1} \circ \cdots \circ \varphi_{\lambda_n} \Sigma$ and from this we get

$$\mathrm{Cl}_{L_0(\tau)} f(F) = L_0(\tau) \cap \mathrm{Cl}_\Sigma f(F) \subseteq L_0(\tau) \cap \mathrm{Cl}_\Sigma (\Sigma \setminus \varphi_{\lambda_1} \circ \cdots \circ \varphi_{\lambda_n} \Sigma) =$$

$$L_0(\tau) \cap (\Sigma \setminus \varphi_{\lambda_1} \circ \cdots \circ \varphi_{\lambda_n} \Sigma) \subseteq \Sigma \setminus \varphi_{\lambda_1} \circ \cdots \circ \varphi_{\lambda_n} \Sigma.$$

Since $f(x) \in \varphi_{\lambda_1} \circ \cdots \circ \varphi_{\lambda_n} \Sigma$, we get $f(x) \notin \mathrm{Cl}_{L_0(\tau)} f(F)$.

Case 2 Let $x \in X_0$ and $f(x) = f_0(x) = [\lambda_1, \lambda_2, \ldots]$.

First we prove that there is a $k \in \mathbf{N}$, such that $f_0^{-1}(\varphi_{\lambda_1} \circ \cdots \circ \varphi_{\lambda_k} \Sigma) \cap F = \emptyset$.

Otherwise, choosing an $x_k \in f_0^{-1}(\varphi_{\lambda_1} \circ \cdots \circ \varphi_{\lambda_k} \Sigma) \cap F \subseteq X_0$ for any $k \in \mathbf{N}$, because of compactness of X_0 there would be a convergent subsequence x_{k_n}. Let $\lim x_{k_n} = x_0$. Since both X_0 and F are closed subsets of X, we infer that $x_0 \in X_0 \cap F$.

Let $f_0(x_0) = \lim f_0(x_{k_n}) = [\mu_1, \mu_2, \ldots]$. From $[\mu_1, \mu_2, \ldots] \neq [\lambda_1, \lambda_2, \ldots]$ it would follow that there is an $m \in \mathbf{N}$, such that $\varphi_{\lambda_1} \circ \cdots \circ \varphi_{\lambda_m} \Sigma \cap \varphi_{\mu_1} \circ \cdots \circ \varphi_{\mu_m} \Sigma = \emptyset$. Since $\varphi_{\mu_1} \circ \cdots \circ \varphi_{\mu_m} \Sigma$ is an neighborhood of $[\mu_1, \mu_2, \ldots]$, it would contain almost all $f_0(x_{k_n})$. But that contradicts $f_0(x_{k_n}) \in \varphi_{\lambda_1} \circ \cdots \circ \varphi_{\lambda_{k_n}} \Sigma$.

Therefore $f_0(x_0) = [\lambda_1, \lambda_2, \ldots] = f_0(x)$. Injectivity of f_0 implies $x = x_0$. Therefore $x_0 \in F$ and $x_0 = x \in X \setminus F$ — a contradiction.

Now we choose a fixed $k \in \mathbf{N}$, such that $f_0^{-1}(\varphi_{\lambda_1} \circ \cdots \circ \varphi_{\lambda_k} \Sigma) \cap F = \emptyset$. Because of the compactness of $f_0^{-1}(\varphi_{\lambda_1} \circ \cdots \circ \varphi_{\lambda_k} \Sigma)$ (note that this set is a closed subset of X_0) it follows that $\mathrm{d}(f_0^{-1}(\varphi_{\lambda_1} \circ \cdots \circ \varphi_{\lambda_k} \Sigma), F) > 0$. Then we choose an $n \in \mathbf{N}$ such that

$$\frac{1}{n} < \mathrm{d}(f_0^{-1}(\varphi_{\lambda_1} \circ \cdots \circ \varphi_{\lambda_k} \Sigma), F)$$

and $n > k$. Then

$$f_0^{-1}(\varphi_{\lambda_1} \circ \cdots \circ \varphi_{\lambda_n} \Sigma) \subseteq f_0^{-1}(\varphi_{\lambda_1} \circ \cdots \circ \varphi_{\lambda_k} \Sigma).$$

This implies

$$B(f_0^{-1}(\varphi_{\lambda_1} \circ \cdots \circ \varphi_{\lambda_n} \Sigma), \frac{1}{n}) \subseteq B(f_0^{-1}(\varphi_{\lambda_1} \circ \cdots \circ \varphi_{\lambda_k} \Sigma), \frac{1}{n})$$

which then implies

$$B(f_0^{-1}(\varphi_{\lambda_1} \circ \cdots \circ \varphi_{\lambda_n} \Sigma), \frac{1}{n}) \cap F \subseteq B(f_0^{-1}(\varphi_{\lambda_1} \circ \cdots \circ \varphi_{\lambda_k} \Sigma), \frac{1}{n}) \cap F = \emptyset.$$

Since $x \in V_{(\lambda_1,\ldots,\lambda_n)} \in \mathcal{V}_n'^{(1)}$, and since by (11) $V_{(\lambda_1,\ldots,\lambda_n)} \subseteq B(f_0^{-1}(\varphi_{\lambda_1} \circ \cdots \circ \varphi_{\lambda_n} \Sigma), \frac{1}{n})$, it follows that $V_{(\lambda_1,\ldots,\lambda_n)} \cap F = \emptyset$.

Hence, as in Case 1 we have $f(F) \subseteq \Sigma \setminus \varphi_{\lambda_1} \circ \cdots \circ \varphi_{\lambda_n} \Sigma$ and $f(x) \in \varphi_{\lambda_1} \circ \cdots \circ \varphi_{\lambda_n} \Sigma$, and we complete the proof in the same way as in Case 1. ∎

Remark 2.2 *The compactness condition in Theorem 2.1 cannot be replaced by closedness of X_0 in X, as the following simple example shows:*

$X_0 = L_0(\tau)$, $f_0 = id$, X a disjoint union of X_0 with any 0-dimensional metric space of weight $\leq \tau$.

Therefore, it would be of interest to find other conditions on X_0, and probably on $f_0(X_0)$, still guaranteeing the existence of an embedding extending f_0.

Remark 2.3 *Of course, it would also be of interest do extend our result to higher dimensions.*

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