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APPROXIMATION OF MAPS INTO LIPSCOMB'S SPACE BY EMBEDDINGS

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Approximation of maps into Lipscomb's space by embeddings

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Abstract

Let $\mathcal{J}(\tau)$ be Lipscomb's one-dimensional space (cf. [4]) and $L_n(\tau) = \{x \in \mathcal{J}(\tau)^{n+1}:$ *at least one coordinate of* x *is irrational* $\} \subseteq \mathcal{J}(\tau)^{n+1}$ Lipscomb's n-dimensional universal space of weight $\tau \geq \aleph_0$. Modifying Lipscomb's decompositions constructed in [4] and using the indexing techniques developed in [6, 2, 7] we prove:

Let X *be a metrizable space,* $\dim X \leq n$, $wX \leq \tau$, $f : X \longrightarrow \mathcal{J}(\tau)^{n+1}$ *a continuous map, and* ε *a positive number. Then there is an embedding* $\psi : X \longrightarrow L_n(\tau)$ *such that* $\mathrm{d}(f, \psi) \leq \varepsilon$.

Also, in the separable case an analogous result is obtained, in which the classic triangular Sierpiński curve (homeomorphic to $\mathcal{J}(3)$) is used instead of $\mathcal{J}(\aleph_0)$ (strengthening the results of [2]).

Keywords: covering dimension, (generalized) Sierpiński curve, universal space, Lipscomb's universal space, embedding, decompositions of topological spaces

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1 Introduction and definitions

In [3, 4] S. L. Lipscomb has defined the space $\mathcal{J}(\tau)$ as a factor-space of generalized Baire's 0-dimensional space and used it in his construction of a universal n-dimensional metrizable space of weight τ, $\tau \geq \aleph_0$, which was defined as a subspace of $\mathcal{J}(\tau)^{n+1}$. In [6, 7] it was proved that $\mathcal{J}(\tau)$ is homeomorphic to a generalized Sierpiński curve $\Sigma(\tau)$ (for all cardinal numbers $\tau \neq 0$, including the finite ones; $\tau = 3$ yields the classic triangular Sierpiński curve — compare [5]). Also, Lipscomb's embedding theorem is proved in [6, 7] by the use of a special indexing of certain decompositions of metrizable spaces. Here we use similar techniques to prove the theorem stated in the abstract.

We shall use the notation of [2] which is in turn based on the notation of [1, 3] (with a few slight modifications).

$|X|$ denotes the cardinal number of the set X.

Let $\tau \geq \aleph_0$ be a cardinal number and let Λ be a fixed set such that $|\Lambda| = \tau$. Then Baire's generalized 0-dimensional space (of weight τ) is the set $\Lambda^{\mathbf{N}}$ ($\mathbf{N} = \{1, 2, 3, \ldots\}$) of all sequences of elements of Λ equipped with the product topology (while Λ is equipped with the discrete topology).

For $\lambda = (\lambda_1, \ldots, \lambda_m, \ldots), \mu = (\mu_1, \ldots, \mu_m, \ldots)$ the equivalence relation $\sim$ is defined as follows ([3, 4]):

$\lambda \sim \mu \iff \lambda = \mu \,\text{or}\, \exists j \in \mathbf{N} \,\text{such that}:$

i) $\forall k, k < j \implies \lambda_k = \mu_k,$

ii) $\forall s \in \mathbf{N}, \lambda_j = \mu_{j+s},$

iii) $\forall s \in \mathbf{N}, \lambda_{j+s} = \mu_j.$

In the case $\mu \neq \lambda$ such a j is uniquely determined and is called the *tail index* of λ and μ. We also say that the two sequences are *interwoven.*

We shall make use of an analogous equivalence relation defined on Λ^m by
$(\lambda_1, \ldots, \lambda_m) \sim (\mu_1, \ldots, \mu_m) \iff (\lambda_1, \ldots, \lambda_m) = (\mu_1, \ldots, \mu_m)$
or $\exists j \in \{1, \ldots , m - 1\}$ such that

i) $\forall k, k < j \implies \lambda_k = \mu_k,$

ii) $\forall l \in \{j+1, \ldots, m\}, \lambda_j = \mu_l,$

iii) $\forall l \in \{j+1, \ldots, m\}, \lambda_l = \mu_j.$

Lipscomb's space $\mathcal{J}(\tau)$ is defined as the quotient space $\mathcal{J}(\tau) = \Lambda^{\mathbf{N}}/\sim$.

The equivalence class of $(\lambda_1, \ldots, \lambda_m, \ldots)$ is denoted by $[\lambda_1, \ldots, \lambda_m, \ldots]$. An equivalence class may be a singleton — in which case it is called an *irrational point* of $\mathcal{J}(\tau)$ — or a dyad — in which case it is called a *rational point* of $\mathcal{J}(\tau)$. This construction generalizes the construction of the segment $[0, 1]$ from the Cantor middle-third set by identifying the adjacent end points in the Cantor space. $\mathcal{J}(\tau)$ is a one-dimensional metrizable space of weight τ ([3]).

The *generalized Sierpiński curve* $\Sigma(\tau)$ was defined in [6] as a subspace of the Hilbert space $\ell_2(\tau) = \{(x_\lambda) \in \mathbf{R}^\Lambda : \sum_{\lambda \in \Lambda} x_\lambda^2 < \infty\}$ as follows.

Let e^λ, $\lambda \in \Lambda$, be defined by $e^\lambda_\mu = \delta_{\lambda,\mu}$ (the Kronecker symbol) for $\forall \mu \in \Lambda$, and let $\varphi_\lambda : \ell_2(\tau) \longrightarrow \ell_2(\tau)$ be defined by

$$(\varphi_\lambda(x))_\mu = \begin{cases} (x_\lambda + 1)/2, & \mu = \lambda \\ x_\mu/2, & \mu \neq \lambda \end{cases}$$

be the homotheties with the coefficients $1/2$ and the centers e^λ, $\lambda \in \Lambda$.

Let $\sigma = \{(x_\lambda) \in \ell_2(\tau) : \sum_{\lambda \in \Lambda} x_\lambda = 1 \;\&\; \forall \lambda, 0 \leq x_\lambda \leq 1\}$. Then $\Sigma = \mathrm{Cl}\,\sigma = \{(x_\lambda) \in \ell_2(\tau) : \sum_{\lambda \in \Lambda} x_\lambda \leq 1 \;\&\; \forall \lambda, 0 \leq x_\lambda \leq 1\}$.

Then finally

$$\Sigma_m = \bigcup_{(\lambda_1, \ldots, \lambda_m) \in \Lambda^m} \varphi_{\lambda_1} \circ \cdots \circ \varphi_{\lambda_m} \Sigma$$

and

$$\Sigma(\tau) = \bigcap_{m \in \mathbf{N}} \Sigma_m.$$

Call the images of the points e^λ, $\lambda \in \Lambda$, via all $\varphi_{\lambda_1} \circ \cdots \circ \varphi_{\lambda_n}$ (with the exception of the points e^λ) *the rational points of* $\Sigma(\tau)$, and all other points *the irrational points of* $\Sigma(\tau)$. In [6, 7] it was proved that $\chi : \mathcal{J}(\tau) \longrightarrow \Sigma(\tau)$ defined by

$$\chi([\lambda_1, \ldots, \lambda_m, \ldots]) = \bigcap_{m \in \mathbf{N}} \varphi_{\lambda_1} \circ \cdots \circ \varphi_{\lambda_m} \Sigma \tag{1}$$

is a homeomorphism which sends rational points to rational points and irrational points to irrational points.

In the rest of the paper we shall use the homeomorphism χ to identify the two spaces freely. In particular, $\mathcal{J}(\tau)$ is metrizied by the metric inherited from $\ell_2(\tau)$. $\mathcal{J}(\tau)^{n+1}$ is equipped with the metric $\mathrm{d}(x, y) = \max\{\mathrm{d}(x_j, y_j) : j = 1, \ldots, n+1\}$.

The constructions of $\mathcal{J}(\tau)$ and of $\Sigma(\tau)$ make sense for finite τ, too, when $\ell_2(\tau)$ is replaced by $\mathbf{R}^k$, for an appropriate $k \in \mathbf{N}$. The function χ, defined by (1), is in this case again a homeomorphism preserving rationality/irrationality of the points. Note that $\Sigma(3)$ is the classic triangular Sierpiński curve.

Lipscomb's universal space $L_n(\tau)$ (for the class of metrizable spaces of dimension $\leq n$ and weight $\leq \tau$) is the following subspace of $\mathcal{J}(\tau)^{n+1}$:

$$L_n(\tau) = \{x \in \mathcal{J}(\tau)^{n+1} : \textit{ at least one coordinate of } x \textit{ is irrational }\}.$$

(In this definition, as in all other situations, we shall not distinguish $\mathcal{J}(\tau)$ from $\Sigma(\tau)$, so if it will suit our purposes, we shall interpret $L_n(\tau)$ as $\{x \in \Sigma(\tau)^{n+1} :$ *at least one coordinate of* x *is irrational* $\}$.)

Let $\mathcal{U}$ be a family of subsets of X, $x \in X$. The *local order* of $\mathcal{U}$ at x is defined as $\mathrm{lord}_x\mathcal{U} = \inf\{k : x \text{ has a neighborhood intersecting } k \text{ elements of } \mathcal{U}\} \in \{0, 1, 2, \ldots, \infty\}$. The *local order* of $\mathcal{U}$ is defined as $\mathrm{lord}\mathcal{U} = \sup\{\mathrm{lord}_x\mathcal{U} : x \in X\}$.

$\mathrm{Bd}\mathcal{U} = \bigcup_{U \in \mathcal{U}} \mathrm{Bd}U$, where $\mathrm{Bd}U$ denotes the boundary of the set U; $\mathrm{Cl}\mathcal{U} = \bigcup_{U \in \mathcal{U}} \mathrm{Cl}U$, where $\mathrm{Cl}U$ denotes the closure of the set U.

A *decomposition* of the space X is a pairwise disjoint locally finite family of open nonempty subsets of X whose closures cover X.

2 The main theorem

In this section we are going to state and prove the main result of the paper that for any map from an n-dimensional metrizable space of weight τ into $\mathcal{J}(\tau)^{n+1}$ there is an embedding of that space into $L_n(\tau)$ arbitrarily close to the map.

As in [2] we shall construct special sequences of decompositions and then use an indexing similar to one obtained in [2] in order to construct an approximation of the given map by an embedding.

The main tool enabling us to perform the inductive construction of such sequences of decompositions is the following Lipscomb's lemma (the notation is modified to fit ours; compare [6, 2]):

Lemma 1 *([4, Lemma 4, p.152]) Let* $n \in \{0, 1, 2, \ldots\}$*. Let* X *be a metric space such that* $\dim X = n$*,* $wX = \tau \geq \aleph_0$*.*

Let $X = X_1 \cup X_2 \cup \cdots \cup X_{n+1}$, where $X_1, \ldots, X_{n+1}$ are pairwise disjoint 0-dimensional subsets of X.

Let $\mathcal{T}$ be an arbitrary open covering of X. For each j, $1 \le j \le n+1$, let $\mathcal{V}_j$ be a decomposition of X such that $|\mathcal{V}_j| \le \tau$ and $\operatorname{lord} \mathcal{V}_j \le 2$. Let $\mathcal{F}_j$, $|\mathcal{F}_j| \le \tau$, be a discrete closed family such that

$$\operatorname{Bd} \mathcal{V}_j = \bigcup \mathcal{F}_j, \tag{2}$$

and let for each $k \in \{1, \ldots, n+1\}$ and distinct $j_1, \ldots, j_k \in \{1, \ldots, n+1\}$

$$\dim(\operatorname{Bd} \mathcal{V}_{j_1} \cap \cdots \cap \operatorname{Bd} \mathcal{V}_{j_k}) \le n - k \tag{3}$$

hold.

Let $\mathcal{O}_j = \{O_F : F \in \mathcal{F}_j\}$ be an open family such that $F \subseteq O_F$ for each $F \in \mathcal{F}_j$.

Then for each j, $1 \le j \le n+1$, there are discrete families $\mathcal{W}_j^S$, $\mathcal{W}_j^B$, and $\mathcal{W}_j^Q$ of cardinality $\le \tau$, which are disjoint in pairs, such that

$$\mathcal{W}_j = \mathcal{W}_j^S \cup \mathcal{W}_j^B \cup \mathcal{W}_j^Q$$

is a decomposition of X satisfying (for each j, $1 \le j \le n+1$):
(a) $\operatorname{lord} \mathcal{W}_j \le 2$;
(b) $\{\operatorname{Cl} W : W \in \mathcal{W}_j^S\}$ *refines* $\mathcal{T}$; $\bigcup_{j=1}^{n+1} \mathcal{W}_j^S$ *covers* X;
(c) if $x \in \operatorname{Bd} \mathcal{W}_j$ *then there are distinct elements* W_1, W_2 *in* $\mathcal{W}_j$ *such that* $x \in \operatorname{Bd} W_1 \cap \operatorname{Bd} W_2$;
(d) $\mathcal{W}_j$ *covers* X_j *(hence* $\operatorname{Bd} \mathcal{W}_j$ *misses* X_j*)*;
(e) $\operatorname{Bd} \mathcal{W}_j \cap \operatorname{Bd} \mathcal{V}_j = \emptyset$;
(f) $\mathcal{W}_j^S \cup \mathcal{W}_j^Q$ *refines* $\mathcal{V}_j$;
(g) $\mathcal{W}_j^S \cup \mathcal{W}_j^B$ *is a discrete family;*
(h) $\mathcal{W}_j^B = \{W_F : F \in \mathcal{F}_j\}$ *(the indexing is faithful, i.e. injective) and* $F \subseteq W_F \subseteq \operatorname{Cl} W_F \subseteq O_F$ *for each* $F \in \mathcal{F}_j$.

If X is a metrizable space of dimension $\dim X = n$, it may be presented in the form $X = X_1 \cup X_2 \cup \cdots \cup X_{n+1}$, where $X_1, \ldots, X_{n+1}$ are pairwise disjoint 0-dimensional (or empty) subsets of X. We fix one such presentation and use it throughout the paper.

Besides Lemma 1 the main tool in performing the inductive construction in the proof of the main result, is the following lemma:

Lemma 2 *([2, Lemma 3]) Let families $\mathcal{V}$, $\mathcal{F}$, $\mathcal{W}^B$, $\mathcal{W}^S$ of subsets of X, where*

$$\mathcal{F} = \{\operatorname{Bd} V_1 \cap \operatorname{Bd} V_2 : V_1, V_2 \in \mathcal{V},\ V_1 \ne V_2,\ \operatorname{Bd} V_1 \cap \operatorname{Bd} V_2 \ne \emptyset\}, \tag{4}$$

satisfy

i) $\mathcal{V}$ *is a decomposition,* $|\mathcal{V}| \le \tau$, $\operatorname{lord} \mathcal{V} \le 2$;

ii) $x \in \operatorname{Bd}\mathcal{V} \iff$ *there exist distinct members* V_1, V_2 *of* $\mathcal{V}$, *such that* $x \in \operatorname{Bd} V_1 \cap \operatorname{Bd} V_2$;

iii) $\operatorname{Bd}\mathcal{V} = \bigcup \mathcal{F}$;

iv) $\mathcal{F}$ *is a discrete closed family of cardinality* $\leq \tau$;

v) $\mathcal{W}^B \cup \mathcal{W}^S$ *is an open discrete family,* $\mathcal{W}^B \cap \mathcal{W}^S = \emptyset$;

vi) $(\operatorname{Bd}\mathcal{W}^B \cup \operatorname{Bd}\mathcal{W}^S) \cap \operatorname{Bd}\mathcal{V} = \emptyset$;

vii) $\mathcal{W}^S$ *refines* $\mathcal{V}$;

viii) for each $F = \operatorname{Bd} V_1 \cap \operatorname{Bd} V_2 \in \mathcal{F}$ *there is an element* $W \in \mathcal{W}^B$, *such that* $F \subseteq W \subseteq \operatorname{Cl} W \subseteq V_1 \cup F \cup V_2$ *(since it is uniquely determined we denote it* W_F*);* $\mathcal{W}^B = \{W_F : F \in \mathcal{F}\}$;

ix) $\mathcal{W}^B$, $\mathcal{W}^S$ *are families of cardinality* $\leq \tau$;

x) for any $W \in \mathcal{W}^B \cup \mathcal{W}^S$ *and* $x \in \operatorname{Bd} W$, *and for any neighborhood* U *of* x,

$$U \cap (X \setminus \operatorname{Cl} W) \neq \emptyset.$$

Then, if we define

$$\mathcal{W}^R = \{V \setminus \operatorname{Cl}(\mathcal{W}^S \cup \mathcal{W}^B) : V \in \mathcal{V},\ V \setminus \operatorname{Cl}(\mathcal{W}^S \cup \mathcal{W}^B) \neq \emptyset\},$$

$$\mathcal{W} = \mathcal{W}^S \cup \mathcal{W}^B \cup \mathcal{W}^R,$$

$$\tilde{\mathcal{V}} = \{V \cap W : V \in \mathcal{V},\ W \in \mathcal{W},\ V \cap W \neq \emptyset\},$$

$$\tilde{\mathcal{F}} = \{\operatorname{Bd} V_1 \cap \operatorname{Bd} V_2 : V_1, V_2 \in \tilde{\mathcal{V}},\ V_1 \neq V_2,\ \operatorname{Bd} V_1 \cap \operatorname{Bd} V_2 \neq \emptyset\},$$

it holds true that $\tilde{\mathcal{V}}$, $\tilde{\mathcal{F}}$ *satisfy i) – iv), as well as the additional properties:*

$$\tilde{\mathcal{F}} = \mathcal{F} \cup \{\operatorname{Bd} W : W \in \mathcal{W}^S\} \cup \{(\operatorname{Bd} W_F) \cap V_k,\ k = 1, 2 :$$

$$F \in \mathcal{F}, W_F \in \mathcal{W}^B, V_1, V_2 \in \mathcal{V}, V_1 \neq V_2, W_F \subseteq V_1 \cup F \cup V_2\},$$

and

$$\tilde{\mathcal{V}} = \mathcal{W}^S \cup \mathcal{W}^R \cup \{V \cap W : V \in \mathcal{V},\ W \in \mathcal{W}^B,\ V \cap W \neq \emptyset\},$$

and $\mathcal{W}$ *satisfies*

a) $\mathcal{W}$ *is a decomposition and* $\operatorname{lord}\mathcal{W} \leq 2$ *[compare D1],*

b) $x \in \operatorname{Bd}\mathcal{W} \iff$ *there exist distinct members* W_1, W_2 *of* $\mathcal{W}$, *such that* $x \in \operatorname{Bd} W_1 \cap \operatorname{Bd} W_2$ *[compare D2],*

c) $\mathcal{W}^R \cap (\mathcal{W}^S \cup \mathcal{W}^B) = \emptyset$ *and* $\mathcal{W}^R$ *is an open discrete family [compare D3],*

d) $\operatorname{Bd}\mathcal{W} \cap \operatorname{Bd}\mathcal{V} = \emptyset$ *[compare D4],* *e)* $\mathcal{W}$ *of cardinality* $\leq \tau$ *[compare D10]. (The list of the properties D1 – D13 appears on the page 6.)*

Proof. Though the lemma was formulated and proved in [2] for the countable case only, it plainly holds true for any $\tau \geq \aleph_0$. The same proof works word by word after replacing the countability conditions by the condition, that the appropriate families are of cardinality $\leq \tau$. The cardinality bounds are then obviously satisfied, and in the proofs of all other properties separability has not been used. ∎

The same comment applies to other citations of results from [2], when used in the current paper.

The plan for the proof of the main result is this: using Lemmas 1, 2 and starting from appropriate decompositions $\mathcal{V}_{1,j}$ and families of closed sets $\mathcal{F}_{1,j}$, as well as appropriate coverings $\mathcal{T}_{i,j}$ (all open balls of sufficiently small radii) and families $\mathcal{O}_{i,j}$ (sufficiently narrow neighborhoods of elements of $\mathcal{F}_{i,j}$, i.e. the balls of small radii around the sets), we shall inductively get the decompositions $\mathcal{V}_{i,j}$, $\mathcal{W}_{i,j}$, and the families of closed subsets $\mathcal{F}_{i,j}$, $i \in \mathbf{N}$, $1 \leq j \leq n+1$, such that for all i, j:

D1 $\mathcal{V}_{i,j}$, $\mathcal{W}_{i,j}$ are decompositions of X and $\operatorname{lord} \mathcal{V}_{i,j} \leq 2$, $\operatorname{lord} \mathcal{W}_{i,j} \leq 2$;

D2 $x \in \operatorname{Bd} \mathcal{W}_{i,j} \iff$ there exist distinct members W_1, W_2 of $\mathcal{W}_{i,j}$, such that $x \in \operatorname{Bd} W_1 \cap \operatorname{Bd} W_2$;

D3 $\mathcal{W}_{i,j} = \mathcal{W}^S_{i,j} \cup \mathcal{W}^B_{i,j} \cup \mathcal{W}^R_{i,j}$, where $\mathcal{W}^S_{i,j}$, $\mathcal{W}^B_{i,j}$, $\mathcal{W}^R_{i,j}$ are discrete families which are disjoint in pairs, and $\mathcal{W}^S_{i,j} \cup \mathcal{W}^B_{i,j}$ is a discrete family (superscripts S, B, R come from *small*, *boundary* and *remnant*, and those are what we call the elements of the families — the terminology is motivated by their properties);

D4 $\operatorname{Bd} \mathcal{W}_{i,j} \cap \operatorname{Bd} \mathcal{V}_{i,j} = \emptyset$;

D5 $\mathcal{W}^S_{i,j}$ refines $\mathcal{V}_{i,j}$, i.e. every element of $\mathcal{W}^S_{i,j}$ is a subset of an element of $\mathcal{V}_{i,j}$;

D6 $\mathcal{W}^R_{i,j} = \{V \setminus \operatorname{Cl}(\mathcal{W}^S_{i,j} \cup \mathcal{W}^B_{i,j}) : V \in \mathcal{V}_{i,j},\ V \setminus \operatorname{Cl}(\mathcal{W}^S_{i,j} \cup \mathcal{W}^B_{i,j}) \neq \emptyset\}$;

D7 $\mathcal{F}_{i,j} = \{\operatorname{Bd} V_1 \cap \operatorname{Bd} V_2 : V_1, V_2 \in \mathcal{V}_{i,j},\ V_1 \neq V_2,\ \operatorname{Bd} V_1 \cap \operatorname{Bd} V_2 \neq \emptyset\}$;

D8 for each $F = \operatorname{Bd} V_1 \cap \operatorname{Bd} V_2 \in \mathcal{F}_{i,j}$ there is an element $W \in \mathcal{W}^B_{i,j}$, such that $F \subseteq W \subseteq \operatorname{Cl} W \subseteq V_1 \cup F \cup V_2$ (since it is uniquely determined we denote it by W_F); $\mathcal{W}^B_{i,j} = \{W_F : F \in \mathcal{F}_{i,j}\}$;

D9 $\mathcal{V}_{i+1,j} = \{V \cap W : V \in \mathcal{V}_{i,j},\ W \in \mathcal{W}_{i,j},\ V \cap W \neq \emptyset\}$;

D10 all the families have at most τ elements;

D11 the intersection of the elements from $\mathcal{W}^B_{k,j}$, $k \geq i$, containing a fixed $F \in \mathcal{F}_{i,j}$, is F;

D12 $\operatorname{Bd} \mathcal{W}_{i,j} \cap X_j = \emptyset$, for all $i \in \mathbf{N}$ and all j, $1 \leq j \leq n+1$;

D13 $\cup_{j=1}^{n+1}\mathcal{W}_{i,j}^{S}$ covers X, for all $i \in \mathbf{N}$;

D14 $\operatorname{diam} W < 1/i$, for all $i \in \mathbf{N}$, all j, $1 \leq j \leq n+1$, and all $W \in \mathcal{W}_{i,j}^{S}$.

The decompositions $\mathcal{V}_{i,j}$ play an essential role — using an appropriate indexing of the families $\mathcal{V}_{i,j}$, $i \in \mathbf{N}$, for a fixed j, we shall define a function $\psi_j : X \longrightarrow \Sigma(\tau)$, such that $\psi = (\psi_1, \ldots, \psi_{n+1})$ will be the required embedding $\psi : X \longrightarrow L_n(\tau) \subseteq \Sigma(\tau)^{n+1}$, $\mathrm{d}(f, \psi) \leq \varepsilon$.

In [6, 2, 7] no control on closeness of the embedding to a given map was required, and therefore it was sufficient to take $\mathcal{V}_{1,j} = \{X\}$ and $\mathcal{F}_{1,j} = \emptyset$. Our main — and most difficult — task in this paper is to construct $\mathcal{V}_{1,j}$ and $\mathcal{F}_{1,j}$ in such a way that $\mathrm{d}(\psi, f) \leq \varepsilon$ will be obtained at the end.

Now we are able to prove

Theorem 3 *Let X be a metrizable space, $\dim X = n$, $wX \leq \tau$, $f : X \longrightarrow \mathcal{J}(\tau)^{n+1}$ a continuous map, and ε a positive number. Then there is an embedding $\psi : X \longrightarrow L_n(\tau)$ such that $\mathrm{d}(f, \psi) \leq \varepsilon$.*

Proof. Recall that $\Sigma = \operatorname{Cl} \sigma \subseteq \ell_2$. The equality $\operatorname{diam} \varphi_{\lambda_1} \circ \cdots \circ \varphi_{\lambda_m} \Sigma = \operatorname{diam} \Sigma / 2^m$ is a simple consequence of the fact that mappings φ_λ are homotheties with the coefficient $1/2$ (for details see [6]).

Choose m so that $\operatorname{diam} \Sigma / 2^m < \varepsilon/2$. Then for any λ_1, λ_2, $\ldots$,λ_m from Λ it is true that $\operatorname{diam} \varphi_{\lambda_1} \circ \cdots \circ \varphi_{\lambda_m} \Sigma < \varepsilon/2$.

For each $j = 1, \ldots, n+1$ let $f_j = p_j \circ f$, where $p_j : \mathcal{J}(\tau)^{n+1} \longrightarrow \mathcal{J}(\tau)$ denotes the projection onto the jth factor.

We are going to modify the construction of decompositions and their indexing from [2] in such a way, that we will obtain mappings $\psi_j : X \longrightarrow \mathcal{J}(\tau)$, which will satisfy the inequalities $\mathrm{d}(f_j, \psi_j) \leq \varepsilon$, and such that $\psi = (\psi_1, \ldots, \psi_{n+1})$ will be an embedding of X into $L_n(\tau)$.

Recall that $\Sigma_m = \cup_{(\lambda_1, \ldots, \lambda_m) \in \Lambda^m} \varphi_{\lambda_1} \circ \cdots \circ \varphi_{\lambda_m} \Sigma$. We shall interpret the functions f_j as functions into Σ_m. Also, Cl, Int, etc. will apply to Σ_m if they are used in the range space.

In order to simplify notation, let us denote $K = \Lambda^m$. If $\kappa = (\lambda_1, \ldots, \lambda_m)$, let S_κ denote

$$S_\kappa = \varphi_{\lambda_1} \circ \cdots \circ \varphi_{\lambda_m} \Sigma.$$

Also, for any $\kappa \in K$, let T_κ denote

$$T_\kappa = \Sigma_m \setminus \bigcup_{\chi \in K \setminus \{\kappa\}} S_\chi = S_\kappa \setminus \bigcup_{\chi \in K \setminus \{\kappa\}} S_\chi.$$

Using Lemmas 4, 11, 12, and 13 of [6] we see that all S_κ are closed in Σ (and hence in Σ_m, too) and form a locally finite family and that all T_κ are open in Σ_m. The same Lemmas imply that each T_κ is obtained from S_κ by removing all the mth level vertices[1] (with the exception of vertices e^λ of Σ) and that $\mathcal{T} = \{T_\kappa : \kappa \in K\}$ is a decomposition of Σ_m (since $\operatorname{Cl} T_\kappa = S_\kappa$) with $\operatorname{lord} \mathcal{T} = 2$.

[1]Points of the form $\varphi_{\lambda_1} \circ \cdots \circ \varphi_{\lambda_m} e^\mu$.

In fact, exactly the mth level vertices have the local order 2; all other points have the local order 1.

Also, $\operatorname{lord}\{\operatorname{Cl} T_\kappa : \kappa \in K\} = 2$, hence, for each j,

$$\operatorname{lord}\{f_j^{-1}(\operatorname{Cl} T_\kappa) : \kappa \in K\} \le 2. \tag{5}$$

Note that all we are doing now is done for a fixed j, but, to keep notation as simple as possible, we will not use that j as an additional index until the very end of the proof, where we are going to use all indices simultaneously. The only exceptions are f_j's and X_j's, where the omission of j could cause an ambiguity (f and X already have its meaning).

For any *nonempty* set $f_j^{-1}(\operatorname{Cl} T_\kappa \cap \operatorname{Cl} T_{\kappa'})$, $\kappa \ne \kappa'$, we want to introduce an open subset $\Omega_{\kappa,\kappa'}$ of X, in such a way that all $\operatorname{Cl} \Omega_{\kappa,\kappa'}$ would form a discrete family, and that

$$\begin{gathered} f_j^{-1}(\operatorname{Cl} T_\kappa \cap \operatorname{Cl} T_{\kappa'}) \subseteq \Omega_{\kappa,\kappa'} \subseteq \\ (f_j^{-1}(\operatorname{Cl} T_\kappa) \cap f_j^{-1}(\operatorname{Cl} T_{\kappa'})) \cup f_j^{-1}(T_\kappa) \cup f_j^{-1}(T_{\kappa'}) \subseteq \\ f_j^{-1}(\operatorname{Cl} T_\kappa \cup \operatorname{Cl} T_{\kappa'}) \end{gathered} \tag{6}$$

would hold true.

Let us explain in some detail how this can be done, since it is of the fundamental importance for our work.

In Lemma 4 of [6] it is proved that

a) $\varphi_{\lambda_1} \circ \cdots \circ \varphi_{\lambda_n} \Sigma \cap \varphi_{\mu_1} \circ \cdots \circ \varphi_{\mu_n} \Sigma \ne \emptyset \iff (\lambda_1, \ldots, \lambda_n) \sim (\mu_1, \ldots, \mu_n)$ or $\lambda_1 = \mu_1, \cdots, \lambda_{n-1} = \mu_{n-1}$.

b) If two different n-tuples $(\lambda_1, \ldots, \lambda_n), (\mu_1, \ldots, \mu_n)$ are equivalent, with the tail index $k \le n-1$, then $\varphi_{\lambda_1} \circ \cdots \circ \varphi_{\lambda_k} e^{\lambda_{k+1}} = \varphi_{\lambda_1} \circ \cdots \circ \varphi_{\lambda_{k-1}} \varphi_{\lambda_{k+1}} e^{\lambda_k}$ is the only point of $\varphi_{\lambda_1} \circ \cdots \circ \varphi_{\lambda_n} \Sigma \cap \varphi_{\mu_1} \circ \cdots \circ \varphi_{\mu_n} \Sigma$.

c) If two different n-tuples $(\lambda_1, \ldots, \lambda_n)$,$(\mu_1, \ldots, \mu_n)$ satisfying a) are not equivalent then $\varphi_{\lambda_1} \circ \cdots \circ \varphi_{\lambda_n} e^{\mu_n} = \varphi_{\lambda_1} \circ \cdots \circ \varphi_{\lambda_{n-1}} \varphi_{\mu_n} e^{\lambda_n}$ is the only point of $\varphi_{\lambda_1} \circ \cdots \circ \varphi_{\lambda_n} \Sigma \cap \varphi_{\mu_1} \circ \cdots \circ \varphi_{\mu_n} \Sigma$.

Therefore, for any $\kappa \ne \kappa'$, $\kappa = (\lambda_1, \ldots, \lambda_n)$, $\kappa' = (\mu_1, \ldots, \mu_n)$, with nonempty $f_j^{-1}(\operatorname{Cl} T_\kappa \cap \operatorname{Cl} T_{\kappa'})$, it holds true that $(\lambda_1, \ldots, \lambda_{n+2}) = (\lambda_1, \ldots, \lambda_n, \lambda_n, \lambda_n)$ and $(\mu_1, \ldots, \mu_{n+2}) = (\mu_1, \ldots, \mu_n, \mu_n, \mu_n)$ are interwoven in case b), and that $(\lambda_1, \ldots, \lambda_{n+2}) = (\lambda_1, \ldots, \lambda_n, \mu_n, \mu_n)$ and $(\mu_1, \ldots, \mu_{n+2}) = (\mu_1, \ldots, \mu_n, \lambda_n, \lambda_n)$ are interwoven in case c). In both cases $\varphi_{\lambda_1} \circ \cdots \circ \varphi_{\lambda_{n+2}} \Sigma \cup \varphi_{\mu_1} \circ \cdots \circ \varphi_{\mu_{n+2}} \Sigma$ is a closed neighborhood of $\operatorname{Cl} T_\kappa \cap \operatorname{Cl} T_{\kappa'}$.

Defining

$$\Omega_{\kappa,\kappa'} = f_j^{-1}(\operatorname{Int}_{\Sigma_m}(\varphi_{\lambda_1} \circ \cdots \circ \varphi_{\lambda_{n+2}} \Sigma \cup \varphi_{\mu_1} \circ \cdots \circ \varphi_{\mu_{n+2}} \Sigma)),$$

we obtain a discrete open family, satisfying (6).

This follows from the fact that the sets $\varphi_{\lambda_1} \circ \cdots \circ \varphi_{\lambda_{n+2}} \Sigma \cup \varphi_{\mu_1} \circ \cdots \circ \varphi_{\mu_{n+2}} \Sigma$ form a discrete family and from the fact that

$$\begin{gathered} \operatorname{Cl} T_\kappa \cap \operatorname{Cl} T_{\kappa'} \subseteq \varphi_{\lambda_1} \circ \cdots \circ \varphi_{\lambda_{n+2}} \Sigma \cap \varphi_{\mu_1} \circ \cdots \circ \varphi_{\mu_{n+2}} \Sigma \subseteq \\ \operatorname{Int}_{\Sigma_m}(\varphi_{\lambda_1} \circ \cdots \circ \varphi_{\lambda_{n+2}} \Sigma \cup \varphi_{\mu_1} \circ \cdots \circ \varphi_{\mu_{n+2}} \Sigma) \subseteq \end{gathered}$$

$$\varphi_{\lambda_1} \circ \cdots \circ \varphi_{\lambda_{n+2}} \Sigma \cup \varphi_{\mu_1} \circ \cdots \circ \varphi_{\mu_{n+2}} \Sigma \subseteq$$
$$(\mathrm{Cl}\, T_\kappa \cap \mathrm{Cl}\, T_{\kappa'}) \cup T_\kappa \cup T_{\kappa'} \subseteq \mathrm{Cl}\, T_\kappa \cup \mathrm{Cl}\, T_{\kappa'}.$$

See Fig. 1, where the shaded regions represent the originals in the case $m = 1$).

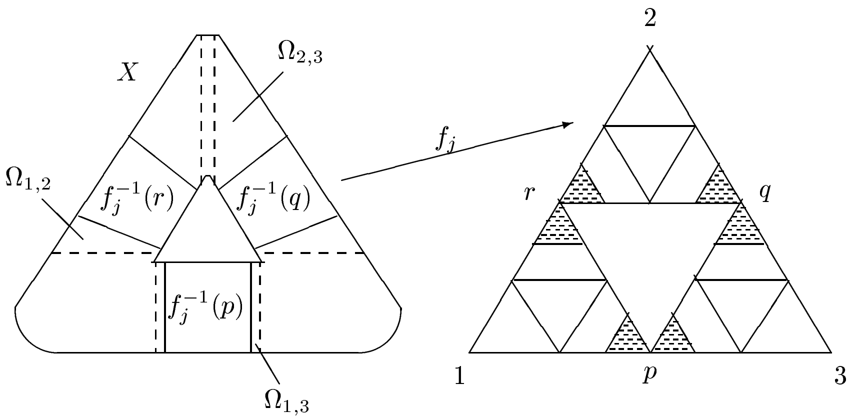

Figure 1: Defining $\Omega_{\kappa,\kappa'}$ for $m = 1$.

In doing this, we will not distinguish $\Omega_{\kappa,\kappa'}$ from $\Omega_{\kappa',\kappa}$ (i.e. we shall choose the same set in both cases).

Let us now introduce the following convention: if $f_j^{-1}(\mathrm{Cl}\, T_\kappa \cap \mathrm{Cl}\, T_{\kappa'})$ is an empty set, then we define $\Omega_{\kappa,\kappa'} = \emptyset$. Obviously, the above property (6) is still satisfied.

For each pair of distinct indices κ, κ' (using 0-dimensionality of X_j) we choose an open set $O_{\kappa,\kappa'}$ satisfying:

$$f_j^{-1}(\mathrm{Cl}\, T_\kappa \cap \mathrm{Cl}\, T_{\kappa'}) \subseteq O_{\kappa,\kappa'} \subseteq \mathrm{Cl}\, O_{\kappa,\kappa'} \subseteq \Omega_{\kappa,\kappa'} \tag{7}$$

and

$$\mathrm{Bd}\, O_{\kappa,\kappa'} \cap X_j = \emptyset.$$

Again we take $O_{\kappa,\kappa'} = O_{\kappa',\kappa}$, and if $f_j^{-1}(\mathrm{Cl}\, T_\kappa \cap \mathrm{Cl}\, T_{\kappa'}) = \emptyset$, then we put $O_{\kappa,\kappa'} = \emptyset$.

Obviously, the sets $O_{\kappa,\kappa'}$ form a discrete family

$$\mathcal{N} = \{O_{\kappa,\kappa'} : \kappa, \kappa' \in K,\, \kappa \neq \kappa'\}. \tag{8}$$

Define

$$U_\kappa = f_j^{-1}(T_\kappa) \cup \bigcup_{\kappa' \neq \kappa} O_{\kappa,\kappa'}. \tag{9}$$

The discreteness of $\mathcal{N}$ implies

$$\mathrm{Cl}\, U_\kappa = \mathrm{Cl}\, f_j^{-1}(T_\kappa) \cup \bigcup_{\kappa' \neq \kappa} \mathrm{Cl}\, O_{\kappa,\kappa'}.$$

In fact

$$\operatorname{Cl} U_\kappa = f_j^{-1}(T_\kappa) \cup \bigcup_{\kappa' \neq \kappa} \operatorname{Cl} O_{\kappa,\kappa'} \tag{10}$$

since

$$\operatorname{Cl} f_j^{-1}(T_\kappa) \setminus f_j^{-1}(T_\kappa) \subseteq f_j^{-1}(\operatorname{Cl} T_\kappa) \setminus f_j^{-1}(T_\kappa) =$$

$$f_j^{-1}(\operatorname{Cl} T_\kappa \setminus T_\kappa) = f_j^{-1}\Big(\bigcup_{\kappa' \neq \kappa} (\operatorname{Cl} T_\kappa \cap \operatorname{Cl} T_{\kappa'})\Big) =$$

$$\bigcup_{\kappa' \neq \kappa} f_j^{-1}(\operatorname{Cl} T_\kappa \cap \operatorname{Cl} T_{\kappa'}) \subseteq \bigcup_{\kappa' \neq \kappa} \operatorname{Cl} O_{\kappa,\kappa'}.$$

The family $\mathcal{U} = \{U_\kappa : \kappa \in K\}$ is an open cover of X, since

$$f_j^{-1}(\operatorname{Cl} T_\kappa) = f_j^{-1}(\operatorname{Cl} T_\kappa \setminus T_\kappa) \cup f_j^{-1}(T_\kappa) =$$

$$f_j^{-1}(T_\kappa) \cup \bigcup_{\kappa' \neq \kappa} f_j^{-1}(\operatorname{Cl} T_\kappa \cap \operatorname{Cl} T_{\kappa'}) \subseteq f_j^{-1}(T_\kappa) \cup \bigcup_{\kappa' \neq \kappa} O_{\kappa,\kappa'} = U_\kappa$$

and $\bigcup_{\kappa \in K} \operatorname{Cl} T_\kappa = \Sigma_m$.

For any nonempty $\operatorname{Cl} O_{\kappa,\kappa'}$, the set $\Omega_{\kappa,\kappa'}$ is the neighborhood of $\operatorname{Cl} O_{\kappa,\kappa'}$ meeting exactly two elements of the family $\mathcal{U}$. If a point x of X does not belong to any $\operatorname{Cl} O_{\kappa,\kappa'}$, then for some $\kappa \in K$ it belongs to

$$U_\kappa \setminus \bigcup_{\kappa' \neq \kappa} \operatorname{Cl} O_{\kappa,\kappa'} = \operatorname{Cl} U_\kappa \setminus \bigcup_{\kappa' \neq \kappa} \operatorname{Cl} O_{\kappa,\kappa'}. \tag{11}$$

This set is a neighborhood of x, intersecting exactly one member of the family $\mathcal{U}$ (namely, U_κ). Therefore,

$$\operatorname{lord} \mathcal{U} \leq 2. \tag{12}$$

Let $\kappa \neq \kappa'$. Then $(\bigcup_{\mu \neq \kappa'} O_{\kappa',\mu}) \cap (\bigcup_{\nu \neq \kappa} O_{\kappa,\nu}) = O_{\kappa,\kappa'}$, because of the discreteness of $\mathcal{N}$ (see (8)). Therefore, $O_{\kappa,\kappa'} \subseteq U_\kappa \cap U_{\kappa'}$. Also $f_j^{-1}(T_\kappa \cap T_{\kappa'}) \subseteq O_{\kappa,\kappa'}$ by (7). Further, for $\mu \neq \kappa, \kappa'$, by (6) and (7), $f_j^{-1}(T_\kappa) \cap O_{\kappa',\mu} \subseteq f_j^{-1}(T_\kappa) \cap \big((f_j^{-1}(\operatorname{Cl} T_{\kappa'}) \cap f_j^{-1}(\operatorname{Cl} T_\mu)) \cup f_j^{-1}(T_{\kappa'}) \cup f_j^{-1}(\operatorname{Cl} T_\mu)\big) \subseteq f_j^{-1}(T_\kappa \cap \operatorname{Cl} T_{\kappa'} \cap \operatorname{Cl} T_\mu) \cup f_j^{-1}(T_\kappa \cap T_{\kappa'}) \cup f_j^{-1}(T_\kappa \cap T_\mu) = \emptyset$. Therefore $f_j^{-1}(T_\kappa) \cap \bigcup_{\mu \neq \kappa'} O_{\kappa',\mu} \subseteq O_{\kappa,\kappa'}$. Analogously $f_j^{-1}(T_{\kappa'}) \cap \bigcup_{\lambda \neq \kappa} O_{\kappa,\lambda} \subseteq O_{\kappa,\kappa'}$. Therefore, $U_\kappa \cap U_{\kappa'} \subseteq O_{\kappa,\kappa'}$, hence we have established

$$U_\kappa \cap U_{\kappa'} = O_{\kappa,\kappa'}. \tag{13}$$

Using analogous reasoning, one can see that also

$$\operatorname{Cl} U_\kappa \cap \operatorname{Cl} U_{\kappa'} = \operatorname{Cl} O_{\kappa,\kappa'}. \tag{14}$$

Now, fix a well order $<$ on K. After that we define

$$V_\kappa = U_\kappa \setminus \bigcup_{\nu < \kappa} \operatorname{Cl} O_{\kappa,\nu}. \tag{15}$$

V_κ is open, due to the discreteness of $\mathcal{N}$.

Let

$$\mathcal{V} = \{V_\kappa : \kappa \in K,\, V_\kappa \neq \emptyset\}. \tag{16}$$

Since $V_\kappa \subseteq U_\kappa$, it follows $\operatorname{lord} \mathcal{V} \leq 2$.

Let $\kappa' < \kappa$. Then $V_\kappa \cap V_{\kappa'} \subseteq U_\kappa \cap U_{\kappa'} = O_{\kappa,\kappa'}$, by (13). Since $V_\kappa \cap O_{\kappa,\kappa'} = \emptyset$ by (15), it follows $V_\kappa \cap V_{\kappa'} = \emptyset$. Therefore, the elements of $\mathcal{V}$ are disjoint in pairs. To prove that $\mathcal{V}$ is a decomposition of X, it remains only to prove that

$$\operatorname{Cl} \mathcal{V} = X. \tag{17}$$

Simultaneously we shall prove that

$$\operatorname{Cl} V_\kappa \cap \operatorname{Cl} V_{\kappa'} = \operatorname{Bd} V_\kappa \cap \operatorname{Bd} V_{\kappa'} \subseteq \operatorname{Bd} O_{\kappa,\kappa'} \tag{18}$$

holds if $\kappa \neq \kappa'$.

Let x be an element of X and let

$$\kappa = \min\{\nu \in K : x \in \operatorname{Cl} U_\nu\}. \tag{19}$$

The minimum exists, since $\operatorname{Cl} \mathcal{U} = X$, therefore

$$x \in \operatorname{Cl} U_\kappa,$$

and $x \notin \operatorname{Cl} U_\nu$, for every $\nu < \kappa$. Consequently, $x \notin \operatorname{Cl} O_{\kappa,\nu}$ by (10). Hence, if $x \in U_\kappa$, it follows $x \in V_\kappa$ by (15).

Suppose $x \notin V_\kappa$. Then, by (9), (10), and (15), we see that $x \in (\operatorname{Cl} U_\kappa) \setminus V_\kappa = \left(U_\kappa \cup \bigcup_{\kappa' \neq \kappa} \operatorname{Cl} O_{\kappa,\kappa'}\right) \setminus \left(U_\kappa \setminus \bigcup_{\kappa' < \kappa} \operatorname{Cl} O_{\kappa,\kappa'}\right) \subseteq \bigcup_{\kappa' \neq \kappa} \operatorname{Cl} O_{\kappa,\kappa'}$. Because of the discreteness of $\mathcal{N}$ (8), there is a uniquely determined $\kappa' \neq \kappa$, such that $x \in \operatorname{Cl} O_{\kappa,\kappa'}$. Then, by (10), $x \in \operatorname{Cl} U_{\kappa'}$, and by the minimality of κ (19), it follows $\kappa < \kappa'$. Since, by (9) and (15), $O_{\kappa,\kappa'} \subseteq V_\kappa$, it follows

$$\operatorname{Cl} O_{\kappa,\kappa'} \subseteq \operatorname{Cl} V_\kappa, \tag{20}$$

hence $x \in \operatorname{Cl} V_\kappa$. In this case $x \in \operatorname{Bd} V_\kappa$.

Therefore, we have proved that $\mathcal{V}$ is a decomposition of X.

Since $V_\kappa \cap V_{\kappa'} = \emptyset$, it follows $\operatorname{Cl} V_\kappa \cap \operatorname{Cl} V_{\kappa'} = \operatorname{Bd} V_\kappa \cap \operatorname{Bd} V_{\kappa'}$, for any $\kappa' \neq \kappa$.

Let $x \in \operatorname{Bd} V_\kappa \cap \operatorname{Bd} V_{\kappa'}$, for $\kappa' \neq \kappa$. Without loss of generality, we may assume $\kappa < \kappa'$. Then $\kappa = \min\{\nu \in K : x \in \operatorname{Cl} U_\nu\}$ because of (12), since $x \in \operatorname{Cl} U_\kappa$ and $x \in \operatorname{Cl} U_{\kappa'}$. Therefore, as above it follows $x \in \operatorname{Cl} O_{\kappa,\kappa'}$. From $x \in O_{\kappa,\kappa'}$ it follows $x \in U_\kappa$ and finally $x \in V_\kappa$, a contradiction with $x \in \operatorname{Bd} V_\kappa$. Therefore $x \in \operatorname{Bd} O_{\kappa,\kappa'}$.

The last inclusion in (18) is important, since it implies that

$$\operatorname{Bd} \mathcal{V} \cap X_j = \emptyset, \tag{21}$$

and this in turn implies that property (3) of Lemma 1 will be satisfied for the families we are now constructing (more detailed explanation follows after a few lines).

There is only one more modification needed. Replacing any $V \in \mathcal{V}$ by

$$V^+ = \operatorname{Int} \operatorname{Cl} V$$

we preserve all properties of $\mathcal{V}$ mentioned above, including (21), while making (2) fulfilled.

Now the families $\mathcal{V}$ (obtained for different $j = 1, \ldots, n+1$) may play the role of $\mathcal{V}_{1,j}$, $j = 1, \ldots, n+1$, i.e. they may be taken as the initial stage in the inductive construction announced on the page 6.

This means that the families $\mathcal{V}_{1,j}$, $j = 1, \ldots, n+1$, satisfy the properties of the families $\mathcal{V}_j$ of Lemma 1, and that if we take $\{\operatorname{Bd} V_1 \cap \operatorname{Bd} V_2 \neq \emptyset : V_1, V_2 \in \mathcal{V}_{1,j}\}$ for $\mathcal{F}_j$, so obtained families satisfy all the conditions of Lemma 1. Taking all open balls of the diameter less then 1 for $\mathcal{T}_j$, and open balls of the diameter 1 around all the elements of $\mathcal{F}_j$ for $\mathcal{O}_j$, applying Lemma 1 we get $\mathcal{W}^S_{1,j}$, $\mathcal{W}^B_{1,j}$.

It may again be necesary to replace the elements W of $\mathcal{W}^S_{1,j}$, $\mathcal{W}^B_{1,j}$ by $W^+ = \operatorname{Int} \operatorname{Cl} W$ (see Lemma 6 of [2] for the details of the proof why this may be done) in order to make sure that all the required properties of $\mathcal{V}$, $\mathcal{F}$, $\mathcal{W}^B$, $\mathcal{W}^S$ in Lemma 2 will be satisfied if we take them to be $\mathcal{V}_{1,j}$, $\mathcal{F}_{1,j}$, $\mathcal{W}^B_{1,j}$, $\mathcal{W}^S_{1,j}$, respectively. (While replacing W by W^+ we shall preserve $\mathcal{W}^S_{1,j}$, $\mathcal{W}^B_{1,j}$ as the symbols for the new families, in order to simplify the notation.)

Now Lemma 2 gives us families $\mathcal{W}^R$, $\mathcal{W}$, $\tilde{\mathcal{V}}$, $\tilde{\mathcal{F}}$, which we rename to $\mathcal{W}^R_{1,j}$, $\mathcal{W}_{1,j}$, $\mathcal{V}_{2,j}$, $\mathcal{F}_{2,j}$.

So obtained $\mathcal{V}_{2,j}$, $\mathcal{F}_{2,j}$ (in place of $\mathcal{V}$, $\mathcal{F}$) now satisfy all the requirements of Lemma 1, and the proces continues that way while the whole infinite sequences are obtained. (Doing this we take all open balls of the diameter less then $1/i$ for $\mathcal{T}_j$, and open balls of the diameter $1/i$ around all the elements of $\mathcal{F}_j$ for $\mathcal{O}_j$.)

It can easily be seen that the families $\mathcal{V}_{i,j}$, $\mathcal{F}_{i,j}$, and $\mathcal{W}_{i,j}$, $i \in \mathbf{N}$, $j = 1, \ldots, n+1$, obtained that way satisfy the properties D1–D14. It is also very important for our goal (proving the theorem) that they are construced inductively beginning from the families $\mathcal{V}_{1,j}$, obtained by the above construction.

Our next goal is to construct an indexing of the elements of the families $\mathcal{V}_{i,j}$ satisfying the following properties:

I1 Each element of $\mathcal{V}_{1,j}$ is indexed by an element $\kappa \in K = \Lambda^m$. Each element of $\mathcal{V}_{i,j}$, $i \geq 2$, is indexed by an element of Λ^{m+i-1}.

I2 Let i be the first index such that $F \in \mathcal{F}_{i,j}$ (i.e. $F \in \mathcal{F}_{i,j} \setminus \mathcal{F}_{i-1,j}$ if $i \geq 2$, or $F \in \mathcal{F}_{1,j}$ if $i = 1$). If $F = \operatorname{Bd} V_1 \cap \operatorname{Bd} V_2 \neq \emptyset$, for $V_1, V_2 \in \mathcal{V}_{i,j}$, $V_1 \neq V_2$, then V_1, V_2 are indexed by $(\lambda_1, \ldots, \lambda_k, \mu)$, $(\lambda_1, \ldots, \lambda_k, \nu)$, $\mu \neq \nu$.

For any $l > i$, $F \in \mathcal{F}_{l,j}$. Let $F = \operatorname{Bd} \tilde{V}_1 \cap \operatorname{Bd} \tilde{V}_2$, $\tilde{V}_1, \tilde{V}_2 \in \mathcal{V}_{l,j}$.

Suppose $\tilde{V}_1 \subseteq V_{(\lambda_1,\ldots,\lambda_k,\mu)}$ and $\tilde{V}_2 \subseteq V_{(\lambda_1,\ldots,\lambda_k,\nu)}$. Then $\tilde{V}_1$ is indexed by $(\lambda_1, \ldots, \lambda_k, \mu, \nu, \ldots, \nu) \in \Lambda^{m+l-1}$, and similarly $\tilde{V}_2$ is indexed by the interwoven element $(\lambda_1, \ldots, \lambda_k, \nu, \mu, \ldots, \mu) \in \Lambda^{m+l-1}$.

I3 If $V \in \mathcal{V}_{i,j}$, $i \geq 2$, is indexed by an index having two or more identical ciphers at the end, then there is an $F \in \mathcal{F}_{k,j}$, $k < i$, such that $F = \operatorname{Bd} V_1 \cap \operatorname{Bd} V_2 \neq \emptyset$, $V_1, V_2 \in \mathcal{V}_{i,j}$, $V_1 \neq V_2$, and either $V = V_1$ or $V = V_2$.

I4 If $V \in \mathcal{V}_{i,j}$, $i \geq 1$, is indexed by $(\lambda_1, \ldots, \lambda_l)$, and if $V' \in \mathcal{V}_{k,j}$, $k > i$, is indexed by $(\mu_1, \ldots, \mu_{m+k-1})$, then $V' \subseteq V$ implies $(\lambda_1, \ldots, \lambda_l) = (\mu_1, \ldots, \mu_l)$.

This will be done inductively.

By (16) the elements of $\mathcal{V}_{1,j}$ are indexed by elements of K. Now, the inductive construction goes as follows. Given the indexing of $\mathcal{V}_{i,j}$, the indexing of $\mathcal{V}_{i+1,j}$ is uniquely determined up to the last coordinate of $(m+i)$-tuples, according to the property I4. Then, indices of elements of $\mathcal{V}_{i+1,j}$ obtained from boundary sets of $\mathcal{W}_{i,j}$ are uniquely determined by I2. All other elements V of $\mathcal{V}_{i+1,j}$ may be indexed as follows: if $V \subseteq V_{(\lambda_1,\ldots,\lambda_{m+i-1})} \in \mathcal{V}_{i,j}$, then V is indexed by $(\lambda_1, \ldots, \lambda_{m+i-1}, \lambda)$, where $\lambda \in \Lambda \setminus \{\lambda_1, \ldots, \lambda_{m+i-1}\}$. Since $V_{(\lambda_1,\ldots,\lambda_{m+i-1})}$ contains at most τ small and remnant sets from $\mathcal{V}_{i+1,j}$ and since the cardinality of $\Lambda \setminus \{\lambda_1, \ldots, \lambda_{m+i-1}\}$ is τ, this can be done.

It must be checked only that the interweaving may be done at the initial stage. Let

$$F = \operatorname{Cl} V_{(\lambda_1,\ldots,\lambda_m)} \cap \operatorname{Cl} V_{(\mu_1,\ldots,\mu_m)} \neq \emptyset$$

be an arbitrary element of $\mathcal{F}_{1,j}$. Let us introduce $\kappa = (\lambda_1, \ldots, \lambda_m)$, $\kappa' = (\mu_1, \ldots, \mu_m)$. Then $F \subseteq \operatorname{Bd} O_{\kappa,\kappa'}$ and $\kappa \neq \kappa'$ This implies that $O_{\kappa,\kappa'}$ is nonempty. Then $f_j^{-1}(\operatorname{Cl} T_\kappa \cap \operatorname{Cl} T_{\kappa'}) \neq \emptyset$, and hence $\operatorname{Cl} T_\kappa \cap \operatorname{Cl} T_{\kappa'} \neq \emptyset$. Finally, using Lemma 4 of [6], we conclude that $(\lambda_1, \ldots, \lambda_m) \sim (\mu_1, \ldots, \mu_m)$ (case 1) or $(\lambda_1, \ldots, \lambda_{m-1}) = (\mu_1, \ldots, \mu_{m-1})$ (case 2) (as already mentioned). Now, it is obvious that we may continue interweaving in the first case, and may start a new one in the second.

An element $V_{(\lambda_1,\ldots,\lambda_{m+i-1})}$ of $\mathcal{V}_{i,j}$ will be denoted in the rest of the paper by $V^j_{(\lambda_1,\ldots,\lambda_k)}$, for the sake of emphasis, since we now simultaneously consider all the families.

Let $j \in \{1, 2, \ldots, n+1\}$ be fixed.

A) To a point $x \in X \setminus \bigcup_{i\in\mathbf{N}}(\bigcup \mathcal{F}_{i,j})$, a unique sequence $\lambda_1, \lambda_2, \ldots$ may be assigned in such a way, that $x \in V^j_{(\lambda_1,\ldots,\lambda_k)}$ for any $k \geq m$.

B) If $x \in F \in \mathcal{F}_{i,j} \setminus \mathcal{F}_{i-1,j}$, $i \geq 2$, or $x \in F \in \mathcal{F}_{1,j}$, where $F = \operatorname{Bd} V_1 \cap \operatorname{Bd} V_2 \neq \emptyset$, $V_1, V_2 \in \mathcal{V}_{i,j}$, $V_1 \neq V_2$, and V_1, V_2 are indexed according to I2, then $[\lambda_1, \ldots, \lambda_k, \mu, \nu, \ldots, \nu, \ldots] = [\lambda_1, \ldots, \lambda_k, \nu, \mu, \ldots, \mu, \ldots]$ is assigned to x.

Then, $\psi_j : X \longrightarrow \mathcal{J}(\tau)$ is defined by

$$\psi_j(x) = [\lambda_1, \ldots, \lambda_k, \ldots]$$

in Case A), and

$$\psi_j(x) = [\lambda_1, \ldots, \lambda_k, \mu, \nu, \ldots, \nu, \ldots] = [\lambda_1, \ldots, \lambda_k, \nu, \mu, \ldots, \mu]$$

in Case B).

It is obvious that the function ψ_j is well-defined and that points belonging to $\bigcup_{i\in\mathbf{N}}(\bigcup \mathcal{F}_{i,j}) = \bigcup_{i\in\mathbf{N}} \operatorname{Bd} \mathcal{V}_{i,j}$ are mapped to rational points, and all other points to irrational points of $\mathcal{J}(\tau)$ (by properties I3 and D11).

Exactly as in [6] or [2] (i.e. by proving that the family $\{\psi_1, \cdots, \psi_{n+1}\}$ separates points and closed sets) it follows that

$$\psi = (\psi_1, \cdots, \psi_{n+1}) : X \longrightarrow L_n(\tau)$$

is an embedding.

It only remains to prove that $\mathrm{d}(f_j, \psi_j) \leq \varepsilon$.

From the definition of ψ_j it follows that

$$x \in V_\kappa \implies \psi_j(x) \in S_\kappa. \tag{22}$$

Let $x \in X$ be an arbitrary point. As before, choose $(\lambda_1, \ldots, \lambda_m) = \kappa = \min\{\nu : x \in \mathrm{Cl}\, U_\nu\}$.

If $x \in f_j^{-1}(T_\kappa)$, then $x \in U_\kappa$ and by (15) it follows $x \in V_\kappa$. Therefore by (22)

$$\psi_j(x), f_j(x) \in S_\kappa,$$

and $\mathrm{d}(\psi_j(x), f_j(x)) \leq \varepsilon$ follows from the fact that $\mathrm{diam}\, S_\kappa = \mathrm{diam}\, \Sigma / 2^m < \varepsilon$.

If $x \notin f_j^{-1}(T_\kappa)$, then there is a $\kappa' > \kappa$ such that $x \in \mathrm{Cl}\, O_{\kappa,\kappa'} \neq \emptyset$, and since $\mathrm{Cl}\, O_{\kappa,\kappa'} \subseteq \Omega_{\kappa,\kappa'} \subseteq (f_j^{-1}(\mathrm{Cl}\, T_\kappa) \cap f_j^{-1}(\mathrm{Cl}\, T_{\kappa'})) \cup f_j^{-1}(T_\kappa) \cup f_j^{-1}(T_{\kappa'}) \subseteq f_j^{-1}(\mathrm{Cl}\, T_\kappa \cup \mathrm{Cl}\, T_{\kappa'})$ it follows that $f_j(x) \in S_\kappa \cup S_{\kappa'}$. From $\mathrm{Cl}\, O_{\kappa,\kappa'} \subseteq \mathrm{Cl}\, V_\kappa$, by (22) if follows $\psi_j(x) \in S_\kappa$. Hence

$$\psi_j(x), f_j(x) \in S_\kappa \cup S_{\kappa'}.$$

But S_κ and $S_{\kappa'}$ have a point in common, hence $\mathrm{diam}(S_\kappa \cup S_{\kappa'}) \leq 2\, \mathrm{diam}\, S_\kappa \leq 2\, \mathrm{diam}\, \Sigma / 2^n < \varepsilon$. ∎

Corollary 4 *Let X be a metrizable space, $\dim X \leq n$, $wX \leq \tau$, $f : X \longrightarrow \mathcal{J}(\tau)^{n+1}$ a continuous map, and ε a positive number. Then there is an embedding $\psi : X \longrightarrow L_n(\tau)$ such that $\mathrm{d}(f, \psi) \leq \varepsilon$.*

Proof. Apply Theorem 3 to the disjoint union of X and I^n, with f extended to the union by, say, a constant map on I^n. ∎

3 The separable case

$\mathcal{J}(3)$ is the classic triangular Sierpiński curve ([6]). Let $L_n = \{x \in \mathcal{J}(3)^{n+1} :$ at least one coordinate of x is irrational $\}$.

In [2] the following theorem has been announced:

Theorem 5 *Let X be a separable metrizable space, $\dim X \leq n$, $f : X \longrightarrow \mathcal{J}(3)^{n+1}$ a continuous map, and ε a positive number. Then there is an embedding $\psi : X \longrightarrow L_n$ such that $\mathrm{d}(f, \psi) \leq \varepsilon$.*

Proof. The proof may be obtained by combining the proofs of [2] and the present paper. First we follow the construction of the families $\mathcal{W}_{i,j}$, $\mathcal{V}_{i,j}$, $\mathcal{F}_{i,j}$ as described above for the case $\Lambda = \{1, 2, 3\}$ — we identify $\mathcal{J}(3)$ with $\Sigma(3)$.

After that, we modify the families $\mathcal{W}_{i,j}$, $\mathcal{V}_{i,j}$, $\mathcal{F}_{i,j}$ and obtain an indexing of the modified families, as described in [2]. The indexing enables us to define an embedding by the same formula as above (or in [2]), and the same argument as above shows that it satisfies the required properties. The details will appear elsewhere. ∎

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