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THE GENUS OF THE CARTESIAN PRODUCT $K_{m,m,m} \times C_{2n}$

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Abstract

The genus of $K_{m,m,m} \times C_{2n}$ is determined for $m \geq 1$ and for all $n \geq 3$ and $n = 1$. For $n = 2$ both lower and upper bounds are given.

Let $\times$ denote the Cartesian product of graphs.

Theorem 1. *The genus of $K_{m,m,m} \times C_{2n}$ for $m \geq 1, n \geq 3$ is given by the formula:*

$$\gamma(K_{m,m,m} \times C_{2n}) = 1 + m(m-1)n$$

Proof. For $m = 1$ we have $K_{1,1,1} = C_3$ and $C_3 \times C_{2n}$ is obviously toroidal. From here on let $m \geq 2$. We first prove $\gamma(K_{m,m,m} \times C_{2n}) \leq 1 + m(m-1)n$. We start with $2n$ copies of triangulation of $K_{m,m,m}$ in a surface S_g of genus $g = (m-1)(m-2)/2$. For $m = 3$ the surface S_g is a torus as shown in Figure 1. In this particular case the embedding has 6 disjoint patchworks, two of which are indicated. In general there are $2m$ disjoint patchworks, two of which are needed in the construction. Since C_{2n} is a bipartite 2-regular graph we may apply the patchwork method to embed $K_{m,m,m} \times C_{2n}$ into an orientable surface of genus $1 + m(m-1)n$. For explanation of this classical method, see for instance [3, 4, 5]. The two patchworks may be constructed for instance, by taking alternating edges of any Petrie walk of the well-known Hamilton embedding of $K_{n,n}$ in the surface of genus $(m-1)(m-2)/2$ and then augmenting the edges to appropriate triangles of $K_{n,n,n}$ in the same surface. We double-check the genus formula by the following argument.

(1) There are $2n$ copies of S_g, arranged in a circle, each triangulated by a copy of $K_{m,m,m}$.

(2) There are m tubes between any two consecutive S_g, giving the total number of tubes equal to $2nm$.

(3) $(2n-1)$ tubes are needed to connect the $2n$ initial surfaces S_g into a single surface Σ_0. Hence the final surface Σ is homeomorphic to a sphere with $2ng + 2mn - (2n-1) = 1 + m(m-1)n$ handles attached. The embedding consists of $4m(m-1)n$ triangles remaining in the original surfaces S_g and $6mn$ quadrilaterals along the $2mn$ tubes. There are $2m+2$ faces incident with any vertex: $2m-2$ triangles and 4 quadrilaterals.

The proof that $\gamma(K_{m,m,m} \times C_{2n}) \geq 1 + m(m-1)n$ follows.

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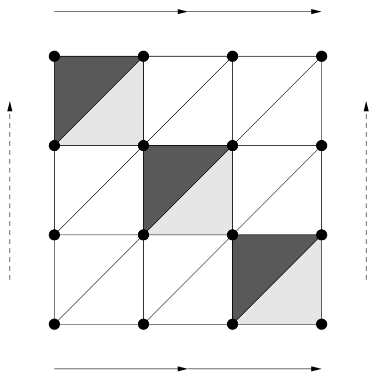

Figure 1: Case $m = 3$. Triangular embedding of $K_{m,m,m}$ in tours with two patchworks indicated.

Let us take an embedding of a graph with vertices $x_1, x_2, ..., x_v$ and a total of f faces. Let f_k denote the total number of faces of size k and let $a_k(x)$ denote the number of faces of size k incident with a given vertex x. Clearly:

$$val(x) = a_3(x) + a_4(x) + ...,$$

$$kf_k = a_k(x_1) + a_k(x_2) + ... + a_k(x_v),$$

and

$$f = f_3 + f_4 + ...$$

For a vertex x define its *face contribution* to be $\phi(x) = a_3(x)/3 + a_4(x)/4 + ...$ If a graph has v vertices, e edges then the genus of Let $\phi_0 = (\phi(x_1)+\phi(x_2)+...+\phi(x_v))/v$ denote the average face contribution. Then $f = \phi(x_1) + \phi(x_2) + ... + \phi(x_v)$. this embedding can be expressed as: $\gamma = 1 + e/2 - v(1 + \phi_0)/2$. Therefore minimizing γ is equivalent to maximizing ϕ_0. In our case, $v = 6mn, e = 6m(m+1)n$. Hence $\gamma(K_{m,m,m} \times C_{2n}) \geq 1 + m(m-1)n$ is equivalent to saying that for any embedding of $K_{m,m,m} \times C_{2n}$ we have $\phi_0 \leq (2m+1)/3$. If we can show this inequality not only for the average face contribution but for the maximal face contribution we are done.

Let $t = a_3(x)$ be the number of triangles incident with a vertex x. Since $val(x) = 2m + 2$ it follows by that $\phi(x) \leq (m+1)/2 + t/12$. Since adjacent vertices in different copies of $K_{m,m,m}$ do not belong to a common triangle $0 \leq t \leq 2m$. The case $t = 2m$ is impossible to attain in an embedding in a surface since the triangles would close-up and the rotation at that vertex would consist of more than one cycle. If $t \leq 2m - 2$ then $\phi(x) \leq (2m+1)/3$ where equality is attained only if $t = 2m - 2$ and the remaining four faces are quadrilaterals. This solution is indeed possible by our 2-patchwork construction in the first half of the proof. In the remaining case ($t = 2m - 1$) we have $2m - 1$ triangular faces and 3 other faces. The triangular faces are necessarily consecutive in the rotation around x, since two of the neighbors of x are not in triangles with x.

There are 4 sub-cases, concerning the number of quadrilateral faces $q = a_4(x)$. We may have $0 \leq q \leq 3$. By an arithmetical argument we rule out the cases $q = 0$ and $q = 1$. Case $q = 3$ is impossible, since $n > 2$ and one face has two edges projecting to C_{2n}. This leaves us with $q = 2$ and the remaining face either pentagonal ($a_5(x) = 1$) or hexagonal ($a_6(x) = 1$). Indeed, if the remaining face has size greater than 6, the value $(2m+1)/3$ cannot be attained. The value $a_6(x) = 1$ gives us exactly $\phi(x) = (2m+1)/3$. The only way that $a_5(x) = 1$ this could occur is to have a string of $2m - 1$ triangles ended on each side by a quadrilateral and

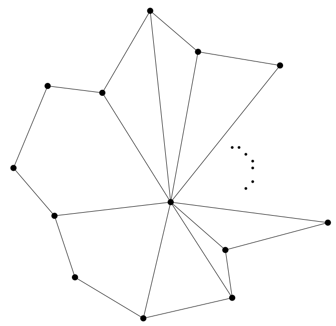

Figure 2: ... a string of $2m-1$ triangles ended on each side by a quadrilateral and the pentagonal face at x.

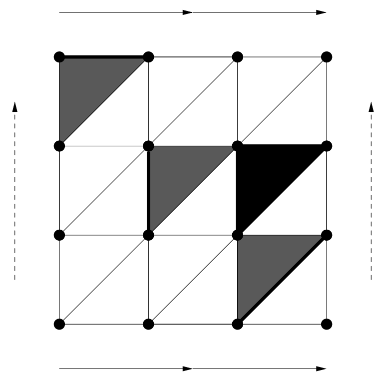

Figure 3: Case $m=3$. The three triangles indicate the patchwork that was used for embedding $K_{3,3,3}\times K_2$. The three thick edges mark the 3 selected quadrilaterals and the black triangle comes in two copies to complete the new patchwork of the embedded $K_{3,3,3}\times K_2$.

the pentagonal face at x has both edges, say xy and xz projecting on C_{2n}. But this is impossible, since the shortest path from y to z not using edge xy and/or xz has length 4. □

Theorem 2. *The genus of $K_{m,m,m}\times C_2, m\geq 1$ is given by the formula:*

$$\gamma(K_{m,m,m}\times C_2)=\gamma(K_{m,m,m}\times K_2)=1-2m+m^2=(m-1)^2$$

Proof. It is easy to see that the two graphs have the same genus embedding and hence consider K_2 instead of C_2. The proof is simpler but analogous to the proof of Theorem 1. In the construction we only need one patchwork. The surface is composed of two surfaces S_g joined by m tubes, hence, it has genus $(m-1)^2$. The converse is easy since each vertex must necessarily contribute only $2m-1$ triangles, and 2 additional quadrilaterals is the best one can hope for. □

Theorem 3. *In general the genus of $K_{m,m,m}\times C_4$ is bounded as follows:*

$$\lceil 2m^2-5m/2+1\rceil\leq\gamma(K_{m,m,m}\times C_4)\leq 1+2m(m-1)=2m^2-2m+1.$$

In particular,

1. for $m = 1$ *the genus is given by*

$$\gamma(K_{1,1,1} \times C_4) = 1$$

2. for $m = 2$ *the genus is given by*

$$\gamma(K_{2,2,2} \times C_4) = 5$$

3. for $m = 3$ *the genus is given by*

$$\gamma(K_{3,3,3} \times C_4) = 12$$

Proof. The upper bound $1 + 2m(m-1)$ is obtained from construction of Theorem 1. The lower bound also follows from the argument in the proof of Theorem 1. Namely, here we cannot rule out the possibility that $\phi_0 = (m+1)/2 + (2m-1)/12 = (8m+5)/12$ that would arise if $2m - 1$ triangles and 3 quadrilaterals are incident with each vertex. For $m = 1$ the two bounds coincide. For $m = 2$ the genus is between 4 and 5 and one can easily check that no genus 4 orientable embedding exists. For $m = 3$ the lower bound is $\lceil 11.5 \rceil = 12$. In order to lower the upper bound to 12 we may use the fact that $K_{m,m,m} \times C_4$ is isomorphic to $K_{m,m,m} \times K_2 \times K_2$. We start with the genus embedding of $K_{m,m,m} \times K_2$ described in the previous Theorem. It contains a patchwork consisting of 2 triangles and 3 quadrilaterals. Using this patchwork one can produce an embedding of $K_{m,m,m} \times K_2 \times K_2$ that has 56 triangular and 30 quadrilateral faces and is therefore genus 12 embedding. The same idea could be explored for more general values of m. It would slightly improve the upper bound at least for m that is divisible by 3. □

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