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&0. Introduction.

One of the central problems in the analysis of holomorphic mappings $h: Z \to X$ is the existence of holomorphic sections, i.e., of maps $f: X \to Z$ satisfying $h(f(x)) = x$ for all $x \in X$. An obvious necessary condition is the existence of continuous sections, and the question appears whether this condition is also sufficient for a particular h. We say that sections of h satisfy (the basic form of) the *Oka principle* if every continuous section of h is homotopic to a holomorphic section. The stronger *parametric Oka principle* means that the inclusion of the space of holomorphic sections to the space of continuous sections is a weak homotopy equivalence.

The terminology stems from Oka's theorem of 1939 to the effect that the second Cousin problem on a Stein manifold is holomorphically solvable provided that it is continuously solvable [Oka]; this is equivalent to the Oka principle stated above for sections of holomorphic principal $\mathbb{C}^*$-bundles over Stein manifolds. In 1957 Grauert [Gra1, Gra2] proved the Oka principle for sections of holomorphic fiber bundles with complex homogeneous fibers over reduced Stein spaces; see also [Car, FR, HL]. In 1989 Gromov [Gro] extended it to the case when h is a submersion onto a Stein manifold X whose fibers admit (locally over small open sets in X) holomorphic families of dominating entire maps from a Euclidean space; such maps are called *sprays* (see Definition 2 below). Complete proofs of Gromov's result and some extensions can be found in [FP1, FP2, FP3] (see also [Lar]).

In the present paper we consider the same problem for maps with *branch points*, i.e., points $z \in Z$ at which the differential $dh_z: T_z Z \to T_{h(z)} X$ is not surjective. The set of all branch points of h is called the *branch locus* br_h. In general there need not exist any local sections of h (not even continuous ones) passing through a branch point. For instance, the map $h(z) = z^2$ $(z \in \mathbb{C})$ has no local sections at 0, but it has a double-valued section $x \to \sqrt{x}$. Similarly, if $w = (w', w'') \in \mathbb{C}^m$ and $h(z, w) = (z^d, w')$, we have d-valued local sections at $(0, 0)$ of the form $(x, w') \to (x^{1/d}, w', g(x, w'))$.

These examples indicate that the natural objects to look for near branch points are *multi-valued sections* of h. Let $d \in \mathbb{N}$ be an integer. A *d-valued section* (or d-section for short) of $h: Z \to X$ is a correspondence which associates to each point $x \in X$ an unordered d-tuple of points $F(x) = [f_1(x), \ldots, f_d(x)]$ (not necessarily distinct) in the fiber $Z_x := h^{-1}(x)$. We consider F as a map from X to the *d-fold symmetric power* $Z^d_{\rm sym}$, the quotient of the Cartesian power Z^d by the action of the permutation group on d elements which permutes the entries of $(z_1, \ldots, z_d) \in Z^d$. A point $z = [z_1, \ldots, z_d] \in Z^d_{\rm sym}$ is an unordered d-tuple of points $z_1, \ldots, z_d \in Z$. $Z^d_{\rm sym}$ inherits from Z^d the structure of a complex space and hence we may speak of continuous resp. holomorphic maps $X \to Z^d_{\rm sym}$. This allows us to define the notion of continuity resp. holomorphicity of d-sections of $h\colon Z \to X$.

Our main result (Theorem 1.1) is a basic version of the Oka principle for multi-valued sections of branched maps over a Stein base which are 'elliptic' submersions over the complement of the critical set. We give some applications (Corollary 1.2) and prove a related Oka principle for liftings of holomorphic maps (Theorem 1.3). As an application of Theorem 1.3 we construct families of nondegenerate entire maps F_x of $\mathbb{C}^d$, depending holomorphically on a parameter x in a Stein manifold, such that the image of F_x omits a given algebraic subvariety $\Sigma_x \subset \mathbb{C}^d$ of codimension at least two (Corollary 1.4).

We hope that these results will find applications in analytic and algebraic geometry where one cannot avoid considering branched maps.

&1. The results.

All complex spaces in this paper are assumed to be reduced and finite dimensional. $X_{\rm sing}$ denotes the singular locus of a complex space X and $X_{\rm reg} = X \backslash X_{\rm sing}$. We denote by ${\rm br}_h \subset Z$ the *branch locus* of a holomorphic map $h\colon Z \to X$; by definition ${\rm br}_h$ contains $Z_{\rm sing} \cup h^{-1}(X_{\rm sing})$.

Definition 1. *Let $h\colon Z \to X$ be a holomorphic map of complex spaces. For any $d \in \mathbb{N}$ we denote by $Z^d_{\rm sym}$ the d-fold symmetric power of Z (section 4).*

*(a) A d-**valued section** of h (or d-section) is map $F\colon X \to Z^d_{\rm sym}$ such that $F(x) \subset h^{-1}(x)$ for each $x \in X$. The number d is called the* **degree** *of F. F is continuous (respectively holomorphic) if it continuous (resp. holomorphic) as a map of X to $Z^d_{\rm sym}$. We identify d-valued maps of X to Z with d-sections of the projection $X \times Z \to X$, $(x, z) \to x$.*

(b) A F is **non-branched** *at $x \in X$ if there is a neighborhood $U \subset X$ of x such that $F|_U$ is a union of d continuous (resp. holomorphic) sections of h over U. We denote by ${\rm br}_F$ the* **branch locus** *of F.*

(c) Denote by $\#F(x)$ the number of distinct points in the set $F(x)$ and let

$$\mu_F = \max\{\#F(x) \colon x \in X\}, \quad \delta_F = \{x \in X \colon \#F(x) < \mu_F\}.$$

The set δ_F is called the **discriminant locus** *of F.*

Clearly ${\rm br}_F \subset \delta_F$ and both sets are closed when F is continuous. If F is holomorphic then ${\rm br}_F$ and δ_F are closed complex subvarieties of X.

The following notion was introduced by Gromov [Gro, 1.1.B].

Definition 2. *A holomorphic map $h: Z \to X$ between complex spaces is an* **elliptic submersion** *over an open subset $\Omega \subset X_{\rm reg}$ if the restriction $h: h^{-1}(\Omega) \to \Omega$ is a submersion of complex manifolds and each point $x \in \Omega$ has an open neighborhood U such that there exist a holomorphic vector bundle $p: E \to Z|_U := h^{-1}(U)$ and a holomorphic map $s: E \to Z|_U$ satisfying the following conditions for each $z \in Z|_U$:*

(i) $s(E_z) \subset Z_{h(z)}$ *(equivalently, $hs = hp$),*

(ii) $s(0_z) = z$, *and*

(iii) the derivative $ds: T_{0_z}E \to T_zZ$ maps the subspace $E_z \subset T_{0_z}E$ surjectively onto the vertical tangent space $VT_z(Z) := \ker dh_z$.

Any tripple (E, p, s) as above is called a (dominating) **spray** associated to the submersion $h: Z|_U \to U$; (iii) is the 'domination property' of s. Examples of elliptic submersions can be found in [Gro, FP1, FP2].

The notation $\mathcal{H}^k(A) = 0$ for a subset $A \subset X$ means that A has the k-dimensional Hausdorff measure equal to zero; this notion is well-defined when X is a complex space.

1.1 Theorem. (Oka's principle for multi-valued sections.) *Let $h: Z \to X$ be a holomorphic map of a complex space Z onto an irreducible n-dimensional Stein space X. Assume that X_0 is a closed complex subvariety of X containing $X_{\rm sing} \cup h({\rm br}_h)$ and h is an elliptic submersion over $X \backslash X_0$. Let F be a continuous d-section of h which is holomorphic in an open set containing $X' := X_0 \cup {\rm br}_F$ and satisfies $\mathcal{H}^{2n-1}(\delta_F) = 0$. Then there exists a homotopy $F_t: X \to Z^d_{\rm sym}$ $(t \in [0,1])$ of continuous d-sections of h such that $F_0 = F$, each F_t is holomorphic in a neighborhood of X', non-branched over $X \backslash X'$ and satisfies $F_t(x) = F(x)$ for $x \in X'$, and F_1 is holomorphic on X.*

Remarks. 1. The space Z in Theorem 1.1 need not be Stein (the fibers of h may even be compact !). When h admits branch points Theorem 1.1 is new even for single-valued sections. Under suitable hypothesis the result also holds over a reducible base X; in this case one may consider multi-valued sections which have different degrees over different irreducible components of X (we shall not go into the details). In particular, the result holds for single-valued sections ($d = 1$) over arbitrary Stein space X without any further hypothesis (in this case we clearly have ${\rm br}_F = \delta_F = \emptyset$).

2. Theorem 1.1 holds with the usual additions described in [Gro, FP2, FP3]. For instance, if F is holomorphic in a neighborhood of $K \cup X'$ for some compact, holomorphically convex subset $K \subset X$ and if h is elliptic over $X \backslash (K \cup X')$ then the homotopy may be chosen such that, in addition to the stated properties, each F_t is holomorphic in a neighborhood of $K \cup X'$, it is uniformly close to F on the set K, and F_t agrees with F to a given finite order along the subvariety X'. (See Theorem 1.4 in [FP3] for a precise statement.)

3. Assume that h has fibers of dimension at least one (which is the only case of interest). A transversality argument shows that for a generic smooth d-section F of $h: Z \to X$ the set $\delta_F \backslash X_0$ is a smooth real submanifold of codimension at least two in $X \backslash X_0$ and hence $\mathcal{H}^{2n-1}(\delta_F) = 0$. This condition guaranties that the set δ_F is nowhere dense in X and $X_{reg} \backslash \delta_F$ is path connected and locally path connected. The latter condition insures the decomposition of multi-valued sections into irreducible components which is used in the proof of Theorem 1.1.

4. Graphs of holomorphic multi-valued sections of $h: Z \to X$ are in bijective correspondence with analytic n-cycles $V \subset Z$ (i.e., finite linear combinations $V = \sum m_j V_j$ of pure n-dimensional complex subvarieties $V_j \subset Z$ with integer coefficients $m_j \in \mathbb{N}$) for which the projection $h: V \to X$ is an analytic cover onto X (see section 4). Hence Theorem 1.1 may be considered as the *Oka principle for analytic covers.* ♠

1.2 Corollary. *Let $h: Z \to X$ be a holomorphic map of a complex space Z onto an irreducible n-dimensional Stein space X. Assume that X_0 is a closed complex subvariety of X containing $X_{\rm sing}$ such that h is a submersion of complex manifolds over $\widetilde{X} := X \backslash X_0$. Set $\widetilde{Z} := Z \backslash h^{-1}(X_0)$. Then the conclusion of Theorem 1.1 holds in each of the following cases:*

(a) each connected component of the fiber Z_x for $x \in \widetilde{X}$ is either a rational curve ($\mathbb{CP}^1$) or an elliptic curve (a complex torus);

(b) the restriction $h: \widetilde{Z} \to \widetilde{X}$ is locally trivial (a holomorphic fiber bundle) and the fiber Z_x is a complex Lie group or a complex homogeneous space;

(c) $\widetilde{Z} = V \backslash \Sigma$ where $h: V \to \widetilde{X}$ is a holomorphic vector bundle over $\widetilde{X}$ of rank $k \geq 2$ and Σ is complex subvariety of the associated bundle $\widetilde{V} \to \widetilde{X}$ with fibers $\widetilde{V}_x \simeq \mathbb{CP}^k$ such that $\dim \Sigma_x \leq k-2$ for all $x \in \widetilde{X}$.

Proof. In each case the restricted submersion $h: \widetilde{Z} \to \widetilde{X}$ is elliptic and hence Theorem 1.1 applies. Case (b) was considered in [Gra1, Gra2] and [FP2], and case (c) was considered in [Gro] and, more explicitly, in [FP2, Theorem 1.7]. In case (c) the fibers Σ_x are algebraic subvarieties of $\widetilde{V}_x \simeq \mathbb{CP}^k$.

In case (a) the connected components of the fiber $Z_x = h^{-1}(x)$ for $x \in \widetilde{X}$ are all of the same type (either $\mathbb{CP}^1$ or elliptic). In the first case the complex structure on Z_x is independent of x and hence $\widetilde{Z} \to \widetilde{X}$ is a fiber bundle with complex homogeneous fiber $\mathbb{CP}^1$, so the result is a special case of (b). If the components of Z_x are elliptic curves $C_{x,j}$ $(1 \leq j \leq j_0)$, the parameter of the complex structure on $C_{x,j}$ is locally a holomorphic function of x. Hence the universal covering maps $\mathbb{C} \to C_{x,j}$ can be chosen to be locally holomorphic in x, and these maps give sprays on $Z|_U$ over small open sets $U \subset \widetilde{X}$. ♠

Part (a) of Corollary 1.2 applies to compact complex surfaces S fibered over a compact Riemann surface such that the generic fiber of S is either $\mathbb{CP}^1$ (in this case S is called a 'ruled surface') or a complex torus (in this case S is an 'elliptic surface'); see [BV] for the theory of such surfaces. Theorem 1.1

applies over any non-compact domain in the base (since any open Riemann surface is a Stein manifold).

The following closely related result extends Theorem 2.1 in [F] to maps h with branch points.

1.3 Theorem. (The Oka principle for liftings.) *Let $h: Z \to X$ be a holomorphic map of a complex space Z onto a a complex space X and let X_0 be a closed complex subvariety of X such that h is an is an elliptic submersion over $X \backslash X_0$. Suppose that Y is a Stein space and $f: Y \to X$ is a holomorphic map such that $f(Y_{\rm sing}) \subset X_0$. If $g_0: Y \to Z$ is a continuous map which is holomorphic in an open set containing $Y_0 := f^{-1}(X_0)$ and satisfies $hg_0 = f$, then for each $k \in \mathbb{N}$ there exists a homotopy of continuous maps $g_t: Y \to Z$ such that for each $t \in [0,1]$ we have $hg_t = f$, g_t and g_0 are tangent to order k along Y_0, and the map g_1 is holomorphic on Y. If in addition g_0 is holomorphic in a neighborhood of a compact holomorphically convex subset $K \subset\subset Y$, the homotopy g_t can be chosen such that it approximates g_0 uniformly on K.*

The spaces Z and X in Theorem 1.3 need not be Stein. It suffices to assume that h is a an elliptic submersion over an open set containing $f(Y \backslash Y_0)$.

We give an application of Theorem 1.3 to entire holomorphic maps avoiding certain complex subvarieties. Let $h: E \to X$ be a holomorphic vector bundle of rank q over a Stein manifold X. For each $x \in X$ we denote by $\widehat{E}_x \cong \mathbb{CP}^q$ the compactification of the fiber $E_x \cong \mathbb{C}^q$ obtained by adding to E_x the hyperplane at infinity $\Lambda_x \cong \mathbb{CP}^{q-1}$. The resulting fiber bundle $\widehat{h}: \widehat{E} \to X$ with fibers $\widehat{E}_x \cong \mathbb{CP}^q$ is again holomorphic. (The essential observation is that the transition maps, which are $\mathbb{C}$-linear automorphisms of fibers E_x, extend to projective linear automorphisms of $\widehat{E}_x$).

1.4 Corollary. *Let $h: E \to X$ be a holomorphic vector bundle of rank q over a Stein manifold X and let $\widehat{E} \to X$ be the associated bundle with fiber $\mathbb{CP}^q$. Let Σ be a closed complex subvariety of $\widehat{E}$ whose fiber Σ_x has complex codimension at least two in $\widehat{E}_x$ and satisfies $0_x \notin \Sigma_x$ for each $x \in X$. Then for each $k \in \mathbb{N}$ there exists a fiber-preserving holomorphic map $F: E \to E \backslash \Sigma$ which is tangent to the identity to order k along the zero section of E.*

The conclusion of Corollary 1.4 can be stated by saying that the entire maps $F_x = F(x, \cdot)$ on the fibers $E_x \cong \mathbb{C}^q$, which depend holomorphically on $x \in X$, are tangent to the identity at the origin $0_x \in E_x$ and the image of F_x misses the subvariety Σ_x for each $x \in X$. Of course 0_x can be replaced by the point $g(x)$ for any holomorphic section $g: X \to E \backslash \Sigma$ which avoids Σ. Note that the fibers Σ_x are projective-algebraic by Chow's theorem. The condition that Σ is an analytic subset of $\widehat{E}$ (and not merely an analytic subset of E with algebraic fibers) is equivalent to a local condition on the behavior of the affine-algebraic fibers $\Sigma_x \cap E_x$ over small open sets in X; see [FP2]. The result is false in general if Σ_x has codimension one in E_x (since its complement may even be Kobayashi hyperbolic).

Proof of Corollary 1.4. Let $Z = E \backslash \Sigma$. The hypothesis on Σ implies that the submersion $h_Z\colon Z \to X$ is elliptic (see Corollary 1.8 in [FP2]). Take $Y = E$ (this is a Stein manifold) and let Y_0 denote the zero section of the bundle $E \to X$ (a complex submanifold of Y). Furthermore let $g_0\colon Y \to Z = E\backslash\Sigma$ be a continuous fiber preserving map which equals the identity in an open set $U \subset Y$ containing Y_0 (g_0 can be obtained by a contraction of the fibers E_x to a small neighborhood of 0_x which does not intersect Σ_x). Since $h_Z \circ g_0 = h$ on $Y = E$, g_0 is a continuous lifting of the holomorphic map $h\colon Y \to X$ with respect to the submersion $h|_Z\colon Z \to X$. By Theorem 1.3 there exists a homotopy of liftings $g_t\colon Y \to Z$, starting at $t = 0$ with g_0 and ending at $t = 1$ with a holomorphic lifting $g_1\colon Y \to Z$, such that the homotopy is fixed to order k along Y_0. The map $F = g_1$ clearly satisfies Corollary 1.4. ♠

Questions. 1. Let E and Σ be as in Corollary 1.4. Is it possible to choose $F\colon E \to E\backslash\Sigma$ as above to be *injective* (i.e., such that F_x is a Fatou-Bieberbach map for each $x \in X$) ? By Proposition 1.4 in [Fo1] the answer is affirmative when X consists of a single point.

2. Let Σ be a nonempty algebraic subvariety of codimension at least two in $\mathbb{C}^d$. Is it possible to find a nondegenerate *polynomial map* on $\mathbb{C}^d$ whose image misses Σ ? This is related to the famous *Jacobian problem* (is every polynomial map on $\mathbb{C}^d$ with constant Jacobian an automorphism of $\mathbb{C}^d$?). It is known that there is no *injective* polynomial map F with this property (injectivity implies that F is onto $\mathbb{C}^d$ and hence and automorphism of $\mathbb{C}^d$). ♠

Remark. Theorems 1.1, 1.3 and Corolary 1.2 also hold in the parametric form, i.e., for families of sections with parameter in a compact Hausdorff space P. We shall not state this extension formally but refer instead to [FP2] and [FP3] for precise formulations. In Theorem 1.1 a suitable hypothesis is that all d-sections in a given family $\{F_p\colon p \in P\}$ are holomorphic in a fixed open neighborhood of a subvariety X_0 and are non-branched in the open set $X\backslash X_0$. In addition we may be given a compact subspace $P_0 \subset P$ such that F_p is holomorphic on X when $p \in P_0$. The goal is to deform the family by homotopy $F_{p,t}$ ($t \in [0,1]$) to a family of holomorphic d-sections $F_{p,1}$ such that the homotopy is fixed for $p \in P_0$ (for this P_0 must be a neighborhood retract in P). The proof of this parametric Oka principle for single-valued sections ($d = 1$) does not require any substantial changes (see the proof of Theorem 2.1 in section 2 below and the proof of Theorem 1.4 in [FP3]). The parametric version of Theorem 1.1 for $d > 1$ is obtained from the following parametric version of Theorem 1.3 above.

Let $h\colon Z \to X$ and $X_0 \subset X$ be as in Theorem 1.3. We are given a family of holomorphic maps $f_p\colon Y \to X$ from a Stein space Y to X, depending continuously on a parameter $p \in P$, such that $f_p(Y_{\rm sing}) \subset X_0$ and f_p is holomorphic in a fixed neighborhood of $f_p^{-1}(X_0)$ for all $p \in P$. The goal is to find a continuous family of holomorphic liftings $g_p\colon Y \to Z$ ($p \in P$) starting from a family of continuous liftings (i.e., we must have $hg_p = f_p$ for all $p \in P$). The construction in section 3 reduces this to the problem of finding a family of single-valued sections $\widehat{g}_p$ of a continuous family of maps $\widehat{h}_p\colon \widehat{Z}_p \to Y$ $(p \in P)$, where $\widehat{h}_p$ is the pull-back of $h\colon Z \to X$ by the map $f_p\colon Y \to X$. Although the Oka principle for families of sections has not been established in the required generality in [FP2], the proof given there (together with the modifications given in section 2 below for maps with branch points) carries over to this case. ♠

In the remainder of this section we explain the organization of the paper and outline the proofs of the stated results.

In section 2 we prove Theorem 1.1 for single-valued sections (Theorem 2.1). The main point is that the proof of Oka's principle for submersions given in [FP2, FP3] extends to the present situation provided that one can construct a local spray around the graph of any holomorphic section of h over a holomorphically convex subset of X. In [FP2, FP3] this was done using the fact that for a submersion $h\colon Z \to X$ the vertical tangent space $VT(Z) = \ker dh$ is a holomorphic vector bundle over Z and hence is generated in any Stein open set by finitely many sections (vertical vector fields). The composition of local flows of these sections is a local spray on the given Stein set in Z. When h has branch points, $VT(Z)$ is no longer a vector bundle but merely a linear space over Z. Nevertheless, germs of holomorphic sections of this space form a coherent analytic sheaf over Z which is locally free over $Z \backslash \mathrm{br}_h$ (Proposition 2.2), and this enables us to complete the proof.

In section 3 we reduce Theorem 1.3 to Theorem 2.1; the main point of this reduction is to associate to any map $f\colon Y \to X$ the pull-back $\widehat{h}\colon \widehat{Z} \to Y$ of $h\colon Z \to X$ such that sections of $\widehat{h}$ over Y are in one-to-one correspondence with maps $g\colon Y \to Z$ satisfying $hg = f$ (i.e., liftings of f). Moreover, each spray on $h^{-1}(U) \to U$ (for U an open subset of X) pulls back to a spray on $\widehat{h}^{-1}(V) \to V$ where $V = f^{-1}(U) \subset Y$.

In section 4 we recall some basic results on multi-valued sections which are used in the proof of Theorem 1.1 for $d > 1$.

In section 5 we deduce the general case of Theorem 1.1 from Theorem 1.3 as follows. We introduce a complex space Y and a continuous map $g_0\colon Y \to Z$ such that

- the composition $f = hg_0\colon Y \to X$ is a d-sheeted analytic cover onto X which is non-branched over $X \backslash X'$,
- g_0 maps the fiber $Y_x = f^{-1}(x)$ onto $F(x)$ for each $x \in X$, and
- g_0 is holomorphic in $f^{-1}(U_0)$ if F is holomorphic in $U_0 \subset X$.

The space Y is just the 'normalized graph' of F where the self-intersections have been removed, and f^{-1} is a holomorphic d-valued section of $f\colon Y \to X$. Since f is a finite map of Y onto a Stein space X, it follows that Y is also Stein. We consider the map $g_0\colon Y \to Z$ as a continuous lifting of $f = hg_0\colon Y \to X$ with respect to the projection $h\colon Z \to X$ and apply Theorem 1.3 to obtain a homotopy of liftings $g_t\colon Y \to Z$ connecting g_0 to a holomorphic lifting g_1. Then

$F_t = g_t f^{-1}$ $(0 \le t \le 1)$ is a homotopy of d-sections of h satisfying Theorem 1.1.

&2. Proof of Theorem 1.1 for single-valued sections.

In this section we prove the following version of Theorem 1.1 for single-valued sections. All complex spaces are assumed to be reduced and finite dimensional.

2.1 Theorem. *Let $h\colon Z \to X$ and $X_0 \subset X$ be as in Theorem 1.1. For any continuous section $F\colon X \to Z$ which is holomorphic in an open set containing X_0 and for any $k \in \mathbb{N}$ there exists a homotopy $F_t\colon X \to Z$ $(t \in [0,1])$ of continuous sections such that $F_0 = F$, for each $t \in [0,1]$ the section F_t is holomorphic in a neighborhood of X_0 and tangent to F_0 to order k along X_0, and the section F_1 is holomorphic on X.*

The same result holds with uniform approximation on compact holomorphically convex sets in X. When Z and X are complex manifolds and h is a surjective submersion (i.e., $\mathrm{br}_h = \emptyset$), Theorem 2.1 is a special case of Theorem 1.4 in [FP3]; when $X_0 = \emptyset$ it is included [Gro, 4.5 Main Theorem] and in [FP2, Theorem 1.5]. The presence of branch points of h requires a refinement of the proof, related to the construction of local sprays in neighborhoods of graphs of holomorphic sections over Stein open sets in X, which we shall now describe.

2.2 Proposition. (Existence of local sprays.) *Let $h\colon Z \to X$ be a holomorphic map of complex spaces. For any open Stein subset Ω of Z there exist an integer $N \in \mathbb{N}$, an open set $V \subset \Omega \times \mathbb{C}^N$ containing $\Omega \times \{0\}^N$, and a holomorphic map $s\colon V \to Z$ satisfying the following:*

(i) $s(z,0) = z$ *for* $z \in \Omega$,

(ii) $h(s(z,t)) = h(z)$ *for* $(z,t) \in V$,

(iii) $s(z,t) = z$ *when* $(z,t) \in V$ *and* $z \in \mathrm{br}_h$,

(iv) for each $z \in \Omega \backslash \mathrm{br}_h$ *the derivative at* $t = 0 \in \mathbb{C}^N$ *of* $t \to s(z,t) \in Z$ *maps* $T_0\mathbb{C}^N$ *surjectively onto the vertical tangent space* $VT_zZ := \ker dh_z$.

A map s satisfying Proposition 2.2 will be called a **local spray** over Ω for the map h. Indeed s satisfies the properties of a spray except that it is not defined globally on $\Omega \times \mathbb{C}^N$ and the domination property (iv) only holds in the complement of the branch locus br_h. When h is a submersion of complex manifolds, the subset $VT(Z) \subset TZ$ with fibers $VT_zZ = \ker dh_z$ is a holomorphic vector subbundle of the tangent bundle TZ (the so called *vertical tangent bundle*); in this case Proposition 2.2 coincides with Lemma 5.3 in [FP1].

Proof of Proposition 2.2. Our reference are Chapters 1 and 2 in [Fis]. Recall that the **tangent space** of a complex space Z is a linear space $\pi\colon TZ \to Z$ (a complex space with linear fibers over Z) obtained as follows. Fix a point $z_0 \in Z$ and represent an open neighborhood $Z_0 \subset Z$ of z_0 as a closed complex subspace of an open subset W of $\mathbb{C}^m$, defined by a sheaf of ideals $\mathcal{I} \subset \mathcal{O}_W$ which is generated by holomorphic functions $f_1, \ldots, f_r \in \mathcal{O}(W)$. Thus the structure sheaf $\mathcal{O}_{Z_0}$ is isomorphic to the quotient $\mathcal{O}_W/\mathcal{I}$ restricted to $\{f = 0\} = Z_0$. If $(w_1, \ldots, w_m, \xi_1, \ldots, \xi_m)$ are coordinates in $W \times \mathbb{C}^m$ then $TZ_0 = TZ|_{Z_0}$ is the closed complex subspace of $W \times \mathbb{C}^m$ generated by the functions

$$f_1, \ldots, f_r \quad \text{and} \quad \frac{\partial f_i}{\partial w_1}\,\xi_1 + \cdots + \frac{\partial f_i}{\partial w_m}\,\xi_m \quad \text{for} \quad i = 1, \ldots, r. \tag{2.1}$$

The projection $TZ_0 \to Z_0$ is induced by $W \times \mathbb{C}^m \to W$, $(w, \xi) \to w$. Different local representations of Z in $\mathbb{C}^m$ give isomorphic representations of the tangent space. Over $Z_{\rm reg}$ the space TZ is the usual tangent bundle of Z.

Suppose furthermore that $h: Z \to X$ is a holomorphic map of complex spaces. The *vertical tangent space* $\pi: VT(Z) \to X$ with respect to h (also called in [Fis] 'the tangent space of Z over X' and denoted $T(Z/X)$) is a linear space over Z (a subspace of TZ) with the following local description. Fix a point $z_0 \in Z$ and let $x_0 = h(z_0) \in X$. Let $Z_0 \subset W \subset \mathbb{C}^m$ be a local representation of a neighborhood of z_0 as above and let $X_0 \subset W' \subset \mathbb{C}^n$ be a local representation of a neighborhood $X_0 \subset X$ of x_0. We may choose these neighborhoods such that $h(Z_0) \subset X_0$ and the restriction $h: Z_0 \to X_0$ extends to a holomorphic map $H = (H_1, \ldots, H_n): W \to \mathbb{C}^n$. One takes $T(Z/X)|_{Z_0}$ to be the closed complex subspace of $W \times \mathbb{C}^m$ generated by the functions (2.1) together with

$$\frac{\partial H_i}{\partial w_1}\,\xi_1 + \cdots + \frac{\partial H_i}{\partial w_m}\,\xi_m \quad \text{for} \quad i = 1, \ldots, n. \tag{2.2}$$

Again the result is independent of the local representations up to isomorphism and the complex space obtained in this way coincides with the usual vertical tangent bundle over $Z \backslash {\rm br}_h$. The spaces TZ and $VT(Z)$ need not be reduced even if Z is, but this will not be important for our purposes.

We denote by $\mathcal{T}_Z$ (resp. $\mathcal{VT}_Z$) the sheaf of germs of holomorphic sections of TZ (resp. of $VT(Z)$). These are $\mathcal{O}_Z$-analytic sheaves which are free over $Z \backslash {\rm br}_h$. Sections of $\mathcal{T}_Z$ are called *vector fields* on Z and sections of $\mathcal{VT}_Z$ are *vertical vector fields* (the latter are vector fields tangent to the fibers of h).

2.3 Lemma. *The $\mathcal{O}_Z$-analytic sheaves $\mathcal{T}_Z$ and $\mathcal{VT}_Z$ are coherent.*

Proof. Indeed the sheaf $\mathcal{L}$ of holomorphic sections of any linear space $L \to Z$ over a complex space Z is a coherent $\mathcal{O}_Z$-analytic sheaf [Fis, p. 53, Corollary]. In the present case this can be seen directly as follows. Using a local representation for $VT(Z)|_{Z_0}$ as above, the sheaf $\mathcal{VT}_{Z_0}$ consists of germs of holomorphic maps $\xi = (\xi_1, \ldots, \xi_m): Z_0 \to \mathbb{C}^m$ satisfying

$$\sum_{k=1}^m \frac{\partial f_i(w)}{\partial w_k}\,\xi_k(w) = 0 \quad (1 \le i \le r); \quad \sum_{k=1}^m \frac{\partial H_j(w)}{\partial w_k}\,\xi_k(w) = 0 \quad (1 \le j \le n).$$

To get $\mathcal{T}_Z$ we only take the first set of equations. Thus both sheaves are locally sheaves of relations and hence coherent [GR, p. 131]. ♠

We continue with the proof of Proposition 2.2. Let $\mathcal{J} \subset \mathcal{O}_Z$ denote the (coherent) sheaf of ideals of the branch locus br_h. For any $k \in \mathbb{N}$ the analytic sheaf $\mathcal{S}_k = \mathcal{J}^k \cdot \mathcal{VT}_Z$ (the product of k copies of $\mathcal{J}$ with $\mathcal{VT}_Z$) is also coherent. Fix k and write $\mathcal{S} = \mathcal{S}_k$. Let Ω be a Stein open subset of Z. We claim that there exist finitely many sections $X_1, \ldots, X_N$ of $\mathcal{S}$ over Ω which generate $\mathcal{S}$ at each point of $\Omega \backslash \mathrm{br}_h$. This can be seen by a standard argument as follows.

Since Ω is Stein, Cartan's Theorem A gives for each point $z \in \Omega$ finitely many sections of $\mathcal{S}$ over Ω which generate $\mathcal{S}$ at z. We can choose a point a_j in each connected component Ω_j of $\Omega \backslash \mathrm{br}_h$ such that the sequence $\{a_j\}$ is discrete in Ω. Since $\mathcal{S}$ is a free sheaf over the complex manifold $\Omega \backslash \mathrm{br}_h$, a simple (and well known) extension of Cartan's Theorem A gives finitely many sections of $\mathcal{S}$ over Ω which generate $\mathcal{S}$ at each point a_j. The exceptional set Z_1, consisting of points in Ω at which these sections fail to generate $\mathcal{S}$, is a complex subspace of Ω satisfying $\dim Z_1 \cap \Omega_j < \dim \Omega_j$ for each j. For each index j for which $Z_1 \cap \Omega_j \neq \emptyset$ we now choose a point $b_j \in Z_1 \cap \Omega_j$ such that $\{b_j\}$ is discrete in Ω. By Cartan's Theorem A There exist finitely many sections of $\mathcal{S}$ over Ω which, together with the sections chosen in the first step, generate $\mathcal{S}$ at each b_j. The exceptional set $Z_2 \subset \Omega$ at which all these sections fail to generate $\mathcal{S}$ now satisfies $\dim Z_2 \cap \Omega_j \le \dim \Omega_j - 2$ for each j.

Continuing this way we obtain in finitely many steps sections $X_1, \ldots, X_N$ of the sheaf $\mathcal{S}$ over Ω which generate $\mathcal{S}$ at each point of $\Omega \backslash \mathrm{br}_h$. (However, the minimal number of generators of $\mathcal{S}$ at branch points $z \in \mathrm{br}_h$ need not be bounded from above and hence there need not exist finitely many sections generating $\mathcal{S}$ over Ω.) By construction $X_1, \ldots, X_N$ are holomorphic vector fields on Ω which are vertical with respect to h, they vanish to order k along br_h, and they generate the vertical tangent space $VT_z Z = \ker dh_z$ at each point $z \in \Omega \backslash \mathrm{br}_h$. Let ϕ^j_t denote the flow of X_j for complex time t. The map

$$s(z, t_1, \ldots, t_N) = \phi^1_{t_1} \circ \cdots \circ \phi^N_{t_N}(z),$$

which is defined and holomorphic in an open neighborhood of $\Omega \times \{0\}^N$ in $\Omega \times \mathbb{C}^N$ and takes values in Z, satisfies Proposition 2.2. Indeed, its partial derivative on t_j at $t = 0$ equals $X_j(z)$, and since these vectors generate $VT_z(Z)$ for $z \in \Omega \backslash \mathrm{br}_h$, s satisfies (iv). The properties (i)–(iii) are clear. ♠

Proof of Theorem 2.1. Proposition 2.2 enables us to prove Theorem 2.1 by following step by step the proof of Theorem 1.4 in [FP3]. We shall point out the places in the proof where a change or remark is needed.

The reader should first look at Theorems 5.1 and 5.2 in [FP3] which describe patching of holomorphic sections (no changes are required before that). We are given a pair of closed sets $A, B \subset X$ (in the proof of Theorem 5.2

the set A contains a subvariety $X_0 \subset X$), open Stein neighborhoods $A \subset \widetilde{A}$ and $B \subset \widetilde{B}$, and holomorphic sections $a\colon \widetilde{A} \to Z$, $b\colon \widetilde{B} \to Z$ which are close to each other over $\widetilde{C} = \widetilde{A} \cap \widetilde{B}$. We must patch them into a single holomorphic section over a neighborhood of $A \cup B$ which is uniformly close to a over A. To accomplish this we need sprays s_1 resp. s_2 over the graphs $a(\widetilde{A})$ resp. $b(\widetilde{B})$.

The second set $\widetilde{B}$ is always assumed to be a sufficiently small subset of $X \backslash X_0$ such that, by hypothesis in the theorem, there exists a spray for the submersion $h\colon h^{-1}(\widetilde{B}) \to \widetilde{B}$. The map s_2 is obtained by restricting this spray to the graph $b(\widetilde{B})$.

The first map s_1 only needs to be a local spray and the domination property (property (iii) in Definition 1) is only required on the set $a(\widetilde{C})$ (since the patching is carried out over a neighborhood of the intersection $C = A \cap B$). The graph $a(\widetilde{A})$ is a Stein subvariety of $h^{-1}(\widetilde{A})$ and hence it has a Stein open neighborhood $\Omega \subset Z$ [Siu]. The map s furnished by Proposition 2.2 satisfies all requirements and may therefore be used as s_1 in the proof of Theorem 5.2 in [FP3]. All other steps in the proof carry over without changes and we see that [FP3, Theorem 5.2] holds in our present situation, with X_0 replaced by X'. (As we point out in the Correction at the end of the present paper, the sets A and B in [FP3, Theorem 5.2] must have a basis of Stein neighborhoods, the assumption which was accidentally omitted in the statement of the theorem.)

Another remark should be made regarding the proof of Proposition 4.2 and Theorem 5.2 in [FP3]. The complex space X may have singularities (included in X'), but the patching of sections is performed only over open sets in $X \backslash X' \subset X_{\mathrm{reg}}$, and the relevant $\overline{\partial}$-problem which arises in this patching has compact support contained in $X \backslash X'$. Such $\overline{\partial}$-problems can be solved by transporting them to $\mathbb{C}^N$ via a holomorphic map $g\colon X \to \mathbb{C}^N$ which is a homeomorphism of X onto a closed complex subvariety $\widetilde{X} \subset \mathbb{C}^N$ and is biholomorphic on X_{reg}. (See section 7 in [FP3].)

We now proceed to section 6 of [FP3] where Theorem 1.4 of that paper is proved. The crucial step is furnished by Proposition 6.1 in [FP3]. To see that the proof of this proposition remains valid in our current situation we need to observe that the sets $A_0, A_1, \ldots, A_n \subset X$, which are chosen at the beginning of the proof of Proposition 6.1 in [FP3], are such that A_0 contains a neighborhood of X' while the sets $A_1, \ldots, A_n$ do not intersect X'. Since h is a submersion of complex manifolds over $X \backslash X'$, the techniques developed in [FP2, FP3] for holomorphic submersions onto Stein manifolds can be applied whenever the first set A_0 is not involved. Using those techniques we can patch any collection of holomorphic sections $a_j\colon \widetilde{A}_j \to Z$ $(1 \le j \le n)$, where $\widetilde{A}_j \subset X \backslash X_0$ is a small open neighborhood of A_j over which h admits a spray, into a single holomorphic section b over a neighborhood of $A^n = A_1 \cup A_2 \cup \ldots \cup A_n$, provided that the sections a_j belong to a holomorphic complex. (We are referring to the transformation of a holomorphic complex associated to the Cartan string $(A_1, \ldots, A_n)$ into a holomorphic section over their union A^n; the details of this procedure are explained in [FP2, Proposition 5.1].) The same procedure also gives a homotopy of holomorphic sections over a neighborhood of $A_0 \cap A^n$ connecting a and b.

It remain to patch a and b into a single holomorphic section over a neighborhood of $A_0 \cup A^n$. This is accomplished as in [FP3] by combining the homotopy version of the Oka-Weil approximation theorem (see e.g. Theorem 2.1 in [FP3]) with Theorem 5.2 in [FP3] which holds in the current situation as explained above. The proof of Theorem 2.1 can now be concluded by the globalization procedure given in [FP3] (proof of Theorem 1.4).

&3. Proof of Theorem 1.3.

In this section we reduce Theorem 1.3 to Theorem 2.1. Let $h: Z \to X$ and $f: Y \to X$ be as in Theorem 1.3. Set

$$\begin{aligned} \widehat{Z} &= \{(y,z): y \in Y,\ z \in Z,\ f(y) = h(z)\}, \\ \widehat{h}(y,z) &= y \in Y, \quad \sigma(y,z) = z \in Z. \end{aligned} \tag{3.1}$$

Clearly $\widehat{Z}$ is a closed complex subspace of $Y \times Z$, the maps $\widehat{h}: \widehat{Z} \to Y$ and $\sigma: \widehat{Z} \to Z$ are holomorphic, and we have $f\widehat{h} = h\sigma$.

By assumption the set $Y_0 = f^{-1}(X_0) \subset Y$ contains the singular locus $Y_{\rm sing}$. For each $y \in Y \backslash Y_0$ we have $f(y) \in X \backslash X_0$ and hence h is a submersion over an open neighborhood $U \subset X \backslash X_0$ of $f(y)$. Setting $V = f^{-1}(U)$ it follows that $\widehat{h}: \widehat{h}^{-1}(V) \to V$ is a submersion. Thus $\widehat{h}$ is a surjective submersion over $Y \backslash Y_0$. For any section $\widehat{g}: Y \to \widehat{Z}$ of $\widehat{h}: \widehat{Z} \to Y$ the map $g = \sigma\widehat{g}: Y \to Z$ is a lifting of f with respect to h:

$$hg = h(\sigma\widehat{g}) = (h\sigma)\widehat{g} = (f\widehat{h})\widehat{g} = f(\widehat{h}\widehat{g}) = f.$$

Moreover, any lifting g of f is of this form: from $h(g(y)) = f(y)$ $(y \in Y)$ it follows that the point $\widehat{g}(y) := (y, g(y)) \in Y \times Z$ belongs to the subset $\widehat{Z} \subset Y \times Z$ (3.1) and hence $\widehat{g}: Y \to \widehat{Z}$ is a section of $\widehat{h}$. Furthermore, $\sigma(\widehat{g}(y)) = \sigma(y, g(y)) = g(y)$ whence g is obtained from the section $\widehat{g}: Y \to \widehat{Z}$. Therefore Theorem 1.3 follows immediately from Theorem 2.1 and the following lemma.

3.1 Lemma. **(Pull-back sprays.)** *Let $f: Y \to X$ and $h: Z \to X$ be holomorphic maps. Assume that $U \subset X$ is an open set such that $h: Z|_U = h^{-1}(U) \to U$ is a submersion which admits a spray. Then the map $\widehat{h}: \widehat{Z} \to Y$ defined by (3.1) is a submersion with spray over $V = f^{-1}(U) \subset Y$.*

Proof. Let (E, p, s) be a spray associated to the submersion $h: Z|_U \to U$ (Definition 1). Set $V = f^{-1}(U) \subset Y$ and observe that σ maps $\widehat{Z}|_V = \widehat{h}^{-1}(V)$ to $Z|_U$. Let $\widehat{p}: \widehat{E} \to \widehat{Z}|_V$ denote the pull-back of the holomorphic vector bundle

$p: E \to Z|_U$ by the map $\sigma: \widehat{Z}|_V \to Z|_U$. Explicitly, we have

$$\begin{aligned}\widehat{E} &= \{(\widehat{z}, e): \widehat{z} \in \widehat{Z}|_V,\ e \in E;\ \sigma(\widehat{z}) = p(e)\} \\ &= \{(y, z, e): y \in V,\ z \in Z,\ e \in E;\ f(y) = h(z),\ p(e) = z\}; \\ \widehat{p}(\widehat{z}, e) &= \widehat{z}.\end{aligned}$$

Consider the map $\widehat{s}: \widehat{E} \to \widehat{Z}|_V$, $s(y, z, e) = (y, s(e))$. We claim that $(\widehat{E}, \widehat{p}, \widehat{s})$ is a spray associated to the submersion $\widehat{h}: \widehat{Z}|_V \to V$. We first check that $\widehat{s}$ is well defined. If $(y, z, e) \in \widehat{E}$ then $p(e) = z$ and $h(z) = f(y)$. Since s is a spray for h, we have $h(s(e)) = h(z) = f(y)$ which shows that the point $\widehat{s}(y, z, e) = (y, s(e)) \in Y \times Z$ belongs to the fiber $\widehat{Z}_y$. This verifies property (i) in Definition 1. Clearly $\widehat{s}(y, z, 0_{(y,z)}) = (y, s(0_z)) = (y, z)$ which verifies property (ii) in Definition 1. It is also immediate that $\widehat{s}$ satisfies property (iii) provided that s does since the vertical derivatives of the two maps coincide under the identification $\widehat{Z}_y \cong Z_{f(y)}$ and $\widehat{E}_{(y,z)} \cong E_z$. This proves Lemma 3.1.

&4. Multi-valued sections and analytic covers.

In this section we recall some results on symmetric products and multi-valued sections which will be used in the proof of Theorem 1.1. Our reference is [Whi, Appendix V].

Denote by Z^d the d-fold Cartesian power of a set Z. The group Π_d of all permutations on d elements acts on Z^d by permuting the entries, and we denote this action by ρ. The quotient space is called the **d-fold symmetric power** of Z and is denoted by $Z^d_{\rm sym}$. For $z = (z_1, \ldots, z_d) \in Z^d$ we write $\pi(z) = [z] = [z_1, \ldots, z_d] \in Z^d_{\rm sym}$. Similarly we write maps $F: X \to Z^d_{\rm sym}$ in the form $F(x) = [f_1(x), \ldots, f_d(x)]$; although the components f_j are not well defined as individual maps into Z, they are locally well defined (up to a permutation) outside the branch locus br_F (see Definition 1 in section 1). The number d is called the **degree** of F and denoted $\deg F$.

Assume from now on that X and Z are reduced complex spaces. Then $\pi: Z^d \to Z^d_{\rm sym}$ induces a natural (quotient) complex structure on $Z^d_{\rm sym}$ such that holomorphic functions on $Z^d_{\rm sym}$ correspond to ρ-invariant holomorphic functions on Z^d. In particular, if $F = [f_1, \ldots, f_d]: X \to Z^d_{\rm sym}$ is a holomorphic map and if P is a ρ-invariant holomorphic function on Z^d then $P(f_1, \ldots, f_d)$ is a well defined holomorphic function on X.

We recall some natural operations on symmetric products:

1. If Z is a complex subspace of another complex space S then $Z^d_{\rm sym}$ is in a natural way a subspace of $S^d_{\rm sym}$, and any map $X \to S^d_{\rm sym}$ whose image belongs to $Z^d_{\rm sym}$ may also be considered as a map $X \to Z^d_{\rm sym}$. More generally, any holomorphic map $g: Z \to S$ induces a holomorphic map $\widehat{g}: Z^d_{\rm sym} \to S^d_{\rm sym}$.

2. For any pair of integers $d, k \in \mathbb{N}$ we have a natural holomorphic map $\tau: Z^d_{\mathrm{sym}} \times Z^k_{\mathrm{sym}} \to Z^{d+k}_{\mathrm{sym}}$ induced by the identification $Z^d \times Z^k = Z^{d+k}$. Given a pair of maps $F_1: X \to Z^d_{\mathrm{sym}}$ and $F_2: X \to \times Z^k_{\mathrm{sym}}$, we denote

$$F_1 \oplus F_2 = \tau(F_1, F_2): X \to Z^{d+k}_{\mathrm{sym}}.$$

The direct sum operation generalizes to several terms and we write $F = \oplus_j m_j F_j$, where the F_j's are multi-valued maps of X to Z and the m_j's are non-negative integers (this means that F_j appears with the multiplicity m_j in F). Clearly we have $\deg F = \sum_j m_j \deg F_j$. A map $F: X \to Z^d_{\mathrm{sym}}$ is called **irreducible** if it cannot be decomposed as a direct sum of multi-valued maps of smaller degrees.

The following will be used in the proof of Theorem 1.1.

4.1 Proposition. *Assume that $F: X \to Z^d_{\mathrm{sym}}$ is a continuous (resp. holo) d-valued map such that br_F is nowhere dense in X and its complement $X_F = X \backslash \mathrm{br}_F$ is pathwise connected and locally pathwise connected. Then F has a decomposition $\deg F = \sum_{j=1}^{j_0} m_j \deg F_j$ into irreducible continuous (resp. holomorphic) multi-valued maps $F_j: X \to Z^{d_j}_{\mathrm{sym}}$. The decomposition is unique up to the order of terms. This holds in particular if X is an irreducible n-dimensional complex space and the branch locus of F satisfies $\mathcal{H}^{2n-1}(\mathrm{br}_F) = 0$.*

Proof. The first part is proved in [Whi, Appendix V]. The last part follows from the well known fact that the complement of a closed set with $(2n-1)$-dimensional Hausdorff measure zero in an irreducible n-dimensional complex space is pathwise connected and locally pathwise connected. ♠

Assume now that $h: Z \to X$ is a surjective holomorphic map of reduced complex spaces. A map $F = [f_1, \ldots, f_d]: X \to Z^d_{\mathrm{sym}}$ such that $f_j(x) \in Z_x = h^{-1}(x)$ for all $x \in X$ and all j is called a ***d*-section** of h. Its graph is the multiplicity subset of Z defined by

$$V(F) = \{f_i(x) \in Z: x \in X,\ 1 \le i \le d\},$$

where a point $z \in h^{-1}(x) \subset Z$ has multiplicity k if $z = f_i(x)$ for precisely k values of $i \in \{1, \ldots, d\}$. (Points not on $V(F)$ have multiplicity zero.) The projection $h: V(F) \to X$ is a proper continuous d-sheeted map (counting the multiplicities). The direct sum operation defined above naturally extends to multi-valued sections and so does Proposition 4.1.

Recall that a **chain** in Z is a locally finite combination $V = \sum m_j V_j$ of closed complex subvarieties $V_j \subset Z$ with integer coefficients; if $m_j \ge 0$ for all j then the chain is called **effective**. If $\dim V_j = n$ for all j then V is said to be n-dimensional. In the sense of currents we have $[V] = \sum_j m_j [V_j]$.

4.2 Proposition. *Let $h: Z \to X$ be a holomorphic map of complex spaces. If $F: X \to Z^d_{\rm sym}$ is a holomorphic d-section of h then its graph $V(F)$ is a chain in Z and $h: V(F) \to X$ is a d-sheeted analytic cover. Conversely, if X is an irreducible n-dimensional complex space then any effective n-chain $V \subset Z$ such that $h: V \to X$ is a d-sheeted analytic cover is the graph of a holomorphic d-section of h. In particular, if $h: V \to X$ is a d-sheeted analytic cover onto X then $h^{-1}: X \to V^d_{\rm sym}$ is a d-valued holomorphic section of h.*

Remarks. 1. For the definition and general properties of analytic covers see [GR] or [Whi]. The converse part of Proposition 4.2 is false on reducible complex spaces X.

2. If F is a holomorphic multi-valued section which decomposes as $F = \oplus m_j F_j$ then $V(F) = \sum_j m_j V(F_j)$ as chains. Furthermore, if X is irreducible then the decomposition of F into its irreducible components (in the sense of multi-valued sections) correponds with the decomposition of its graph $V(F)$ into irreducible complex subvarieties.

Proof of Proposition 4.2. We include the proof only for the sake of completness since this is essentially contained in [Whi]. Fix a point $x_0 \in X$ and let $z_1, \ldots, z_k$ be the distinct points in $F(x_0)$, where z_j appears with multiplicity d_j (so $\sum_{j=1}^k d_j = d$). Choose pair-wise disjoint neighborhoods $U_j \subset Z$ of z_j for $j = 1, \ldots, k$. By continuity of F there is a neighborhood U of x_0 in X such that for each $x \in U_0$ we have $F(x) \subset \cup_{j=1}^k U_j$ and U_j contains precisely d_j of the points in the d-tuple $F(x) = [f_1(x), \ldots, f_d(x)]$. These points define a d_j-valued section $F_j: U \to Z|_U$ of $h: Z|_U \to U$ with values in U_j. To prove Proposition 4.2 it suffices to consider the maps F_j separately.

Thus we may assume that $F(x_0)$ consists of a single point z_0 of multiplicity d, $\widetilde U$ is an open neighborhood of z_0 in Z embedded as a closed complex subvariety in some open set in $\mathbb{C}^N$, and $U \subset X$ is an open neighborhood of x_0 such that $F(x) \subset \widetilde U$ for all $x \in U$. Write $F = [f_1, \ldots, f_d]$ where $f_j(x)$ is thought of as a point in $\mathbb{C}^N$ for each $x \in U$ and each j. Denote by $z = (z_1, \ldots, z_N)$ the coordinates on $\mathbb{C}^N$. Let $W = \{W_1, \ldots, W_d\}$ be any d-tuple of points $W_k \in \mathbb{C}^N$. There exist finitely many holomorphic polynomials of the form

$$\Phi_j(z, W) = \sum_\alpha P_{j,\alpha}(W) z^\alpha, \quad z \in \mathbb{C}^N,\ W \in (\mathbb{C}^N)^d,\ 1 \le j \le j_0$$

such that (i) the coefficients $P_{j,\alpha}: (\mathbb{C}^N)^d \to \mathbb{C}$ are symmetric polynomials in the components $W_k \in \mathbb{C}^N$ of W (i.e., they are invariant under permutations of the W_k's), and (ii) a point $w \in \mathbb{C}^N$ belongs to the d-tuple W if and only if $\Phi_j(w, W) = 0$ for all $j = 1, \ldots, j_0$. (Any such collection of polynomials is called a set of *canonical defining functions* for d-tuples in $\mathbb{C}^N$ (see e.g. [Whi]). Set $W = F(x)$ for some $x \in U$. Then the function $x \to P_{j,\alpha}(F(x))$ is well defined and holomorphic on U. Thus the set

$$\Gamma(F|_U) = \{(x, z) \in U \times \mathbb{C}^N : \Phi_j(z, F(x)) = 0,\ 1 \le j \le j_0\}$$

is a closed complex subvariety of $U \times \mathbb{C}^N$ contained in $U \times \widetilde{U}$. By construction we have $(x, z) \in \Gamma(F|_U)$ if and only if $z \in F(x)$, and hence $\Gamma(F|_U)$ is the graph of $F|_U$ provided that we identify Z with the subvariety $\{(x, z) \in X \times Z : h(z) = x\}$ of $X \times Z$.

This shows that $V(F)$ is a closed complex subvariety of Z (at this point we consider $V(F)$ as a set without multiplicities). Clearly we have $\dim_z V(F) = \dim_{h(z)} X$ for each $z \in V(F)$. Let $V(F) = \cup_{j=1}^k V_j$ be the decomposition into irreducible components. Then for each $j = 1, \ldots, k$, $h: V_j \to X$ is a proper finite map of V_j onto an irreducible component $X_{k(j)}$ of X. A standard argument shows that F has constant multiplicity along V_j which we denote by m_j, and hence $V(F) = \sum_{j=1}^k m_j V_j$ as chains. If X is irreducible we also have $d = \sum m_j d_j$ (in general this need not be the case since different components V_j may project onto different components of X). This proves the first part of Proposition 4.2.

To prove the converse we assume that X is irreducible and $V = \sum_{j=1}^k m_j V_j$ is a complex n-chain in Z such that $h: V_j \to X$ is a d_j-sheeted analytic cover for each $j = 1, \ldots, k$. For any $x \in X$ we denote the points in the fiber $V_j \cap h^{-1}(x)$ by $f_{j,m}(x)$, $1 \le m \le d_j$, listed according to algebraic multiplicities. Then $x \in X \to F_j(x) = [f_{j,1}(x), \ldots, f_{j,d_j}(x)] \in Z^{d_j}_{\text{sym}}$ defines a d_j-valued section of $h: Z \to X$.

It remains to prove that each F_j is holomorphic. Let $z_0 \in F_j(x_0)$ for some $x_0 \in X$. Let $I = \{i_1, \ldots, i_s\}$ be precisely those indices $i \in \{1, \ldots, d_j\}$ for which $f_{j,i}(x_0) = z_0$. Let g be a germ of a holomorphic function on Z^d at the point $(z_0, \ldots, z_0) \in Z^s$ which is invariant under the action of the permutation group on s elements. By the elementary properties of analytic covers (see e.g. [Whi]) the function $x \to g(f_{j,i_1}(x), \ldots, f_{j,i_s}(x))$ is well defined and holomorphic for x near x_0. This means that the local branches of F_j which at $x = x_0$ pass through z_0 define a holomorphic s-valued section near x_0. Since this holds for every point $z_0 \in F_j(x_0)$, F_j is holomorphic. ♠

The following lemma shows that local multi-valued sections of h exist at each point where h has maximal rank.

4.3 Lemma. *Let $h: Z \to X$ be a holomorphic map of complex spaces. Suppose that Z (resp. X) is locally irreducible at the point $z_0 \in Z$ (resp. at $x_0 := h(z_0) \in X$) and*

$$\dim_{z_0} h^{-1}(x_0) = \dim_{z_0} Z - \dim_{x_0} X.$$

Then there exist an integer $d \in \mathbb{N}$ and a local holomorphic d-valued section F of h in a neighborhood of x_0 such that $F(x_0) = z_0$ with multiplicity d.

The number $k := \dim_{z_0} h^{-1}(x_0)$ is called the **corank** of the map h at z_0, and $\dim_{z_0} Z - k$ is the **rank** of h at z_0. Clearly the rank cannot exceed $\dim_{x_0} X$, and the hypothesis in the lemma is that h has maximal rank at z_0.

Proof. Since $h^{-1}(x_0)$ is a complex subvariety of Z whose dimension at z_0 equals k, there exists a germ of an irreducible complex subvariety $V \subset Z$ at z_0 such that $\dim V + k = \dim_{z_0} Z$ and z_0 is an isolated point of $V \cap h^{-1}(x_0)$. By a standard localization argument we obtain open neighborhoods $U \subset X$ of x_0 and $\widetilde{U} \subset Z$ of z_0 such that $h: V \cap \widetilde{U} \to U$ is a proper finite map. The rank hypothesis implies $\dim V = \dim_{x_0} X$ and hence $h(V \cap \widetilde{U}) = U$ provided that U is irreducible (as we may assume to be the case). By Proposition 4.2 the set $V \cap \widetilde{U}$ is the graph of a holomorphic multi-valued section of h over U. ♠

&5. Proof of Theorem 1.1.

Let $F: X \to Z$ be a d-section of $h: Z \to X$ satisfying the hypotheses of Theorem 1.1. Thus F is assumed to be holomorphic in an open set $U_0 \subset X$ containing a complex subvariety $X' \subset X$ and non-branched over $X \backslash X'$. The hypothesis $\mathcal{H}^{2n-1}(\delta_F) = 0$ implies by Proposition 4.2 that $F = \oplus m_j F_j$ for some irreducible multi-valued sections F_j which are holomorphic over U_0, non-branched over $X \backslash X'$ and satisfy $\mathcal{H}^{2n-1}(\delta_{F_j}) = 0$. It suffices to prove the result for each F_j.

Thus we may assume without loss of generality that F is an irreducible d-section which is non-branched over $X \backslash X'$ and satisfies $\mu_F = d$. The heart of the proof is the following

5.1 Lemma. *There exists a normal complex space Y and a continuous map $g_0: Y \to Z$ such that, setting $f = hg_0: Y \to X$, we have:*

(a) g_0 maps the fiber $Y_x = f^{-1}(x)$ onto $F(x)$ for each $x \in X$ and is holo in the open set $f^{-1}(U_0) \subset Y$,

(b) $f: Y \to X$ is a d-sheeted analytic cover onto X which is non-branched (a d-sheeted holomorphic covering space) over $X \backslash X'$.

The proper way to think about Y is as the 'normalized graph' of F where the self-intersections over $X \backslash X'$ have been removed.

Proof. Over the set U_0 the d-section F is holomorphic and hence its graph $V(F|_{U_0})$ is a complex subvariety of $h^{-1}(U_0)$. Let $g_0: Y_0 \to V(F|_{U_0})$ denote its normalization; then g_0 and $f_0 := hg_0: Y_0 \to U_0$ clearly satisfy the required properties over U_0.

We now extend Y_0 as follows. Let $U \subset X \backslash X'$ be any open set such that $F|_U = \oplus_{j=1}^d F_j$ where $F_j: U \to Z$ are continuous sections of h over U. By Proposition 4.2 the hypothesis $\mathcal{H}^{2n-1}(\delta_F) = 0$ implies that the F_j's are unique up to reordering (and there are no repetitions since $\mu_F = d$). Let $\mathbb{N}_d = \{1, 2, \ldots, d\}$. Set $Y_U = U \times \mathbb{N}_d$ (the disjoint sum of d copies of U) and define the map $g_U: Y_U \to Z$ by $g_U(x, j) = F_j(x)$. We introduce a complex structure on Y_U by requiring that the (trivial) d-sheeted projection $f_U := hg_U: Y_U \to U$ is biholomorphic on each sheet $U \times \{j\}$.

If $U' \subset X\backslash X'$ is another open subset such that $F|_{U'} = \oplus_{j=1}^{d} F'_j$, it follows from $\mathcal{H}^{2n-1}(\delta_F) = 0$ that for each connected component Ω of $U \cap U'$ there is a permutation σ on $\mathbb{N}_p$ (depending on Ω) such that $F_j(x) = F'_{\sigma(j)}(x)$ for $x \in \Omega$ and $j = 1, \ldots, d$. This defines a transition map

$$\sigma_{U,U'}: Y_U|_\Omega \to Y_{U'}|_\Omega, \quad (x, j) \to (x, \sigma(j)).$$

Clearly $\sigma_{U,U'}$ is biholomorphic with respect to the complex structures on Y_U and $Y_{U'}$ and we have $f_U = f_{U'} \circ \sigma_{U,U'}$.

Since F is non-branched over $X\backslash X'$, we may cover $X\backslash X'$ with open sets U as above and use the transition maps $\sigma_{U,U'}$ to define a complex n-dimensional manifold $\widetilde{Y}$, together with a continuous map $\widetilde{g}: \widetilde{Y} \to Z$, such that the composition $\widetilde{f} = h\widetilde{g}: \widetilde{Y} \to X\backslash X'$ is a d-sheeted holomorphic covering projection onto $X\backslash X'$.

It is immediate that the analytic covers $f_0: Y_0 \to U_0$ and $\widetilde{f}: \widetilde{Y} \to X\backslash X'$ restrict to isomorphic covering projections over the set $U_0\backslash X'$ and may be identified to obtain the required space Y which contains both Y_0 and $\widetilde{Y}$ as open domains. The stated properties are clear from the construction. ♠

We continue with the proof of Theorem 1.1. Since $f: Y \to X$ is a finite map of Y onto a Stein space X, it follows that the space Y is also Stein [GRe, LeB]. The inverse f^{-1} is a d-valued holomorphic section of $f: Y \to X$. Consider the map $g_0: Y \to Z$ as a continuous lifting of $f = hg_0: Y \to X$ with respect to $h: Z \to X$. Theorem 1.3 furnishes a homotopy of liftings $g_t: Y \to Z$ connecting g_0 to a holomorphic lifting g_1. Then $F_t = g_t f^{-1}$ $(0 \le t \le 1)$ is a homotopy of d-sections of h satisfying Theorem 1.1. ♠

A correction to [FP3]. We take this occasion to point out the following omission in the hypothesis of Theorem 5.2 in [FP3]: *The sets A and B in the statement of that theorem must have a basis of open Stein neighborhoods in X.* (This property was assumed in the closely related Theorem 5.1 in [FP3], but was accidentally omitted in Theorem 5.2.) The reader can observe that in all applications of Theorem 5.2 in [FP3] this additional hypothesis is satisfied.

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