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Abstract

A generalization of some of the Folkman's constructions [13] of the so called semisymmetric graphs, that is regular graphs which are edge- but not vertex-transitive, was given in [22] together with a natural connection of graphs admitting $\frac{1}{2}$ -arc-transitive group actions and certain graphs admitting semisymmetric group actions. This connection is studied in more detail in this paper. In Theorem 2.1 a sufficient condition for the semisymmetry of the so called generalized Folkman graphs arising from certain graphs admitting a $\frac{1}{2}$ -arc-transitive group action is given. Furthermore, the concepts of alter-sequence and alter-exponent is introduced and studied in great detail and then used to study the interplay of three classes of graphs: cubic graphs admitting a one-regular group action, the corresponding line graphs which admit a $\frac{1}{2}$ -arc-transitive action of the same group and the associated generalized Folkman graphs (Section 4). At the end an open problem is posed, suggesting an in-depth analysis of the structure of tetravalent $\frac{1}{2}$ -arc-transitive graphs with alter-exponent 2.

Key words: Graph, semisymmetric graph, half-arc-transitive graph, Folkman graph, cubic arc-transitive graph, alter-exponent.

Ključne besede: Graf, semisimetričen graf, poltranzitiven graf, Folkmanov graf, kubičen ločno-tranzitiven graf, alter-eksponent.

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1 Introductory remarks

A *digraph* $D = (V, A)$ consists of a finite *set of vertices* $V(D) = V$ and a *set of arcs* $A(D) = A \subseteq (V \times V) \setminus \{(v, v) \mid v \in V\}$. For an arc (u, v) of a digraph D we say that u and v are *the vertices of* (u, v) , more precisely, u is the *tail* and v is the *head* of (u, v) . Also, we say that v is an *out-neighbor* of u and that u is an *in-neighbor* of v . The symbols $N^+(v)$, $N^-(v)$ and $N(v)$ denote the set of out-neighbors of v , the set of in-neighbors of v and the union $N^+(v) \cup N^-(v)$, respectively, while the symbols $\deg^+(v) = |N^+(v)|$ and $\deg^-(v) = |N^-(v)|$ denote the *out-degree* and the *in-degree* of v , respectively. The minimal out-degree and in-degree of a digraph D are denoted by $\delta^+(D)$ and $\delta^-(D)$, respectively. If $\deg^-(v)$ is constant for all $v \in V(D)$, we say that D is *in-regular*. Similarly, if $\deg^+(v)$ is constant for all $v \in V(D)$, we say that D is *out-regular*. A digraph D is *regular* if it is both in- and out-regular. If $\deg^-(v) = \deg^+(v)$ for all $v \in V(D)$, we say that D is *balanced*. Note that a regular digraph is always balanced. If for all $u, v \in V(D)$, we have that $(u, v) \in A(D)$ whenever $(v, u) \in A(D)$, we say that D is a *graph*. An *edge* of a graph is an unordered pair $\{u, v\}$ (also denoted by uv) such that (u, v) is an arc of the graph. The set of edges of the graph X is denoted by $E(X)$.

We refer the reader to [9, 25, 28] for group-theoretic concepts not defined here. Let X be a graph and G a subgroup of the automorphism group $\text{Aut}X$ of X . We say that X is *G -vertex-transitive*, *G -edge-transitive* and *G -arc-transitive* if G acts transitively on $V(X)$, $E(X)$ and $A(X)$, respectively. Furthermore, X is said to be *$(G, \frac{1}{2})$ -arc-transitive* if it is G -vertex-transitive, G -edge-transitive but not G -arc-transitive. In the special case when $G = \text{Aut}X$ we say that X is *vertex-transitive*, *edge-transitive*, *arc-transitive*, and *$\frac{1}{2}$ -arc-transitive* if it is $(\text{Aut}X)$ -vertex-, $(\text{Aut}X)$ -edge-, $(\text{Aut}X)$ -arc-transitive, and $(\text{Aut}X, \frac{1}{2})$ -arc-transitive, respectively.

Let G be a subgroup of $\text{Aut}X$ such that X is G -edge-transitive but not G -vertex-transitive. Then X is necessarily bipartite, where the two parts of the bipartition are orbits of G . Clearly, if X is regular then these two parts have equal cardinality. The graph X is said to be *G -semisymmetric* if it is a regular G -edge- but not G -vertex-transitive graph, and it is said to be *semisymmetric* in the special case $G = \text{Aut}X$. The study of semisymmetric graphs was initiated by Folkman [13] who gave a construction of several infinite families of such graphs. The smallest graph in his construction has 20 vertices and happens to be also the smallest semisymmetric graph. At the end of his paper several open problems were posed, most of which have already been solved (see [2, 3, 10, 11, 15, 16, 17]). Folkman's construction of semisymmetric graphs was generalized by the authors in [22], where an interesting connection with $\frac{1}{2}$ -arc-transitive group actions was suggested. It is precisely the interplay of semisymmetric and $\frac{1}{2}$ -arc-transitive group actions that constitutes the central topic of this article.

Let H be a transitive permutation group acting on a set V and let $v \in V$. There is a 1-1 correspondence between the set of *suborbits* of H , that is, the set of orbits of the stabilizer H_v on V , and the set of *orbitals* of H , that is, the set of orbits in the natural action of H on $V \times V$, with the trivial suborbit $\{v\}$ corresponding to the diagonal $\{(v, v) : v \in V\}$. For an orbital Γ we let $S_{\Gamma, v} = \{w \mid (v, w) \in \Gamma\}$ denote the suborbit of H (relative to v) associated with Γ . Conversely, for a suborbit S of H relative to v we let $\Gamma_{S, v}$ be the

associated orbital in the above 1-1 correspondence.

The paired orbital Γ^{-1} of an orbital Γ is the orbital $\{(v, w) : (w, v) \in \Gamma\}$. If $\Gamma^{-1} = \Gamma$ we say that Γ is *self-paired*. Similarly, for a suborbit S of H relative to v we let $S^{-1} = S_{\Gamma^{-1}, v}$ denote the *paired suborbit* of S . If $S^{-1} = S$ we say that S is self-paired. The *orbital digraph* $\vec{X}(H, V; \Gamma)$ of (H, V) relative to Γ , is the digraph with vertex set V and arc set Γ . The underlying undirected graph of $\vec{X}(H, V; \Gamma)$ will be called the *orbital graph of (H, V) relative to Γ* and will be denoted by $X(H, V; \Gamma)$. If $\Gamma = \Gamma^{-1}$ is a self-paired orbital then $X(H, V; \Gamma)$ admits a vertex- and arc-transitive action of H . On the other hand, if Γ is not self-paired then $X(H, V; \Gamma)$ admits a vertex- and edge- but not arc-transitive action of H , in short, a $\frac{1}{2}$ -arc-transitive action of H .

For a permutation τ of V contained in the normalizer of the permutation group H in the symmetric group $\text{Sym}V$ we let Γ^τ denote the set $\{(x^\tau, y^\tau) \mid (x, y) \in \Gamma\}$. Since τ normalizes H , the set Γ^τ is also an orbital of H . If $v \in V$ is left fixed by τ and S is the suborbit of H corresponding to the orbital Γ relative to the vertex v , then the set $S^\tau = \{s^\tau \mid s \in S\}$ is a suborbit of H , which corresponds to the orbital Γ^τ relative to the vertex v .

The following definition of a generalized Folkman graph, introduced in [22], generalizes one of the original Folkman's constructions of semisymmetric graphs [13].

Definition 1.1 Let H be a transitive permutation group on the set V , Γ its orbital, let $k \geq 2$ be an integer and let τ be a permutation of V contained in the normalizer of H in $\text{Sym}V$ such that $\tau^k \in H$. Let $\mathcal{B} = \{B_x \mid x \in V\}$ and $V_{0j} = \{x_{0j} \mid x \in V, j \in \mathbb{Z}_k\}$ be $k + 1$ copies of the set V . Let $Y = Y(H, V, \Gamma, \tau, k)$ denote the graph with vertex set $\mathcal{B} \cup V_{00} \cup \dots \cup V_{0k-1}$ and edge set $\{x_{0j}B_y \mid (x, y) \in \Gamma^{\tau^j}\}$. The *generalized Folkman graph* $\mathcal{F}(H, V, \Gamma, \tau, k)$ is obtained from $Y(H, V, \Gamma, \tau, k)$ by expanding each B_x to a k -tuple of vertices $x_{10}, x_{11}, \dots, x_{1, k-1}$ each retaining the neighbors' set of B_x . For $j \in \mathbb{Z}_k$ we let V_{1j} denote the set $\{x_{1j} \mid x \in V\}$.

The generalized Folkman graph $\mathcal{F} = \mathcal{F}(H, V, \Gamma, \tau, k)$ is always G -semisymmetric for some group $G \leq \text{Aut}\mathcal{F}$ (see [22] for details). The following simple proposition [22, Proposition 1.2] gives a sufficient condition for semisymmetry of a generalized Folkman graph.

Proposition 1.2 *If no k distinct vertices in $\bigcup_{j \in \mathbb{Z}_k} V_{0j}$ have the same set of neighbors in the graph $Y(H, V, \Gamma, \tau, k)$, then the generalized Folkman graph $\mathcal{F}(H, V, \Gamma, \tau, k)$ is semisymmetric.*

The main purpose of this article is to study the natural connection of graphs admitting semisymmetric group actions and graphs admitting $\frac{1}{2}$ -arc-transitive group actions as suggested in [22]. In Section 2 we give a sufficient condition for the semisymmetry of generalized Folkman graphs arising from certain graphs admitting a $\frac{1}{2}$ -arc-transitive group action (Theorem 2.1). In particular, such a generalized Folkman graph is necessarily semisymmetric provided the $\frac{1}{2}$ -arc-transitive subgroup is primitive and of index 2 in an arc-transitive group of automorphisms. In Section 3 we introduce and study in great detail the concepts of alter-sequence and alter-exponent for a general digraph. These tools are then used in Section 4 to study the interplay of three classes of graphs: cubic

graphs admitting a one-regular group action, the corresponding line graphs which admit a $\frac{1}{2}$ -arc-transitive action of the same group, and the associated generalized Folkman graphs. At the end we pose an open problem, suggesting an in-depth analysis of the structure of tetravalent $\frac{1}{2}$ -arc-transitive graphs with alter-exponent 2.

2 Generalized Folkman graphs arising from $\frac{1}{2}$ -arc-transitive actions

Let X be a $(H, \frac{1}{2})$ -arc-transitive graph admitting arc-transitive action of a group G and let H be a subgroup in G of index 2. Then there exists a non-self-paired orbital Γ of H such that $X = X(H, V(X); \Gamma)$ and $\Gamma \cup \Gamma^{-1}$ is an orbital of G . The two orbital digraphs $\vec{X}(H, V(X); \Gamma)$ and $\vec{X}(H, V(X); \Gamma^{-1})$ may be obtained from X by orienting the edges of X according to the action of H . We shall use the notation $D_H(X)$ for both of these graphs. Let τ be an arbitrary element in $G \setminus H$. Then $\Gamma^\tau = \Gamma^{-1}$ and $\tau^2 \in H$. We can thus construct a generalized Folkman graph $\mathcal{F}(H, V(X), \Gamma, \tau, 2)$, which will be denoted by $\mathcal{F}(X; H)$. This construction of generalized Folkman graphs arising from graphs admitting $\frac{1}{2}$ -arc-transitive group action was first introduced in [22, Examples 2.6 and 2.7]. One of the purposes of this article is a detailed analysis of these graphs.

A quadruple (v_1, v_2, v_3, v_4) of vertices of a digraph D is a *parallel 4-cycle* of D if (v_1, v_2) , (v_2, v_3) , (v_1, v_4) and (v_4, v_3) are arcs of D , and is an *alternating 4-cycle* of D if (v_1, v_2) , (v_3, v_2) , (v_3, v_4) and (v_1, v_4) are arcs of D . The following result gives a sufficient condition for semisymmetry of generalized Folkman graphs arising from $\frac{1}{2}$ -arc-transitive group actions.

Theorem 2.1 *Let X be a connected graph, which is neither a cycle nor a complete graph, admitting an arc-transitive action of a subgroup $G \leq \text{Aut} X$ and a $\frac{1}{2}$ -arc-transitive action of a subgroup H of index 2 of G . Let Γ be a non-self-paired orbital of H such that $X = X(H, V(X); \Gamma)$ and $\Gamma \cup \Gamma^{-1}$ is an orbital of G , and let $D = \vec{X}(H, V(X); \Gamma)$. If for every pair of distinct vertices $u, v \in V(X)$ neither $N_D^+(u) = N_D^+(v)$ nor $N_D^+(u) = N_D^-(v)$, then $\mathcal{F}(X; H)$ is semisymmetric. In particular,*

- (i) *if D has neither alternating 4-cycles nor parallel 4-cycles, then $\mathcal{F}(X; H)$ is semisymmetric.*
- (ii) *if H acts primitively on $V(X)$, then $\mathcal{F}(X; H)$ is semisymmetric.*

PROOF. The general statement of this theorem is an immediate consequence of Proposition 1.2. To prove (i) observe that nonexistence of alternating 4-cycles and parallel 4-cycles excludes the existence of pairs of distinct vertices $u, v \in V(X)$ satisfying, respectively, $N_D^+(u) = N_D^+(v)$ and $N_D^+(u) = N_D^-(v)$. Similarly, to prove (ii) it suffices to see that no pair of distinct vertices $u, v \in V(X)$, satisfying one of the above two equalities, exists. Assume first that there are distinct vertices $u, v \in V(X)$ such that $N_D^+(u) = N_D^+(v)$. Clearly, the set $\{w \mid N_D^+(w) = N_D^+(u)\}$ is a block of the action of H on $V(X)$. Since this block contains at least two vertices and H acts primitively, it follows that it is the whole of $V(X)$. In particular, for $w \in N_D^+(u)$ we have that $w \in N_D^+(w)$, which is clearly

impossible. Similarly, it may be proved that there is no pair u, v of distinct vertices of D such that $N^-(u) = N^-(v)$.

Suppose now that there exists a pair of distinct vertices $u, v \in V(X)$ such that $N_D^+(u) = N_D^-(v)$. Since H is transitive on $V(X)$ we may therefore assume (in view of the results of the previous paragraph) that for each vertex $w \in V(X)$ there exists a unique vertex $w^+ \in V(X)$ and a unique vertex $w^- \in V(X)$ such that $N^+(w) = N^-(w^+)$ and $N^-(w) = N^+(w^-)$. Let ρ be an element of H mapping u to u^+ . For each integer i let $u_i = u^{\rho^i}$. We claim that the orbit $O(u) = \{u_i \mid i \in \mathbb{Z}\}$ of $\langle \rho \rangle$ is a block of H . Namely, take an arbitrary element $\sigma \in H$ fixing the vertex u and observe that $(u_1)^\sigma = (u^+)^\sigma$ is either u_1 or $u_{-1} = u^-$, and inductively $(u_i)^\sigma$ is either u_i or u_{-i} . In other words, $O(u)$ is preserved by σ and is thus a block of H . But H acts primitively on $V(X)$, forcing $V(X) = O(u)$. Let $n = |V(X)|$. Since $\langle \rho \rangle$ is a regular cyclic subgroup of H , the digraph D is the directed circulant with symbol $S = \{s \in \mathbb{Z}_n \setminus \{0\} \mid u_s \in N^+(u)\}$. Observe that S and $-S$ are disjoint. Moreover, $N^+(u_0) = N^-(u_1)$ implies that $S = 1 - S$. Next, since cyclic groups of composite orders are B -groups [28, Theorem 25.3], it follows that either H is doubly transitive on $V(X)$ or n is a prime number. The first case is impossible as it gives rise to a complete graph. We can thus assume that n is a prime number. By the well known results on edge-transitive graphs of prime order [5] it follows that there exists $a \in \mathbb{Z}_n^*$ such that $S = aT$ is a coset of a subgroup T of \mathbb{Z}_n^* . Hence $aT = 1 - aT$ and so (using the fact that T is a group) we have $aT = T - aT$, which implies $|T| \in \{0, 1, n\}$ (see [22, Lemma 2.3]), a contradiction. ■

Example 2.2 Let $p \geq 11$ be a prime, let $H = PSL(2, p)$, and let K be a subgroup of H isomorphic to the dihedral group D_{p+1} of order $p+1$ (or to the dihedral group D_{p-1}). Let V be the set of right cosets of H . It may be seen that H acts primitively on V and that some of the orbitals of this action are non-self-paired (see [26, Lemma 4.4] for details). Let X be an arbitrary orbital graph associated with one of the above non-self-paired orbitals. In view of [26, Lemma 4.4] the automorphism group $\text{Aut } X$ coincides with $PGL(2, p)$ and acts arc-transitively on X . By Theorem 2.1 it follows that $\mathcal{F}(X)$ is semisymmetric.

Example 2.3 There is a connection between regular maps and $\frac{1}{2}$ -arc-transitive group actions on graphs of valency 4 and consequently the corresponding generalized Folkman graphs. Let M be a regular map and Y be its medial graph. By [20, Theorem 4.1], it follows that M is regular and reflexible if and only if Y admits a $\frac{1}{2}$ -arc-transitive action of some $H \leq \text{Aut } Y$ with vertex-stabilizer \mathbb{Z}_2 and a 1-arc-transitive action of a subgroup G , such that $H \leq G \leq \text{Aut } Y$, with vertex-stabilizers isomorphic either to \mathbb{Z}_2^2 or to \mathbb{Z}_4 . (Note that the corresponding map M is reflexible and positively self-dual, respectively.) The sufficient condition for the semisymmetry of the associated generalized Folkman graph $\mathcal{F}(Y; H)$ given in part (i) of Theorem 2.1 may be checked using [21, Theorem 4.1], which characterizes graphs of valency 4 and girth 4 admitting $\frac{1}{2}$ -arc-transitive group actions.

We end this section with a couple of remarks regarding the connectedness of the generalized Folkman graph $\mathcal{F}(X; H)$ arising from the $\frac{1}{2}$ -arc-transitive action of a group H on a graph X .

Let V_{ij} , $i, j \in \mathbb{Z}_2$, be the four orbits of the group H in its action on the vertices of the generalized Folkman graph $\mathcal{F} = \mathcal{F}(X, H)$, where $V_{00} \cup V_{01}$ is one of the two bipartition sets of \mathcal{F} . Clearly, \mathcal{F} is connected if and only if for any two vertices $u, v \in V_{00}$ there is a path in \mathcal{F} from u to v . Orient the edges in \mathcal{F} from V_{00} to V_{1j} and from V_{1j} to V_{01} for all $j \in \mathbb{Z}_2$. Observe that this orientation is coherent with the orientation of the arcs in $D_H(X)$. A walk from u to v in \mathcal{F} has the property that the number of arcs travelled with the orientation is the same as the number of arcs travelled against the orientation. Moreover, for any subwalk starting at u , the difference between these two number is one of 0, 1, or 2. Such a walk has a counterpart in the digraph $D_H(X)$, which satisfies the same condition. The connectedness of the graph \mathcal{F} may therefore be read from the digraph $D_H(X)$. In fact, the graph \mathcal{F} is connected if and only if the set of endvertices of walks originating at u and satisfying the above property, coincides with the whole of $V(X)$.

This motivates the study of the above described walks in a more general setting: first, for any digraph and second, allowing the difference between the number of arcs travelled with the orientation and the number of arcs travelled against the orientation in a subwalk to be inside any prescribed interval of integers. This the content of the next section.

3 Alter-exponent of a digraph

Let D be a digraph, $\{v_0, v_1, \dots, v_n\} \subseteq V(D)$ and $\{a_1, a_2, \dots, a_n\} \subseteq A(D)$. A sequence $W = (v_0, a_1, v_1, a_2, v_2, \dots, v_{n-1}, a_n, v_n)$ is a *walk of length n in D from v_0 to v_n* if for all $i \in \{1, 2, \dots, n\}$ either $a_i = (v_{i-1}, v_i)$ or $a_i = (v_i, v_{i-1})$. In the first case a_i is *positively oriented* in W , and is *negatively oriented* in the second case. The *sum $s(W)$ of the walk W* is the difference between the number of positively oriented arcs in the walk and the number of negatively oriented arcs in the walk. The k^{th} *partial sum $s_k(W)$ of the walk W* is the sum of the walk $(v_0, a_1, v_1, a_2, v_2, \dots, v_{k-1}, a_k, v_k)$. In addition, we let $s_0(W) = 0$. The *tolerance of the walk W* is the set $\{s_k(W) \mid k \in \{0, 1, \dots, n\}\}$. Observe that the tolerance of a walk is always an interval of integers containing 0. We say that two vertices u and v of a digraph D are *alter-equivalent with tolerance I* (and write $u \text{ alt}_I v$) if there is a walk from u to v with sum 0 and tolerance J , $J \subseteq I$. It is not difficult to see that the relation alt_I is an equivalence relation for any interval $I \subseteq \mathbb{Z}$ containing 0. The corresponding equivalence class containing a vertex $u \in V(D)$ and the corresponding partition of the set $V(D)$ will be denoted by $\mathcal{B}_I(u; D)$ and by $\mathcal{B}_I(D)$, respectively. For $a, b \in \mathbb{Z} \cup \{-\infty, \infty\}$, $a \leq 0 \leq b$, the abbreviations alt_a , alt_b , $\mathcal{B}_a(D)$ and $\mathcal{B}_b(D)$ will be used for $\text{alt}_{[a,0]}$, $\text{alt}_{[0,b]}$, $\mathcal{B}_{[a,0]}(D)$ and $\mathcal{B}_{[0,b]}(D)$, respectively. The cardinality of the set $\mathcal{B}_\infty(D)$ will be referred to as *alter-perimeter* of D . If the set $\mathcal{B}_\infty(D)$ consists of one class only (that is, if the alter-perimeter equals 1), we say that D is *alter-complete*. A digraph is *alter-incomplete*, if it is not alter-complete.

Let D be a digraph, $u \in V(D)$ and $v \in \mathcal{B}_\infty(u; D)$. The smallest integer t , for which $v \in \mathcal{B}_t(u; D)$, will be denoted by $\text{exp}_D(u, v)$, and the maximum of the set $\{\text{exp}_D(u, v) \mid v \in \mathcal{B}_\infty(u; D)\}$ by $\text{exp}_D(u)$. The *alter-exponent of a digraph D* is the maximum of the set $\{\text{exp}_D(u) \mid u \in V(D)\}$ and is denoted by $\text{exp}(D)$. The alter-exponent of a digraph D is thus the smallest positive integer t , for which $\mathcal{B}_t(D) = \mathcal{B}_\infty(D)$.

Proposition 3.1 *Let D be a digraph and let k and i be positive integers. If $\mathcal{B}_k(D) = \mathcal{B}_{k+i}(D)$, then $\exp(D) \leq k$.*

PROOF. It suffices to show that for any integer k the equality $\mathcal{B}_k(D) = \mathcal{B}_{k+1}(D)$ implies the equality $\mathcal{B}_{k+1}(D) = \mathcal{B}_{k+2}(D)$. Suppose that $\mathcal{B}_k(D) = \mathcal{B}_{k+1}(D)$, let u and w be two vertices of D , and suppose that there is a walk $W = (u, a_1, v_1, \dots, v_{n-1}, a_n, v)$ in D of sum 0 and tolerance contained in the interval $[0, k+2]$. We have to show that there is also a walk in D from u to v with sum 0 and tolerance contained in $[0, k+1]$. Clearly, $a_1 = (u, v_1)$ and $a_n = (v, v_{n-1})$, for otherwise the tolerance of W would contain negative integers. But then $(v_1, a_2, v_2, \dots, v_{n-2}, a_{n-1}, v_{n-1})$ is a walk in D with sum 0 and tolerance contained in $[0, k+1]$. By the assumption, there is also a walk $(v_1, b_2, v'_2, \dots, v'_{n-2}, b_{n-1}, v_{n-1})$ in D with sum 0 and tolerance contained in $[0, k]$. The walk $(u, a_1, v_1, b_2, v'_2, \dots, v'_{n-2}, b_{n-1}, v_{n-1}, a_n, v)$ has sum 0 and its tolerance is contained in the interval $[0, k+1]$. ■

Let $n \geq 2$ be a positive integer. A digraph with vertex set \mathbb{Z}_n and arc set $\{(k, k+1) \mid k \in \mathbb{Z}_n\}$ is called the *oriented cycle of length n* and is denoted by \vec{C}_n . (Note, that by this definition, an oriented cycle \vec{C}_2 is isomorphic to the complete graph K_2 .)

Let \mathcal{B} be a partition of the vertex set V of a digraph D . The *quotient digraph* $D_{\mathcal{B}}$ is defined to have vertex set \mathcal{B} , with a pair $(B, C) \subseteq \mathcal{B} \times \mathcal{B}$, $B \neq C$, being an arc of $D_{\mathcal{B}}$, if and only if there is an arc (u, v) of D , such that $u \in B$ and $v \in C$. For a digraph D and $B \subseteq V(D)$ let $D[B]$ denote the digraph with vertex set B and arc set $(B \times B) \cap A(D)$. Similarly, for two disjoint subsets $B, C \subseteq V(D)$ let $D[B, C]$ denote the digraph with vertex set $B \cup C$ and arc set $(B \times C \cup C \times B) \cap A(D)$. The following proposition justifies the choice of the term *alter-perimeter* to denote the cardinality of the set $\mathcal{B}_{\infty}(D)$.

Proposition 3.2 *Let D be a connected alter-incomplete digraph (that is $|\mathcal{B}_{\infty}(D)| > 1$) with $\delta^+(D) > 0$ and $\delta^-(D) > 0$. Then the quotient digraphs $D_{\mathcal{B}_{\infty}(D)}$ and $D_{\mathcal{B}_{-\infty}(D)}$ are isomorphic to the oriented cycle \vec{C}_n for some positive integer n .*

PROOF. Let $\mathcal{B} = \mathcal{B}_{\infty}(D)$ and $\overline{D} = D_{\mathcal{B}_{\infty}(D)}$. Let $B \in \mathcal{B}$, $u, v \in B$ and $u', v' \in V(D)$, such that $a = (u', u)$ and $b = (v', v)$ are arcs of D . Vertices u' and v' do not belong to B , for otherwise D would be alter-complete. Since u and v belong to the same member of \mathcal{B} , there exist a positive integer t and a walk $(u, a_1, v_1, \dots, v_{k-1}, a_k, v)$ in D with sum 0 and tolerance $I = [0, t]$. But then $(u', a, u, a_1, v_1, \dots, v_{k-1}, a_k, v, b, v')$ is a walk with sum 0 and tolerance $[0, t+1]$ in D from u' to v' . This implies that u' and v' belong to the same member of \mathcal{B} and shows that $\deg_{\overline{D}}^-(B) \leq 1$ for each $B \in \mathcal{B}$. But since $\deg_D^-(v) > 0$ for each $v \in V(D)$, the inequality $\deg_{\overline{D}}^-(B) > 0$ also holds for each $B \in \mathcal{B}$, and thus $\deg_{\overline{D}}^-(B) = 1$ for each $B \in \mathcal{B}$. Similarly, $\deg_{\overline{D}}^+(B) > 0$ for each $B \in \mathcal{B}$ and since $\sum_{B \in \mathcal{B}} \deg_{\overline{D}}^+(B) = \sum_{B \in \mathcal{B}} \deg_{\overline{D}}^-(B)$, we can deduce that $\deg_{\overline{D}}^+(B) = 1$ for each $B \in \mathcal{B}$. Since \overline{D} is connected, it is isomorphic to the oriented cycle \vec{C}_n , where $n = |\mathcal{B}|$. Similarly, by considering the digraph with vertex set $V(D)$ and arc set $\{(u, v) \mid (v, u) \in A(D)\}$ we can also deduce that $D_{\mathcal{B}_{-\infty}}$ is also isomorphic to \vec{C}_n . ■

The product W_1W_2 of a walk $W_1 = (u, a_1, u_1, \dots, u_{m-1}, a_m, w)$ from u to w and a walk $W_2 = (w, b_1, v_1, \dots, v_{n-1}, b_n, v)$ from w to v is defined to be the walk $W_1W_2 = (u, a_1, u_1, \dots, a_{m-1}, a_m, w, b_1, v_1, \dots, v_{n-1}, b_n, v)$ from u to v . The inverse walk W_1^{-1} of the walk W_1 is the walk $(w, a_m, u_{m-1}, \dots, u_1, a_1, u)$. The next two propositions, leading to a characterization of the alter-exponent of a vertex-transitive digraph, are consequences of Proposition 3.2.

Proposition 3.3 *Let D be a connected digraph such that $\delta^+(D) > 0$ and $\delta^-(D) > 0$. If $\mathcal{B}_t(D) = \mathcal{B}_{-t}(D)$ for some positive integer t then $\mathcal{B}_{[-t,t]}(D) = \mathcal{B}_t(D)$.*

PROOF. Let $u, v \in V(D)$ be such that $u \text{ alt}_{[-t,t]} v$. Then there is a walk $W = (u = v_0, a_1, v_1, \dots, v_{n-1}, a_n, v_n = v)$ in D with sum 0 and tolerance contained in the interval $[-t, t]$. Let $0 = j_0 < j_1 < \dots < j_k = n$ be exactly those indices $j_s \in \{1, \dots, n\}$ for which the sum of the walk $(u, a_1, v_1, \dots, v_{j_s-1}, a_{j_s}, v_{j_s})$ equals 0. For any $s \in \{1, \dots, k\}$, the tolerance of the walk $W_s = (v_{j_{s-1}}, a_{j_{s-1}+1}, v_{j_{s-1}+1}, \dots, v_{j_s-1}, a_{j_s}, v_{j_s})$ is contained either in $[0, t]$ or in $[-t, 0]$. Clearly, $W = W_1W_2 \dots W_k$. Let S be the set of those indices s for which the tolerance of the walk W_s is contained in $[-t, 0]$. Since $\mathcal{B}_t(D) = \mathcal{B}_{-t}(D)$, there is a walk W'_s in D from $v_{j_{s-1}}$ to v_{j_s} with sum 0 and tolerance contained in $[0, t]$ for every $s \in S$. By substituting the walk W_s with the walk W'_s for every index $s \in S$ in the product $W_1W_2 \dots W_k$ we obtain a walk from u to v with sum 0 and tolerance contained in $[0, t]$. ■

Proposition 3.4 *Let D be a connected digraph such that $\delta^+(D) > 0$ and $\delta^-(D) > 0$, and let $e = \exp(D)$. Then $\mathcal{B}_e(D) = \mathcal{B}_{-e}(D) = \mathcal{B}_{[-e,e]}(D)$.*

PROOF. Let $u \in V(D)$ be arbitrary and let $v \in \mathcal{B}_e(u; D)$. Since $\deg^-(w) > 0$ for each $w \in V(D)$, we can inductively construct a walk $(u, a_1, u_1, \dots, u_{e-1}, a_e, u_e)$ and a walk $(v, b_1, v_1, \dots, v_{e-1}, b_e, v_e)$ in such a way that all the arcs a_i and b_i , $i \in \{1, \dots, e\}$, are negatively oriented in the above two walks. Since u and v are contained in the same member of the set $\mathcal{B}_e(D)$, the same holds for u_e and v_e . This is clearly the case if D is alter-complete, for then $\mathcal{B}_e(D)$ consists of a single class. On the other hand, if D is alter-incomplete the statement can be deduced from the fact that, in view of Proposition 3.2, $D_{\mathcal{B}_e(D)}$ is isomorphic to \vec{C}_n for some positive integer n . There is therefore a walk $(u_e, c_1, w_1, \dots, w_{t-1}, c_t, v_e)$ in D with sum 0 and tolerance contained in $[0, e]$. But then the walk $(u, a_1, u_1, \dots, u_{e-1}, a_e, u_e, c_1, w_1, \dots, w_{t-1}, c_t, v_e, b_e, v_{e-1}, \dots, b_1, v)$ is a walk from u to v with sum 0 and tolerance contained in $[-e, 0]$. We have thus proved that $\mathcal{B}_e(u; D) \subseteq \mathcal{B}_{-e}(u; D)$. Similarly, $\mathcal{B}_{-e}(u; D) \subseteq \mathcal{B}_e(u; D)$, and thus $\mathcal{B}_e(D) = \mathcal{B}_{-e}(D)$. By Proposition 3.3 the result follows. ■

The study of the alter-exponent is of a particular interest in the case of vertex-transitive digraphs. Namely, let D be a vertex-transitive digraph of exponent e . Clearly, for every $k \in \{1, \dots, e\}$, the partition $\mathcal{B}_k(D)$ is a complete system of imprimitivity for the action of $\text{Aut} D$ on $V(D)$. To every vertex-transitive digraph D we may thus assign a sequence $(\mathcal{B}_1(D), \mathcal{B}_2(D), \dots, \mathcal{B}_{\exp(D)}(D))$ of complete systems of imprimitivity for the action of $\text{Aut} D$ on $V(D)$, each partition being a refinement of the next one in the sequence. The sequence

($|B_1(u; D)|, |B_2(u; D)|, \dots, |B_{\exp(D)}(u; D)|$) of the cardinalities of the members of the above systems of imprimitivity will be called *the alter-sequence of D* .

If G is a group acting transitively on a set V and K is a normal subgroup of G , then the orbits of K on V form a complete system of imprimitivity of G . In general it is not true that every complete system of imprimitivity arises in this way. However, there are special instances with particular interest to us where this is the case.

Proposition 3.5 *Let D be a connected vertex-transitive digraph and K a subgroup of $\text{Aut}D$ consisting of those automorphisms which fix every member of $\mathcal{B}_{\exp(D)}(D)$ setwise. Then K is normal in $\text{Aut}D$ and consequently the members of $\mathcal{B}_{\exp(D)}(D)$ are orbits of K .*

PROOF. Let $\mathcal{B} = \mathcal{B}_{\exp(D)}(D)$, let $G = \text{Aut}D$ and let B be an arbitrary element of \mathcal{B} . If D is alter-complete, the statement of the above proposition is trivially true since $K = G$. We can thus assume that D is alter-incomplete. Since K is the kernel of the action of D on \mathcal{B} , we have that K is normal in G . Let $H = G_B$ be the setwise stabilizer of B in G . Since G acts transitively on $V(D)$ and B is a block of G , it follows that H is transitive on B . We claim that $H = K$. Clearly, $K \subseteq H$. Consider the action of the quotient group G/K on the quotient digraph $D_{\mathcal{B}}$. Clearly, G/K is a subgroup of the group of automorphisms of $D_{\mathcal{B}}$ and H/K is the vertex stabilizer in G/K . But by Proposition 3.2, $D_{\mathcal{B}}$ is isomorphic to an oriented cycle. Therefore H/K is trivial and so $H = K$. It follows that B is an orbit of K . ■

The following definition, lemma and corollary will be needed in the proof of Proposition 3.9, which gives an alternative definition of the alter-exponent in the case of connected vertex-transitive digraphs.

Let D be a digraph, let $u \in V(D)$, and let a, b and k be integers such that $a \leq 0 \leq b$, $a \leq k \leq b$ holds. Then let the symbol $S_{[a,b]}^k(u; D)$ denote the set of those vertices w , for which a walk from u to w with sum k and tolerance contained in $[a, b]$ exists.

Lemma 3.6 *Let D be a digraph, u and w its vertices, and let integer a, b, k be as above. Then the following statements are equivalent:*

- (i) $w \in S_{[a,b]}^k(u; D)$,
- (ii) $\mathcal{B}_{[a,b]}(u; D) = S_{[a-k,b-k]}^{-k}(w; D)$,
- (iii) $u \in S_{[a-k,b-k]}^{-k}(w; D)$,
- (iv) $\mathcal{B}_{[a-k,b-k]}(w; D) = S_{[a,b]}^k(u; D)$.

PROOF. Observe first that implications $[(ii) \Rightarrow (iii)]$ and $[(iv) \Rightarrow (i)]$ are trivial, since $u \in \mathcal{B}_{[a,b]}(u; D)$, as well as $w \in \mathcal{B}_{[a-k,b-k]}(w; D)$. Furthermore, by a substitution $a \mapsto a-k$, $b \mapsto b-k$, $k \mapsto -k$, $u \mapsto w$ and $w \mapsto u$ one can see that the implication $[(i) \Rightarrow (ii)]$ forces also the implication $[(iii) \Rightarrow (iv)]$. It therefore suffices to prove the implication $[(i) \Rightarrow (ii)]$.

Let us therefore assume that $w \in S_{[a,b]}^k(u; D)$ and that W is a walk from u to w with sum k and tolerance contained in $[a, b]$. Suppose that $z \in \mathcal{B}_{[a,b]}(u; D)$. Then there is a walk Z from u to z with sum 0 and tolerance contained in $[a, b]$. But then the walk $W^{-1}Z$ from w to z has sum $-k$ and tolerance contained in $[a - k, b - k]$, which proves that $\mathcal{B}_{[a,b]}(u; D) \subseteq S_{[a-k, b-k]}^{-k}(w; D)$. Suppose now that $z \in S_{[a-k, b-k]}^{-k}(w; D)$. Then there is a walk Z from w to z with sum $-k$ and tolerance contained in $[a - k, b - k]$. But then the walk WZ from u to z has sum 0 and tolerance contained in $[a, b]$, which proves the equality $\mathcal{B}_{[a,b]}(u; D) = S_{[a-k, b-k]}^{-k}(w; D)$. ■

Corollary 3.7 *Let D be a vertex-transitive digraph and a, b integers, such that $a \leq 0 \leq b$. Then the cardinality of the set $S_{[a,b]}^k(u; D)$ does not depend on the choice of a vertex $u \in V(D)$ and an integer $k \in [a, b]$.*

PROOF. Let k be an arbitrary integer contained in the interval $[a, b]$ and let u be an arbitrary vertex of D . Since the sets $\mathcal{B}_{[a,b]}(w; D)$, $w \in V(D)$, form a complete system of imprimitivity of the group $\text{Aut}D$ (and are therefore of equal cardinality), it suffices to see that $|S_{[a,b]}^k(u; D)| = |\mathcal{B}_{[a,b]}(u; D)|$. Let D' be the digraph with vertex-set $V(D)$ and arc-set $\{(w_1, w_2) \mid w_2 \in S_{[a,b]}^k(w_1; D)\}$. In view of the equivalence [(i) \Leftrightarrow (iii)] in Lemma 3.6, it follows that $N_{D'}^-(v) = S_{[a-k, b-k]}^{-k}(v; D)$ for every vertex $v \in V(D)$. Clearly, $\text{Aut}D \subseteq \text{Aut}D'$, which shows that D' is vertex-transitive, and thus regular and balanced. Therefore

$$|S_{[a,b]}^k(v'; D)| = |S_{[a,b]}^k(v; D)| = |S_{[a-k, b-k]}^{-k}(v; D)|$$

for every pair of vertices $v, v' \in V(D)$. Let w be an arbitrary element of the set $N_{D'}^+(u) = S_{[a,b]}^k(u; D)$. Since D and D' are vertex-transitive, it follows that $|S_{[a,b]}^k(u; D)| = |S_{[a,b]}^k(w; D)|$. Combining these facts with Lemma 3.6 we can deduce that

$$|S_{[a,b]}^k(u; D)| = |S_{[a-k, b-k]}^{-k}(w; D)| = |\mathcal{B}_{[a,b]}(u; D)|,$$

as required. ■

Lemma 3.8 *Let D be a vertex-transitive digraph, let $u \in V(D)$, and let a, b and k be integers such that $a \leq 0 \leq b$ and $a \leq k \leq b$. Then the sets $\mathcal{B}_{[a,b]}(u; D)$ and $\mathcal{B}_{[a-k, b-k]}(u; D)$ have equal cardinalities. In particular, $|\mathcal{B}_k(u; D)| = |\mathcal{B}_{-k}(u; D)|$ for every integer k .*

PROOF. Let w be an arbitrary member of the set $S_{[a-k, b-k]}^{-k}(u; D)$. By Corollary 3.7 it follows that $|\mathcal{B}_{[a,b]}(u; D)| = |S_{[a,b]}^0(u; D)| = |S_{[a,b]}^k(w; D)|$ and by Lemma 3.6 we have that $|S_{[a,b]}^k(w; D)| = |\mathcal{B}_{[a-k, b-k]}(u; D)|$. ■

Proposition 3.9 *Let D be a connected vertex-transitive digraph. Then $\exp(D) = \min\{t \mid \mathcal{B}_{-t}(D) = \mathcal{B}_t(D), t > 0\}$.*

PROOF. By Proposition 3.4, we know that $\mathcal{B}_{-\exp(D)}(D) = \mathcal{B}_{\exp(D)}(D)$. Let t be a positive integer such that $\mathcal{B}_{-t}(D) = \mathcal{B}_t(D)$. Then by Proposition 3.3, $\mathcal{B}_t(D) = \mathcal{B}_{[-t,t]}(D)$. By Lemma 3.8 we then have that $|\mathcal{B}_t(D)| = |\mathcal{B}_{2t}(D)|$. Since $\mathcal{B}_t(D)$ is a refinement of $\mathcal{B}_{2t}(D)$ it follows that $\mathcal{B}_t(D) = \mathcal{B}_{2t}(D)$, and by Proposition 3.1 it follows that $t \geq \exp(D)$. ■

Alter-exponents may be arbitrarily large as is shown by the two examples below, giving constructions of, respectively, alter-incomplete and alter-complete digraphs with prescribed alter-exponents.

Example 3.10 Let $k \geq 2$ be an integer and let $e_i, i \in \{1, 2, \dots, k\}$, be the i^{th} standard basis vector of \mathbb{Z}_2^k . Denote by D_k the digraph with vertex set $V = \mathbb{Z}_2^k \times \mathbb{Z}_{2k}$ and arcs of the form $((v, i), (v, i + 1))$ and $((v, i), (v + e_{(i \bmod k) + 1}, i + 1))$ for all $v \in \mathbb{Z}_2^k$ and all $i \in \{1, 2, \dots, k\}$. It is not difficult to see, that $D/\mathcal{B}_\infty \cong \vec{\mathcal{C}}_{2k}$ and that $\exp(D) = k$. Note, that this digraph is a 2^k -fold regular cover of an oriented (multi)cycle with vertex set $\{u_0, \dots, u_{2k-1}\}$ and two arcs from u_i to u_{i+1} for each $i \in \mathbb{Z}_{2k}$ and with \mathbb{Z}_2^k as the group of covering transformations (the voltage group). The voltages on the pair of arcs between u_i and u_{i+1} are 0 and $e_{i'}$, where $i' = (i \bmod k) + 1$.

Example 3.11 Let $k \geq 2$ be an integer, and let A_{2k} be the alternating group acting (for reasons of convenience) on the set \mathbb{Z}_{2k} . Let $r = (0, 1, \dots, 2k - 3, 2k - 2)$ and $t = (0, 1, \dots, 2k - 3, 2k - 1)$ be two elements of A_{2k} . It may be seen that the Cayley digraph $\text{Cay}(A_{2k}; \{r, t\})$ is an arc-transitive digraph of exponent $2k - 2$ and $\mathcal{B}_\infty(D) = \{V(D)\}$. Note that the smallest member of this family was first given in [22, Example 2.6].

We conclude the section by stating the criterion for connectivity of the generalized Folkman graphs, as announced at the end of the previous section.

Proposition 3.12 *Let X and H have the meaning described in the first paragraph of Section 2. Then the generalized Folkman graph $\mathcal{F}(X; H)$ is connected if and only if $D_H(X)$ is alter-complete and $\exp(D_H(X)) \leq 2$. Furthermore, if it is disconnected, it consists of $|\mathcal{B}_2(D)|$ isomorphic connected components.*

It will be convenient to generalize the concepts of alter-exponent and alter-sequence to graphs admitting a $\frac{1}{2}$ -arc-transitive group action via the associated digraphs. Namely, let X be a $(H, \frac{1}{2})$ -transitive graph. We shall use the terms H -alter-exponent and H -alter-sequence of X to denote the alter-exponent and the alter-sequence of the corresponding digraph $D_H(X)$, respectively. In particular, we shall omit the symbol H in the case $H = \text{Aut}X$.

4 Line graphs of cubic graphs and the associated generalized Folkman graphs

There is a natural construction of generalized Folkman graphs via the line graphs of certain cubic arc-transitive graphs. It is the purpose of this section to discuss the semisymmetry (and connectivity) of these graphs.

Let G be a group and X be a connected G -arc-transitive cubic graph. By the well known result of Tutte [27], it follows that G acts regularly on the set of s -arcs of X for some positive integer $s \leq 5$. (We say that G acts s -regularly on X .) Let us now consider the line graph $L(X)$ of X . Note that $\text{Aut}X = \text{Aut}L(X)$. It may be deduced from [24, Proposition 1.1] that G acts 1-regularly on X if and only if it acts $\frac{1}{2}$ -arc-transitively on $L(X)$. In particular, $\text{Aut}X$ acts 1-regularly on X , that is X is 1-regular if and only if $L(X)$ is a $\frac{1}{2}$ -arc-transitive graph of valency 4 and girth 3. Similarly, if G acts 2-regularly on X , then it acts arc-transitively on $L(X)$. Assume that $H \leq G$ are subgroups of $\text{Aut}X$ acting, respectively, 1-regularly and 2-regularly on X . Then we shall say that X is $(1^H, 2^G)$ -regular. In this case H acts $\frac{1}{2}$ -arc-transitively on $L(X)$, and is contained in the arc-transitive group G as a subgroup of index 2. We can thus construct the generalized Folkman graph $\mathcal{F}(L(X); H)$.

The following proposition shows that the alter-exponent of a digraph $D_H(L(X))$, associated with the line graph of a cubic graph X admitting a 1-regular action of a subgroup H of $\text{Aut}X$, is at most 2.

Proposition 4.1 *Let X be a connected cubic graph admitting a 1-regular action of a subgroup H of $\text{Aut}X$. Let $D = D_H(L(X))$ be (one of the two) digraphs obtained from the line graph $L(X)$ by orienting the edges of $L(X)$ in accordance with the $\frac{1}{2}$ -arc-transitive action of H on $L(X)$. Then one of the following occurs:*

- (i) $\exp(D) = 1$, D is alter-incomplete and the alter-perimeter of D equals 3, or
- (ii) $\exp(D) = 2$, D is alter-incomplete and the alter-perimeter of D equals 3, or
- (iii) $\exp(D) = 2$ and D is alter-complete.

PROOF. Let us first prove that alter-exponent of D is at most 2. Observe that in view of the 1-regularity of the action of H on X , every vertex of X gives rise to an oriented 3-cycle of D . Moreover, every arc of D lies on precisely one such oriented 3-cycle of D . This implies that for every arc (u, v) of D there exists a vertex $w \in V(D)$, such that (v, w) and (w, u) are arcs of D . It suffices to show that for any walk in D with sum 0 and tolerance $[0, 3]$, there exists a walk with the same end-vertices having sum 0 and tolerance $[0, 2]$. Assume therefore that $W = (v_0, a_1, v_1, \dots, v_{n-1}, a_n, v_n)$ is a walk in D from v_0 to v_n with sum 0 and tolerance $[0, 3]$. We show that there is also a walk from v_0 to v_n with sum 0 and tolerance $[0, 2]$. Let J be the set of those indices $j \in \{1, 2, \dots, n-1\}$ for which the sum of the walk $(v_0, a_1, v_1, \dots, v_{j-1}, a_j, v_j)$ is 3. Clearly, for every $j \in J$, the arc a_{j-1} is negatively oriented in W , whereas the arc a_j is positively oriented in W . For every $j \in J$ let u_j and w_j denote those vertices of D for which the oriented pairs $b_j^1 = (u_j, v_{j-1})$,

$b_j^2 = (v_j, u_j)$, $c_j^1 = (v_j, w_j)$ and $c_j^2 = (w_j, v_{j+1})$ are arcs of D . For every $j \in J$ substitute the sequence $\dots, a_j, v_j, a_{j+1}, \dots$ in W with the sequence $\dots, b_j^1, u_j, b_j^2, v_j, c_j^1, w_j, c_j^2, \dots$ to obtain a walk in D from v_0 to v_n with sum 0 and tolerance $[0, 2]$.

Suppose now that D is alter-incomplete. We need to show that the alter-perimeter of D equals 3. By Proposition 3.2 some quotient digraph of the digraph D is isomorphic to the oriented cycle \vec{C}_n , where n is the alter-perimeter of D . But since D contains oriented 3-cycles, so does every quotient of D , forcing $n = 3$.

To complete the proof it remains to show that in the case where $\exp(D) = 1$, the digraph D cannot be alter-complete. To this end we use [19, Proposition 2.4 (ii)]. Namely, if D is alter-complete, then (in the terminology of [19]) each H -alternating cycle of $L(X)$ contains all vertices of X . But then by [19, Proposition 2.4 (ii)], there are a positive integer r and some odd $s \in \mathbb{Z}_{2r}^*$ such that X is isomorphic to the circulant $\text{Cir}(2r; \{1, -1, s, -s\})$, contradicting the fact that X contains cycles of length 3. ■

Let X be a cubic s -arc-transitive graph with a sequence of groups $H_1 \leq \dots \leq H_k$, where H_1 is a minimal arc-transitive subgroup of $H_k = \text{Aut}X$ and each H_i is maximal in H_{i+1} . Suppose that for each i , the group H_i is s_i -regular. Then the sequence (s_1, \dots, s_k) is called a *type* of X . It may be deduced by the well-known result of Djoković and Miller [8] that the possible types are as follows: (1), (1, 2), (1, 2, 3), (1, 3), (1, 4), (1, 4, 5), (1, 5), (2), (2, 3), (3), (4), (4, 5) and (5).

In Table 1 we have gathered a comprehensive information on the generalized Folkman graphs arising from line graphs of certain cubic arc-transitive graphs (to be more precise, from $(1^H, 2^G)$ -regular graphs) of order at most 98. Information available from the Foster census [4] and the work of Conder and Dobscanyi [6] was processed with MAGMA [1]. For each cubic arc-transitive graph of order at most 98, Table 1 gives its *Foster code* Fc , meaning that the graph appears in the Foster census under the code c , its *type* and an information about its covers. For the corresponding line graph the alter-sequence is computed in the case of type $(1, \dots)$. Finally, for types (1, 2) and (1, 2, 3), the order, semisymmetry and isomorphism of the connected components of the corresponding generalized Folkman graphs is given.

The smallest connected $(1^H, 2^G)$ -regular cubic graph is K_4 . Its line graph $L(K_4)$ is isomorphic to the lexicographic product $K_3[\vec{K}_2]$. Since the alter-sequence is (2), we have that the generalized Folkman graph $\mathcal{F}(L(K_4); H)$ is disconnected with connected components on 8 vertices. Since the smallest semisymmetric graph has 20 vertices, the latter are not semisymmetric. A similar argument may be applied to the graph $K_{3,3}$ in row 2.

An interesting example arises from the cube Q_3 , the canonical double cover of K_4 . Its automorphism group G acts 2-regularly and contains two non-conjugate 1-regular subgroups H_1 and H_2 . The corresponding digraphs $D_{H_1}(L(Q_3))$ and $D_{H_2}(L(Q_3))$ are both of alter-exponent 2. the first one is alter-incomplete with alter-sequence (2, 4), whereas the second one is alter-complete with alter-sequence (3, 12). As in the previous two examples $\mathcal{F}(L(Q_3); H_1)$ is disconnected and not semisymmetric. However $\mathcal{F}(L(Q_3); H_2)$, a graph of order 48, is semisymmetric; in fact, it is isomorphic to the graph described in [22, Example 2.6].

TABLE 1: Cubic arc-transitive graphs, their line graphs and generalized Folkman graphs.

The names of the columns in the table have the following meanings. CATG:= cubic arc-transitive graph, name:=Foster code, type:= transitivity type, cover:= a cover of which graph; LG:= line graph of CATG, order:=its order, AS:= altersequence, AC:= is it alter-complete or not; GFG:= connected component of the corresponding generalized Folkman graph; order:=its order, SS:=is it semisymmetric, iso:= isomorphism between various GFG.

row	CATG			LG			GFG		
	name	type	cover	order	AS	AC	order	SS	iso
1	F4, K_4	(1, 2)		6	[2]	no	8	no	
2	F6, $K_{3,3}$	(1, 2, 3)		9	[3]	no	12	no	
3	F8, Q_3	(1, 2)	F4	12	[2, 4]	no	16	no	
4	F8, Q_3	(1, 2)	F4	12	[3, 12]	yes	48	yes	row 12
5	F10, Petersen	(2, 3)		15					
6	F14, Heawood	(1, 4)		21	[7]	no			
7	F16, GP(8,3)	(1, 2)	F8	24	[4, 8]	no	32	yes	
8	F16, GP(8,3)	(1, 2)	F8	24	[6, 24]	yes	96	yes	row 22
9	F18, Pappus	(1, 2, 3)	F6	27	[3, 9]	no	36	yes	
10	F20A, Dodec.	(1, 2)	F10	30	[5, 30]	yes	120	yes	row 31
11	F20B, Desarg.	(2, 3)	F10	30					
12	F24, GP(12,5)	(1, 2)	F6, F8	36	[3, 12]	no	48	yes	row 4
13	F24, GP(12,5)	(1, 2)	F6, F8	36	[6, 12]	no	48	yes	
14	F26	(1)		39	[13]	no			
15	F28, Coxeter	(2, 3)		42					
16	F30, Tutte	(4, 5)		45					
17	F32, Dyck	(1, 2)	F8	48	[4, 16]	no	64	yes	
18	F32, Dyck	(1, 2)	F8	48	[3, 48]	yes	192	yes	row 43
19	F38	(1)		57	[19]	no			
20	F40	(1, 2, 3)	F20A, F20B	60	[5, 60]	yes	240	yes	
21	F42	(1)	F6, F14	63	[21]	no			

row	CATG			LG			GFG		
	name	type	cover	order	AS	AC	order	SS	iso
22	F48	(1, 2)	F16, F24	72	[6, 24]	yes	96	yes	row 8
23	F48	(1, 2)	F16, F24	72	[12, 24]	yes	96	yes	
24	F50	(1, 2)		75	[5, 25]	yes	100	yes	
25	F50	(1, 2)		75	[3, 75]	yes	300	yes	
26	F54	(1, 2)	F18	81	[3, 27]	no	108	yes	
27	F54	(1, 2)	F18	81	[9, 27]	no	108	yes	
28	F56A	(1)	F8, F14	84	[14, 28]	no			
29	F56B, Klein	(1, 2)		84	[4, 84]	yes	336	yes	
30	F56C	(2, 3)	F28	84					
31	F60	(1, 2)	F20A	90	[5, 30]	no	120	yes	row 10
32	F62	(1)		93	[31]	no	124	no	
33	F64	(1, 2)	F16, F32	96	[4, 32]	no	128	yes	
34	F64	(1, 2)	F16, F32	96	[6, 96]	yes	384	yes	
35	F72	(1, 2)	F18, F24	108	[3, 36]	no	144	yes	
36	F72	(1, 2)	F18, F24	108	[6, 36]	no	144	yes	
37	F74	(1)		111	[37]	no			
38	F78	(1)	F6, F26	117	[39]	no			
39	F80	(1, 2, 3)	F40	120	[10, 120]	yes	480	yes	
40	F84	(2)		126					
41	F86	(1)		129	[43]	no			
42	F90	(4, 5)	F30	135					
43	F96A	(1, 2)	F24, F32	144	[3, 48]	no	192	yes	row 18
44	F96A	(1, 2)	F24, F32	144	[12, 48]	no	192	yes	
45	F96B	(1, 2, 3)	F24	144	[6, 48]	no	192	yes	
46	F98A	(1)	F14	147	[49]	no			
47	F98B	(1, 2)	F14	147	[7, 49]	yes	196	yes	
48	F98B	(1, 2)	F14	147	[3, 147]	yes	588	yes	

It is not surprising, that a similar bifurcation occurs for covers of Q_3 (see rows 7 and 8, rows 12 and 13, rows 17 and 18, rows 22 and 23, rows 28 and 29, rows 33 and 34, rows 35 and 36, and rows 43 and 44) as well as for graphs in rows 24 and 25, rows 26 and 27, and rows 47 and 48.

Since each of the cubic arc-transitive graphs of types (1, 2) or (1, 2, 3) in rows 5 to 48 has girth greater than 4, the corresponding (connected components of) the generalized Folkman graphs are semisymmetric by Theorem 2.1, (i). As it may be seen from the last column of Table 1, some of these graphs are isomorphic. Note that the corresponding alter-sequences coincide in these cases. We would like to remark that the connected component of the generalized Folkman graph in row 7 is the smallest semisymmetric graph (its order is 32) which has not been mentioned in the literature before.

Finally, the cubic arc-transitive graphs in rows 10 and 29, that is the dodecahedron and the Klein graph, are the smallest members of an infinite family of cubic arc-transitive graphs of order $(p-1)p(p+1)/6$, $p \geq 5$ a prime, and type (1, 2), constructed in [8, Section 13 (pp. 223)]. (The respective values of p are 5 and 7.) A description of this family of graphs and the computations regarding the corresponding line graphs and generalized Folkman graphs is given below. It transpires that the corresponding (oriented) line graphs (of order $(p-1)p(p+1)/4$) are alter-complete and have no 4-cycles, thus giving rise to semisymmetric generalized Folkman graphs of order $(p-1)p(p+1)$ (and valency 4) in view of Theorem 2.1, (i).

For a prime $p \geq 5$ let

$$c = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

be elements of the projective special linear group $PSL_2(p)$. Clearly, $c^3 = y^2 = 1$. Then c and y generate the group $H = PSL_2(p)$. Denote by C and K the subgroups of H generated with elements c and y , respectively, and by $V = H/C$ the set of right cosets of C in H . We define the graph to have vertex-set V and edge-set $E = \{\{Cu, Cydu\} \mid d \in C, u \in H\}$ (that is the orbital graph relative to suborbit $\{Cy, Cyc, Cyc^2\}$). By [8, Proposition 27] the automorphism group $G = \text{Aut}X$ of X is either $PGL_2(p)$ or $PSL(2, p) \times C_2$ and X is $(1^H, 2^G)$ -regular. It can also be easily checked that X contains no cycles of length 4. The corresponding digraph $D = D_H(L(X))$ is isomorphic to the graph with vertex set H/K , the set of right cosets of K in H , and arc set $\{(Ku, Kcku) \mid k \in K, u \in H\}$. Since H is simple, the digraph D is alter-complete, and, by Proposition 4.1, of alter-exponent 2. Combining Theorem 2.1 and Proposition 3.12 we have that the generalized Folkman graph $\mathcal{F}(L(X); H)$ is a connected semisymmetric graph of valency 4 and order $(p-1)p(p+1)$.

To compute the alter-sequence of D we have to compute the cardinality of the set $B_1(K; D)$, which clearly consists of those cosets of K , which are contained in the group generated by the set $Kc^{-1}KcK$. It can be easily checked that the group $\langle Kc^{-1}KcK \rangle$ may also be generated by involutions y and $c^{-1}yc$ and is therefore isomorphic to the dihedral group D_{2r} , where r is the order of the product

$$c^{-1}ycy = \begin{pmatrix} -1 & 1 \\ 1 & -2 \end{pmatrix}$$

in the group $PSL_2(p)$. Observe that for any positive integer k the k^{th} power of the above matrix equals

$$\begin{pmatrix} (-1)^k F_{2k-1} & (-1)^{k+1} F_{2k} \\ (-1)^{k+1} F_{2k} & (-1)^k F_{2k+1} \end{pmatrix},$$

where (F_n) is the Fibonacci sequence defined by $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_n + F_{n+1}$. The order r of the product $c^{-1}ycy$ in the group $PSL_2(p)$ is therefore the least positive integer k for which the prime p divides the $(2k)^{\text{th}}$ term F_{2k} of the Fibonacci sequence. We have thus showed that the alter-sequence of the digraph D equals $(r, \frac{(p-1)p(p+1)}{4})$, where r is as above.

5 Alter-exponent of tetravalent $\frac{1}{2}$ -arc-transitive graphs

The term *tightly attached graphs* was used in [19, 23] for tetravalent graphs admitting a $\frac{1}{2}$ -arc-transitive group action with respect to which the corresponding alter-exponent equals 1. A complete classification of such graphs with alter-sequence $[r]$ was given in [19] and in [23] for r odd and r even, respectively. (The term *radius* was used for the parameter r .) Furthermore, $\frac{1}{2}$ -arc-transitive graphs of alter-exponent 1 and odd radius were classified in [19].

A natural question arises with regards to obtaining a similar classification for graphs of alter-exponent 2 or higher alter-exponent. The line graph of the graph F56A in row 28 in Table 1 is an example of such a graph with alter-exponent 2. In fact, F56A is the smallest member of an infinite family of 1-regular \mathbb{Z}_{k^2+k+1} -covers of Q_3 (given in [12]), for which the corresponding line graphs are $\frac{1}{2}$ -arc-transitive of alter-exponent 2. It may be easily seen that these graphs are alter-incomplete with alter-sequence $[2(k^2+k+1), 4(k^2+k+1)]$. A further generalization of this construction was pointed to us by Malnič and is based on the existence of a 1-regular $\mathbb{Z}_{2(k^2+k+1)}$ -cover of Q_3 for each odd $k \geq 3$ [18]. Moreover, an infinite family of cubic 1-regular graphs arising from alternating and symmetric groups of degree congruent 1 modulo 6 was constructed in [7]. The corresponding line graphs are $\frac{1}{2}$ -arc-transitive of alter-exponent 2 and, since alternating groups are simple, clearly alter-complete. Finally, the line graph of order 648 of the first known 1-regular graph (of order 432 and constructed by Frucht in [14]) is alter-incomplete with alter-sequence $(6, 216)$.

In view of these examples we would like to pose the problem of classifying tetravalent $\frac{1}{2}$ -arc-transitive graphs with alter-exponent 2. An in-depth analysis of the alter-incomplete case seems like a reasonable first step towards obtaining this goal.

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