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EDGE-CRITICAL ISOMETRIC  
SUBGRAPHS OF HYPERCUBES

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# Edge-critical isometric subgraphs of hypercubes

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## Abstract

Isometric subgraphs of hypercubes are known as partial cubes. Edge-critical partial cubes are introduced as the partial cubes  $G$  for which  $G - e$  is not a partial cube for any edge  $e$  of  $G$ . An expansion theorem is proved by means of which one can generate many edge-critical partial cubes. Edge-critical partial cubes are characterized among the Cartesian product graphs. We also show that the 3-cube and the subdivision graph of  $K_4$  are the only edge-critical partial cubes on at most 10 vertices.

## 1 Introduction

Graphs that admit isometric embeddings into hypercubes are known as partial cubes and have been intensively studied by now. They were introduced by Graham and Pollak [8] and soon after characterized by Djoković [5]. For additional characterizations of partial cubes see [2, 3, 19, 20], for different applications of these graphs consult [4, 6, 8, 12, 16], and for the algorithmic point of view we refer to [1, 10]. Partial cubes are presented in detail in the book [11].

Posing some additional condition(s) on partial cubes, one may ask several interesting questions. For instance, which are planar partial cubes and which are regular partial cubes? These two questions are open at the present, in particular the second one seems to be quite difficult. On the other hand, Weichsel [18] succeeded to determine all distance regular partial cubes. In [13] partial cubes are characterized among the subdivision graphs and in [9] it is proved that partial cubes different from cycles are

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3-connected provided that  $|W_{ab}| = |W_{ba}|$  holds for all edges  $ab$ . (For the definition of  $W_{ab}$  see below.)

In this note we introduce the following concept. A partial cube  $G$  is called *edge-critical* if for any edge  $e$  of  $G$ ,  $G - e$  is not a partial cube. In the rest of this section we give necessary definitions and preliminary observations. We follow by an expansion theorem which, from an edge-critical partial cube, produces another such (bigger) graph by means of a peripheral expansion obeying an additional condition. We also characterize edge-critical partial cubes among the Cartesian product graphs. For instance, the Cartesian product of an edge-critical partial cube with an arbitrary partial cube is edge-critical. We conclude by showing that the 3-cube  $Q_3$  and the subdivision graph of  $K_4$ ,  $S(K_4)$ , are the only edge-critical partial cubes on at most 10 vertices.

For a graph  $G$ , the *distance*  $d_G(u, v)$  (or briefly  $d(u, v)$ ) between vertices  $u$  and  $v$  is defined as the number of edges on a shortest  $u, v$ -path. A subgraph  $H$  of  $G$  is called *isometric* if  $d_G(u, v) = d_H(u, v)$  for all  $u, v \in V(H)$ . Isometric subgraphs of hypercubes are called *partial cubes*. A set  $X$  in  $V(G)$  is called *convex* if for all  $u, v \in X$  the vertices of any shortest  $u, v$ -path belong to  $X$ . A subgraph  $H$  in  $G$  is *convex* if its vertex set is convex.

Let  $G = (V, E)$  be a connected, bipartite graph and let  $ab$  be an edge of  $G$ . Set

$$W_{ab} = \{x \in V(G) \mid d(x, a) < d(x, b)\}.$$

Djoković [5] proved that a graph is a partial cube if and only if it is bipartite and the sets  $W_{ab}$  are convex. Two edges  $xy$  and  $uv$  of  $G$  are in the Djoković-Winkler [5, 20] relation  $\Theta$  if

$$d_G(x, u) + d_G(y, v) \neq d_G(x, v) + d_G(y, u).$$

If graph  $G$  is bipartite, then the edges  $xy$  and  $uv$  are in relation  $\Theta$  precisely when  $d(x, u) = d(y, v)$  and  $d(x, v) = d(y, u)$ . Winkler [20] proved that a bipartite graph is a partial cube if and only if  $\Theta$  is transitive.

Let  $G'$  be a connected graph. A *proper cover*  $G'_1, G'_2$  consists of two isometric subgraphs  $G'_1, G'_2$  of  $G'$  such that  $G' = G'_1 \cup G'_2$  and  $G'_0 = G'_1 \cap G'_2$  is a nonempty subgraph, called the *intersection* of the cover. Additionally there are no edges between  $G'_1 \setminus G'_2$  and  $G'_2 \setminus G'_1$ . The *expansion* of  $G'$  with respect to  $G'_1, G'_2$  is the graph  $G$  constructed as follows. Let  $G_i$  be an isomorphic copy of  $G'_i$ , for  $i = 1, 2$ , and, for any vertex  $u'$  in  $G'_0$ , let  $u_i$  be the corresponding vertex in  $G_i$ , for  $i = 1, 2$ . Then  $G$  is obtained from the disjoint union  $G_1 \cup G_2$ , where for each  $u'$  in  $G'_0$  the vertices  $u_1$  and  $u_2$  are joined by an edge. We denote the copy of  $G'_0$  in  $G_i$  by  $G_{0i}$ , for  $i = 1, 2$ . Expansion is called *peripheral* if at least one of the graphs  $G'_1$  or  $G'_2$  is equal to  $G$ . Note that then the other graph equals the intersection that is thus necessarily isometric in  $G$ .

Chepoi [3] proved that a graph is a partial cube if and only if it can be obtained from  $K_1$  by a sequence of expansions. This result was later independently obtained by Fukuda and Handa [6] and is analogous to the convex expansion theorem for median graphs [17].

The *Cartesian product*  $G \square H$  of graphs  $G = (V(G), E(G))$  and  $H = (V(H), E(H))$  is the graph with vertex set  $V(G) \times V(H)$  and vertex  $(a, x)$  is adjacent to vertex  $(b, y)$  in  $E(G \square H)$  whenever  $ab \in E(G)$  and  $x = y$ , or, if  $a = b$  and  $xy \in E(H)$ . For a fixed vertex  $a$  of  $G$ , the vertices  $\{(a, x) \mid x \in V(H)\}$  induce a subgraph isomorphic to  $H$ . We call it an *H-layer*. Analogously we define *G-layers*. The *subdivision graph*  $S(G)$  of a graph  $G$  is obtained from  $G$  by subdividing every edge of  $G$ .

Let  $G$  and  $H$  be connected graphs. Then  $G \square H$  is a partial cube if and only if  $G$  and  $H$  are partial cubes. This observation follows from the fact that the layers of the Cartesian product are convex, which is in turn implied by the fact that the distance function of the product is the sum of distance functions of the factors, cf. [11].

The following observation (probably part of the folklore) will be used in the sequel.

**Lemma 1.1** *If an edge  $e$  of a graph  $G$  lies in a cycle, then  $e$  also lies in an isometric cycle of  $G$ .*

**Proof.** Let  $e = uv$  and let  $P$  be a shortest path connecting the endpoints of  $e$ . Such a path exists since  $e$  lies in a cycle. But then  $C = u \rightarrow \dots P \dots \rightarrow v \rightarrow u$  is an isometric cycle containing  $e$ , for otherwise  $P$  would not be shortest.  $\square$

## 2 Generating edge-critical partial cubes

Recall that  $G$  is an edge-critical partial cube if for any edge  $e$  of  $G$ ,  $G - e$  is not a partial cube. Clearly, an edge-critical partial cube is 2-edge-connected.

One can verify that the generalized Petersen graph  $P(10, 3)$  and a graph of Gedeonova from [7], see Fig. 2.4 in [11]), are edge-critical partial cubes. Two more examples of such graphs are shown in Fig. 1; the graph  $G$  was observed to be edge-critical in [15].

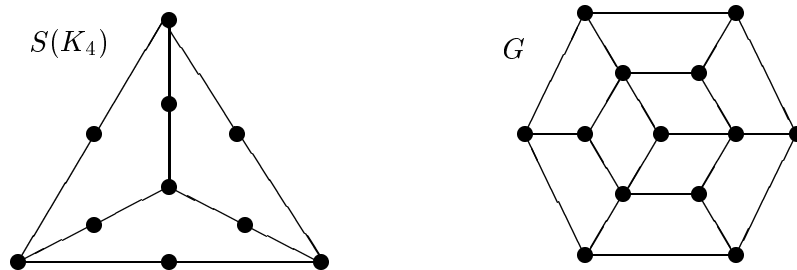


Figure 1: Two edge-critical partial cubes

In order to obtain more such graphs we are going to present an expansion procedure that preserves the property. First a lemma.

**Lemma 2.1** *Let  $G$  be a bipartite graph,  $H = C_{2n} \square K_2$  ( $n \geq 2$ ) an isometric subgraph of  $G$ , and  $e$  an edge of  $H$ . Then  $G - e$  is not a partial cube.*

**Proof.** Let  $v_1, v_2, \dots, v_{2n}$  be the consecutive vertices of the cycle  $C_{2n}$  and let  $V(K_2) = \{1, 2\}$ . Thus vertices of  $H$  are of the form  $(v_i, j)$ ,  $1 \leq i \leq 2n$ ,  $1 \leq j \leq 2$ .

Let  $e$  be an edge of a  $K_2$ -layer of  $H$ . Then we may without loss of generality assume that  $e = (v_2, 1)(v_2, 2)$ . As  $H$  is isometric, we infer that  $(v_1, 1)(v_2, 1)$  is in relation  $\Theta$  in  $G - e$  with  $(v_{n+1}, 1)(v_{n+2}, 1)$ . Clearly,  $(v_{n+1}, 1)(v_{n+2}, 1)$  is in relation  $\Theta$  with  $(v_{n+1}, 2)(v_{n+2}, 2)$  and using isometry again we infer that  $(v_{n+1}, 2)(v_{n+2}, 2)$  is in relation  $\Theta$  with  $(v_1, 2)(v_2, 2)$ . But as  $G$  is bipartite,  $(v_1, 2)(v_2, 2)$  is not in relation  $\Theta$  with  $(v_1, 1)(v_2, 1)$  in  $G - e$ . Hence  $\Theta$  is not transitive in  $G - e$  and so  $G - e$  cannot be a partial cube.

Let  $e$  be an edge of a  $C_{2n}$ -layer. We may assume that  $e = (v_1, 1)(v_2, 1)$ . Then in  $G - e$  we have

$$\begin{aligned} &(v_2, 1)(v_2, 2) \Theta (v_3, 1)(v_3, 2), \\ &(v_3, 1)(v_3, 2) \Theta (v_4, 1)(v_4, 2), \\ &\quad \vdots \\ &(v_{2n}, 1)(v_{2n}, 2) \Theta (v_1, 1)(v_1, 2), \end{aligned}$$

however,  $(v_2, 1)(v_2, 2)$  is not in relation  $\Theta$  with  $(v_1, 1)(v_1, 2)$  in  $G - e$ . Hence  $\Theta$  is again not transitive in  $G - e$ .  $\square$

**Theorem 2.2** *Let  $G'$  be an edge-critical partial cube and let  $G$  be a peripheral expansion of  $G'$  with respect to  $G'_1, G'_2$ , where  $G'_1 = G'$ . Then  $G$  is an edge-critical partial cube if and only if every edge of  $G'_2$  lies in an isometric cycle of  $G'_2$ .*

**Proof.** Suppose that every edge of  $G'_2$  lies in an isometric cycle of  $G'_2$  and let  $e$  be an edge of  $G$ . We wish to show that  $G - e$  is not a partial cube.

Let  $e$  be an edge with at least one endpoint from  $G_1 \setminus G_{01}$ . We claim that  $G_1 - e$  is an isometric subgraph of  $G - e$ . Let  $u$  and  $v$  be any vertices of  $G_1$  and let  $P$  be a shortest  $u, v$ -path in  $G - e$ . Suppose that  $P \cap G_{02} \neq \emptyset$ . Let  $x$  be the first vertex of  $P$  in  $G_{02}$  and  $y$  the last such vertex. Let  $R$  be the  $x, y$ -subpath of  $P$ . We may without loss of generality assume that  $P \cap G_{02} = R$ . Let  $x', y'$  be the neighbors of  $x, y$  in  $G_{01}$  and let  $R'$  be the copy of  $R$  in  $G_{01}$ . Then we can replace the  $x' \rightarrow x \rightarrow \dots \rightarrow R \rightarrow \dots \rightarrow y \rightarrow y'$ -subpath of  $P$  with  $x' \rightarrow \dots \rightarrow R' \rightarrow \dots \rightarrow y'$ . But this new subpath is shorter and does not contain  $e$ , which proves the claim. (In fact, we have proved even more, the convexity of  $G_1 - e$  in  $G - e$ .) Now suppose that  $G - e$  is a partial cube. Then  $G_1 - e$  would also be such. Since  $G_1 - e$  is isomorphic to  $G' - e$ , this contradicts the assumption that  $G'$  is edge-critical.

In the second case,  $e$  is an edge of the subgraph of  $G$  induced by  $G_{01} \cup G_{02}$ . In the first subcase, let  $e \in G_{01}$ . (The case when  $e$  belongs to  $G_{02}$  is treated analogously.) Let  $C$  be an isometric cycle of  $G_{01}$  containing  $e$  and consider the subgraph  $H = C \square K_2$

of  $G_{01} \cup G_{02}$ . Note that  $H$  is an isometric subgraph of  $G$ . Then, using Lemma 2.1 we infer that  $G - e$  is not a partial cube. In the second subcase,  $e$  is an edge between  $G_{01}$  and  $G_{02}$ . Let  $g$  be an edge of  $G_{01}$  adjacent to  $e$ . The edge  $g$  exists since  $G$  is 2-edge-connected and as  $G_{01}$  is isometric in  $G_1$ . Let  $C$  be an isometric cycle of  $G_{01}$  containing  $g$ . Then we have again an isometric subgraph  $H = C \square K_2$  of  $G_{01} \cup G_{02}$  and  $G - e$  is not a partial cube.

For the converse suppose that  $G$  is an edge-critical partial cube and assume that there is an edge  $u'v'$  in  $G'_2$  that does not lie in an isometric cycle of  $G'_2$ . (Note that it may lie in an isometric cycle of  $G'$ .) Let  $u$  and  $v$  be the copies of  $u'$  and  $v'$  in  $G_{01}$  and let  $a$  and  $b$  be the adjacent vertices of  $u$  and  $v$  in  $G_{02}$ , respectively. Let  $e = ab$ . The proof will be complete by showing that  $G - e$  is a partial cube.

Let  $E_1^G$  and  $E_2^G$  be the  $\Theta$ -classes in  $G$  with representatives  $uv$  and  $ua$ , respectively. Let  $f = x_1x_2$  and  $g = y_1y_2$  be edges of an arbitrary  $\Theta$ -class  $E^G$  of  $G$ .

Suppose that  $E^G$  is different from  $E_1^G$  and  $E_2^G$ . We claim that  $f$  is in relation  $\Theta$  with  $g$  also in  $G - e$ . Note first that if  $f$  and  $g$  belong to  $G_1$ , the conclusion follows from the fact that  $G_1$  is convex. Suppose next that  $f \in G_1$  and  $g \in G_{02}$ . Let  $P$  be an  $x_1, y_1$ -geodesic in  $G$  and assume that  $e \in P$ . Let  $w_1$  be the last vertex of  $P$  in  $G_1$  and  $w_2$  its neighbor on  $P$  in  $G_{02}$ . Path  $P$  is of the form  $x_1 \rightarrow \cdots \rightarrow w_1 \rightarrow w_2 \rightarrow \cdots R \cdots \rightarrow y_1$ , where  $R$  contains  $e$ . Let  $w_3$  be the neighbor of  $y_1$  in  $G_{01}$ . Then the subpath  $w_1 \rightarrow w_2 \rightarrow \cdots R \cdots \rightarrow y_1$  can be replaced by a path  $w_1 \rightarrow \cdots R' \cdots \rightarrow w_3 \rightarrow y_1$  of the same length, where  $R'$  is a copy of  $R$  lying in  $G_{01}$ . Hence,  $d_G(x_1, y_1) = d_{G-e}(x_1, y_1)$ . Analogously we get  $d_G(x_2, y_2) = d_{G-e}(x_2, y_2)$ . Hence also in this case  $f$  is in relation  $\Theta$  with  $g$  in  $G - e$ . The last case is when both  $f$  and  $g$  lie in  $G_{02}$ . If  $f$  is not in relation  $\Theta$  with  $g$  in  $G - e$  then we may assume that  $d_G(x_1, y_1) < d_{G-e}(x_1, y_1)$  and  $d_G(x_2, y_2) = d_{G-e}(x_2, y_2)$ . Thus, there is an  $x_1, y_1$ -geodesic in  $G_{02}$  containing  $e$  and an  $x_2, y_2$ -geodesic in  $G_{02}$  not containing  $e$ . Hence there exists a cycle in  $G_{02}$  containing  $e$ . As  $G_{02}$  is convex, by Lemma 1.1 we get an isometric cycle  $C$  in  $G_{02}$  containing  $e$ . Considering the copy  $C'$  of  $C$  in  $G_{01}$  that contains  $uv$  we get a contradiction.

Assume next that  $f$  and  $g$  are edges of  $E_1^G$  different from  $e$ . Then we observe that  $f$  and  $g$  are both in  $G_1$  for otherwise we would find (similarly as above) an isometric cycle in  $G_{02}$  containing  $e$ . Hence by the convexity of  $G_1$ ,  $f$  is in relation  $\Theta$  with  $g$  in  $G - e$ .

Consider finally the class  $E_2^G$ . Let  $x_1, y_1 \in G_{01}$  and  $x_2, y_2 \in G_{02}$ . Set  $E_{21}^{G-e} = E_2^G \cap W_{ab}$  and  $E_{22}^{G-e} = E_2^G \cap W_{ba}$  and assume  $f, g \in E_{21}^{G-e}$ . Then an  $x_2, y_2$ -geodesic does not contain  $e$  (since  $x_2, y_2 \in W_{ab}$ ). Clearly, as  $G_1$  is convex, an  $x_1, y_1$ -geodesic also does not contain  $e$ . Therefore  $f$  is in relation  $\Theta$  with  $g$  also in  $G - e$ . Analogously we see that if the edges  $f$  and  $g$  lie in  $E_{22}^{G-e}$ , then they are in relation  $\Theta$  in  $G - e$ . Let  $f \in E_{21}^{G-e}$  and  $g \in E_{22}^{G-e}$ . We claim that  $f$  is not in relation  $\Theta$  with  $g$ . Let  $k = d_G(x_1, y_1)$ . As  $G_{01}$  is isometric in  $G_1$ ,  $d_{G-e}(x_1, y_1) = k$ . Let  $P$  be a shortest  $x_2, y_2$ -path. Note that  $P$  lies completely in  $G_{02}$ . If  $e$  is not on  $P$ , then  $ab$  lies on a cycle and thus also on an isometric cycle. Therefore,  $uv$  belongs to an isometric cycle of  $G_{01}$ , which is not possible. Hence,  $ab$  is on every  $x_2, y_2$ -geodesic and consequently  $d_{G-e}(x_2, y_2) = k + 2$ .

This proves the claim.

In conclusion,  $G - e$  is a partial cube with the following  $\Theta$ -classes:  $E_1^G - e$  (obtained from  $E_1^G$ ),  $E_{21}^{G-e}$  and  $E_{22}^{G-e}$  (obtained from  $E_2^G$ ), while all the other classes coincide with the remaining classes of  $G$ .  $\square$

Theorem 2.2 can be applied as follows. Take any edge-critical partial cube  $G'$  and an isometric cycle  $C$  of  $G'$ . Then expand  $G'$  with respect to  $G', C$  in order to obtain another edge-critical partial cube  $G$ . Note also that as  $Q_n$  is a peripheral expansion of  $Q_{n-1}$  with respect to  $Q_{n-1}$ ,  $Q_{n-1}$ , Theorem 2.2 also implies that all hypercubes  $Q_n$ ,  $n \geq 3$ , are edge-critical. (That the 3-cube  $Q_3$  is edge-critical is clear.)

Another example of edge-critical partial cubes are the subdivision graphs of complete graphs  $S(K_n)$ ,  $n \geq 4$ . We refer to [13] that they are indeed partial cubes. To see that they are edge-critical consider an arbitrary edge  $e$ . Then  $e$  belongs to an isometric subgraph  $H$  isomorphic to  $S(K_4)$ . Then in  $H - e$  we can find two isometric 6-cycles sharing two edges such that their union is isometric. Now consider these two cycles in  $G - e$  to find out that  $\Theta$  is not transitive. However  $S(K_4)$  can only be obtained by expansion from  $Q_3^-$  which is not an edge-critical partial cube. Thus not all edge-critical partial cubes can be generated using Theorem 2.2.

We next consider the question which Cartesian products of partial cubes are edge-critical.

**Proposition 2.3** *Let  $G$  and  $H$  be partial cubes. Then  $G \square H$  is an edge-critical partial cube if and only if for any pair of edges  $f \in E(G)$ ,  $g \in E(H)$ ,  $f$  or  $g$  lies in a cycle of  $G$  or in cycle of  $H$ .*

**Proof.** Assume first that  $G \square H$  is an edge-critical partial cube and there are edges  $f \in E(G)$  and  $g \in E(H)$  such that neither  $f$  is in a cycle of  $G$  nor  $g$  lies in a cycle of  $H$ . Let  $f = uv$  and  $g = xy$ . Clearly, the edge  $e = (u, x)(v, x)$  lies in exactly one (isometric) square, that is, in  $(u, x)(v, x)(v, y)(u, y)$ . We claim that  $(G \square H) - e$  is a partial cube. Let  $E_1^{G \square H}$  and  $E_2^{G \square H}$  be the  $\Theta$ -classes of  $G \square H$  with representatives  $(u, x)(v, x)$  and  $(u, x)(u, y)$ , respectively. We now argue similarly as in the proof of Theorem 2.2 to infer that  $(G \square H) - e$  is a partial cube with the  $\Theta$ -classes:  $E_1^{G \square H - e}$  (obtained from  $E_1^{G \square H}$ ),  $E_{21}^{(G \square H) - e}$  and  $E_{22}^{(G \square H) - e}$  (obtained from  $E_2^{G \square H}$ ), and the remaining classes that coincide with the remaining  $\Theta$ -classes of  $G \square H$ .

For the converse assume that for any pair of edges  $f \in E(G)$ ,  $g \in E(H)$ , at least one of  $f$  or  $g$  lies in a cycle of  $G$  or in cycle of  $H$ . Let  $e$  be an arbitrary edge of  $G \square H$ . We need to show that  $(G \square H) - e$  is not a partial cube. By the definition of the Cartesian product,  $e$  lies in a square  $S$  of  $G \square H$ . Let  $p_G$  denotes the projection onto  $G$  and let  $p_G(S) = h$ . We may without loss of generality assume that  $h$  lies in a cycle  $C$ . Moreover, by Lemma 1.1 we may in addition assume that  $C$  is isometric. Suppose first that  $p_G(e) = h$ . Then  $e$  lies in subgraph  $C \square K_2$  of  $G \square H$  and by Lemma 2.1 we infer that  $(G \square H) - e$  is not a partial cube. In the other case,  $p_G(e)$  is a vertex. Then

$e$  is a neighbor of an edge  $f$  such that  $p_G(f)$  is an edge lying in an isometric cycle  $C$  of  $G$ . Hence also in this case  $e$  lies in subgraph isomorphic to  $C \square K_2$  and we get the same conclusion.  $\square$

Let  $G$  be 2-edge-connected partial cube and  $H$  an arbitrary partial cube. Then Proposition 2.3 implies that  $G \square H$  is an edge-critical partial cube. For example, the Cartesian products  $C_{2k} \square P_n$ ,  $k, n \geq 2$ , are edge-critical partial cubes.

### 3 Small edge-critical partial cubes

As we already mentioned, Handa [9] proved that partial cubes  $G$  with  $|W_{ab}| = |W_{ba}|$  for any edge  $ab$  are 3-connected if  $G$  has at least two edges and  $G$  is not a cycle. For edge-critical partial cubes we cannot prove neither 3-connectivity nor 3-edge-connectivity, as the example of  $S(K_n)$ ,  $n \geq 4$ , demonstrates. On the other hand, for  $\Theta$ -classes that also form edge cutsets, we can show:

**Lemma 3.1** *Let  $G$  be an edge-critical partial cube and  $F$  a  $\Theta$ -class of  $G$ . Then  $|F| \geq 3$ .*

**Proof.** By 2-edge-connectivity,  $|F| \geq 2$ . Suppose on the contrary that  $F = \{e, f\}$ . Let  $e = uv$  and consider the graph  $G - f$ . Recall that  $W_{uv}$  and  $W_{vu}$  induce convex subgraphs of  $G$  thus they induce partial cubes. Since  $e$  is a cut-edge of  $G - f$  between the blocks induced by  $W_{uv}$  and  $W_{vu}$ ,  $G - f$  is a partial cube and so  $G$  cannot be edge-critical.  $\square$

**Proposition 3.2** *Let  $G$  be an edge-critical partial cube on at most 10 vertices. Then  $G$  is either  $Q_3$  or  $S(K_4)$ .*

**Proof.** Let  $F$  be a  $\Theta$ -class of  $G$  of maximum cardinality. Let  $x_i y_i \in F$ ,  $i = 1, \dots, |F|$ . Let  $G_1$  and  $G_2$  be subgraphs of  $G$  induced by  $W_{x_1 y_1}$  and  $W_{y_1 x_1}$ , and let  $H_1$  and  $H_2$  be subgraphs of  $G$  induced by  $\{x_1, \dots, x_{|F|}\}$  and  $\{y_1, \dots, y_{|F|}\}$ .

**Case 1:**  $|F| = 5$ .

In this case  $V(G) = V(H_1) \cup V(H_2)$ . Hence  $G$  is obtained by an expansion from  $G'$  with respect to  $G'_1 = G'_2 = G'$  which implies that  $G = H_1 \square K_2$ . As there is no partial cube on 5 vertices with the property that every edge would lie in a cycle, it follows from Proposition 2.3 that we get no edge-critical partial cube in this case.

**Case 2.**  $|F| = 4$ .

Assume first that  $H_1$  contains at most two edges. Then  $H_1$  and  $H_2$  are not connected and thus there exist vertices  $x_5 \in G_1 \setminus H_1$  and  $y_5 \in G_2 \setminus H_2$ . It is now easy to see that in no way we can add edges between  $x_5$  and  $x_1, x_2, x_3, x_4$  and between  $y_5$  and  $y_1, y_2, y_3, y_4$  to obtain an edge-critical partial cube.

Let there be three edges in  $H_1$ . Suppose  $H_1 = K_{1,3}$  and let  $x_1$  be the vertex of  $H_1$  of degree 3. As  $K_{1,3} \square K_2$  is not edge-critical, there must be another vertex  $x_5 \in G_2 \setminus H_2$ .



Assume  $|G| = 9$ . By 2-edge-connectivity we may assume that  $x_5$  is adjacent to  $x_3$  and  $x_4$ . If  $x_5$  is adjacent also to  $x_2$ , then  $G$  is not a partial cube, otherwise  $G$  is a partial cube that is not edge-critical. Hence  $|G| = 10$  and let the tenth vertex  $x_6$  belong to  $G_1 \setminus H_1$ . Then in any case  $G - x_2y_2$  is not a partial cube. Thus,  $x_6 \in G_2 \setminus H_2$ . Again  $x_5$  must be adjacent to precisely two vertices, say  $x_2$  and  $x_3$ . If  $x_6$  is adjacent to  $y_2$  and  $y_3$ , then  $G$  is not a partial cube, otherwise  $x_6$  must be adjacent to, say,  $y_2$  and  $y_4$ , in which case  $G$  is not edge-critical.

Suppose  $H_1$  is the path on four vertices  $x_1x_2x_3x_4$ . If any of the vertices  $x_1, x_4, y_1$ , or  $y_4$  is of degree 2, then  $G - x_1y_1$  or  $G - x_4y_4$  is not a partial cube. But we can avoid this only by introducing two new vertices and creating odd cycles.

Finally, assume that there are 4 edges in  $H_1$ . Then  $H_1 = C_4$ . If  $|G| = 8$ , then  $G$  is  $Q_3$  which is edge-critical. We wish to show that there is no other edge-critical partial cube. Assume  $|G| = 9$  and let  $x_5 \in G_1 \setminus H_1$ . By 2-edge-connectivity we may without loss of generality assume that  $x_5$  is adjacent to  $x_1$  and  $x_3$ . As this gives an induced  $K_{2,3}$ ,  $|G| = 10$ . Then we have two adjacent vertices  $x_5, x_6 \in G_1 \setminus H_1$ . Let  $x_5x_1 \in E(G)$ . If  $x_6x_2 \in E(G)$  we would have a  $\Theta$ -class with at least 5 edges. Hence  $x_6x_2 \notin E(G)$  and  $x_6x_4 \in E(G)$  which yields the same conclusion.

**Case 3:**  $|F| = 3$ .

In this case Lemma 3.1 implies that all  $\Theta$ -classes contain 3 edges. Suppose first that there are two edges in  $H_1$ . We may without loss of generality assume that they are  $x_1x_2$  and  $x_2x_3$ . Then  $y_1y_2$  and  $y_2y_3$  are edges of  $H_2$ . None of vertices  $x_1, x_3, y_1$ , and  $y_3$  is of degree 2 in  $G$  for otherwise  $G$  would not be edge-critical. Therefore we have a vertex  $x_4$  in  $G_1 \setminus H_1$  and a vertex  $x_5$  in  $G_2 \setminus H_2$ . By 2-edge-connectivity  $x_4$  is then adjacent to  $x_1$  and  $x_3$ , while  $x_5$  is adjacent to  $y_1$  and  $y_3$ . This gives an induced  $Q_3$  minus an edge in  $G$  that is not a partial cube.

Suppose next that there is only one edge in  $H_1$ , say  $x_1x_2$ . Since  $G_1$  is isometric there should be a geodesic  $P_1$  in  $G_1$  from  $x_3$  to  $x_1$ . Likewise, we have a  $y_3, y_1$ -geodesic  $P_2$  in  $G_2$ . Moreover,  $|P_1| = |P_2|$ . As we consider graphs on at most 10 vertices,  $|P_1| < 4$ . Let  $|P_1| = 3$  and denote the inner vertices on  $P_1$  by  $x_4$  and  $x_5$ , where  $x_4$  is adjacent to  $x_1$ . Let the inner vertices of  $P_2$  be  $y_4$  and  $y_5$ , where  $y_4$  is adjacent to  $y_1$ . Observe that the cycle  $x_1x_4x_5x_3y_3y_5y_4y_1$  is isometric and consider the  $\Theta$ -class of  $x_1x_2$ . It has to have at least three edges, so we may without loss of generality assume that  $x_5$  is adjacent to  $x_2$ . Then  $R = y_1y_4y_5y_3x_3x_5x_2y_2$  is a walk connecting the endpoints of the edge  $y_1y_2$ . Hence  $R$  contains an edge  $g$  that is in relation  $\Theta$  to  $y_1y_2$ , cf. [11, Lemma 2.4]. But now  $x_1x_2, x_4x_5, y_1y_2$ , and  $g$  are different edges belonging to the same  $\Theta$ -class. We conclude that  $|P_1| = |P_2| = 2$ . Let  $x_4$  be the vertex of  $P_1$  adjacent to  $x_1$  and  $x_3$  and  $y_4$  the corresponding vertex of  $P_2$ . Observe that the cycle  $x_1x_4x_3y_3y_4y_1x_1$  is convex. To see this it suffices to observe that  $P_1$  is the unique  $x_1, x_3$ -geodesic, as well as is  $P_2$  the unique  $y_1, y_3$ -geodesic, which in turn holds because otherwise  $\Theta$  would not be transitive. As  $G$  is edge-critical, there is another vertex  $x_5 \in G_1$  adjacent to  $x_2$  and a vertex  $y_5 \in G_2$  adjacent to  $y_2$ . Clearly,  $x_5$  is adjacent neither to  $x_1$  nor to  $x_3$ , so it

must be adjacent to  $x_4$ . Similarly,  $y_5$  must be adjacent to  $y_4$ . But then the edges  $x_4x_5$ ,  $x_1x_2$ ,  $y_1y_2$ , and  $y_4y_5$  belong to the same  $\Theta$ -class.

Finally, we need to consider the case when there are no edges between vertices in  $H_1$ . Suppose that  $d(x_1, x_2) = 3$ . Let  $P_1$  be a  $x_1, x_2$ -geodesic,  $x_4$  and  $x_5$  its inner vertices and  $x_4$  adjacent to  $x_1$ . Similarly, there is a  $y_1, y_2$ -geodesic  $P_2$  in  $G_2$  with inner vertices  $y_4$  and  $y_5$ . To make  $G_1$  connected, we must either add the edge  $x_4x_3$  (and  $y_4y_3$ ) or the edge  $x_5x_3$  (and  $y_5y_3$ ). However, in both cases  $G$  is not a partial cube. Therefore,  $d(x_i, x_j) = d(y_i, y_j) = 2$  for any pair  $i, j$ ,  $i \neq j$ . The conditions  $d(x_i, x_j) = 2$  can be achieved by either adding one or three vertices in  $G_1 \setminus H_1$ , the same holds for the conditions  $d(y_i, y_j) = 2$ . Adding three vertices to  $G_1 \setminus H_1$  and three to  $G_2 \setminus H_2$  gives a graph on 12 vertices, while adding one vertex on each side we get a graph that is not a partial cube. Hence we must add, say, three vertices of degree two to  $G_1 \setminus H_1$  and a vertex of degree three to  $G_2 \setminus H_2$ . The obtained graph is  $S(K_4)$  and is edge-critical. To complete the case observe that adding edges to  $S(K_4)$  yields no further (edge-critical) partial cube.  $\square$

The graphs  $C_4 \square P_3$  and  $C_6 \square P_2$  are edge-critical partial cubes on 12 vertices, while the graph  $G$  from Fig. 1 is an example with 13 vertices. We do not know any such graph on 11 vertices.

## 4 Concluding remark

We have studied partial cubes  $G$  with the property that removing any edge of  $G$  destroys a possibility of its isometric embedding into a hypercubes. A more general question is the following. Let  $e$  be an edge of a partial cube  $G$ . Under which conditions is  $G - e$  a partial cube? This problems seems to be rather involved, but the following concept might be useful in its attack.

Let  $G$  be a connected graph with at least one cycle and let  $\mathcal{C}(G) = \{C^1, C^2, \dots, C^r\}$  be the set of isometric cycles of  $G$ . Let  $G(\mathcal{C})$  be the intersection graph of  $\mathcal{C}(G)$ . More precisely, the vertex set of  $G(\mathcal{C})$  is  $\mathcal{C}(G)$ , and two vertices are adjacent if the corresponding cycles intersect in at least one edge. Label the edges  $C^i C^j$  of  $G(\mathcal{C})$  with  $C^i \cap C^j$  where the cycles are considered as sets of edges and denote the obtained edge-labeled graph  $G_\ell(\mathcal{C})$ . For an edge  $e \in G_\ell(\mathcal{C})$ , let  $l(e)$  denote its label.

We thus pose the question if the structure of  $G_\ell(\mathcal{C})$  suffices to find out whether  $G - e$  a partial cube. For example, one can show:

**Proposition 4.1** *Let  $G$  be a partial cube and  $G(\mathcal{C})$  a tree. Then  $G - e$  is a partial cube for any edge  $e$  of  $G$ .*

**Proof.** Since  $G(\mathcal{C})$  a tree, two isometric cycles of  $G$  intersect in at most one edge, for otherwise  $G$  would not be a partial cube. Moreover, if  $C$  and  $C'$  are isometric cycles of  $G$  sharing an edge, then  $C_1 \cup C_2$  is isometric subgraph of  $G$ . Now it is straightforward to verify the assertion.  $\square$

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