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THE GENERALIZED BALABAN CONFIGURATIONS

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The generalized Balaban configurations*

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Abstract

Symmetry properties of the three 10-cages on 70 vertices are investigated. Being bipartite, these graphs are Levi graphs of triangle-free and quadrangle-free (35_3) configurations. For each of these graphs a Hamilton cycle is given via the associated LCF notation. Furthermore, the automorphism groups of respective orders 80, 120, and 24 are computed.

A special emphasis is given to the Balaban 10-cage, the first known example of a 10-cage [1], and the corresponding *Balaban configuration*. It is shown that the latter is linear, that is, it can be realized as a geometric configuration of points and lines in the Euclidean plane. Finally, based on the Balaban configuration, an infinite series of linear triangle-free and quadrangle-free $((7n)_3)$ configurations is produced for each odd integer $n \geq 5$.

1 Introduction

The motivation for the paper is a connection between cages (graphs) and configurations (incidence structures). In particular, the $2n$-cycles in graphs correspond to n-gons in configurations. We extend the present knowledge about this subject to 10-cages. Also, some additional properties of graphs and configurations are studied.

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*Dedicated to professor Alexandru T. Balaban on the occasion of his 70th birthday.

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1.1 Configurations

A *(symmetric) configuration* (v_r) is an incidence structure of v points and v lines such that there are r lines through each point, r points on each line, and two lines meet in at most one point. We can relate configurations to graphs in the following sense. Let $L = L(\mathcal{C})$ be the bipartite graph with v "black" vertices representing the points of the configuration $\mathcal{C}$, with v "white" vertices representing the lines of $\mathcal{C}$, and with an edge joining two vertices if and only if the corresponding point and line are incident in $\mathcal{C}$. We call L the *Levi graph* of the configuration $\mathcal{C}$. The following proposition characterizes (v_r) configurations in terms of their Levi graphs. Recall that the length of the shortest cycle in a graph G is called the *girth* of G.

Proposition 1. *An incidence structure is a (v_r) configuration if and only if its Levi graph is r-regular and has girth at least 6.*

Proof. See [11]. □

With each configuration $\mathcal{C}$ the *dual* configuration $\mathcal{C}^d$ may be associated by reversing the roles of points and lines in $\mathcal{C}$. Both $\mathcal{C}$ and $\mathcal{C}^d$ share the same Levi graph, only the black-white coloring of its vertices is reversed.

If $\mathcal{C}$ is isomorphic to its dual we say that $\mathcal{C}$ is *self-dual* and a corresponding isomorphism is called a *duality*. A duality of order 2 is called a *polarity*. An isomorphism of $\mathcal{C}$ to itself is called an *automorphism*. Automorphisms of $\mathcal{C}$ form a group denoted by $\mathrm{Aut}_0\,\mathcal{C}$. A configuration $\mathcal{C}$ is called *linear* if it can be realized in the Euclidean plane with points and lines, see [17]. We say that a configuration $\mathcal{C}$ is *triangle-free* if $L(\mathcal{C})$ has girth at least 8, similarly we say that $\mathcal{C}$ is *triangle- and quadrangle-free* if $L(\mathcal{C})$ has girth at least 10, etc. Also, a configuration $\mathcal{C}$ is called *d-gonal* if the girth of $L(\mathcal{C})$ equals $2d$. In this context we say that d is a *normal number* if for each integer n there exists $v \geq n$ such that there is a d-gonal linear (v_3) configuration. For example, 3 is a normal number, see [17].

The search for the smallest n-gonal configuration is therefore equivalent to the search for the smallest bipartite graph with girth $2n$. This is where we meet a well known problem of cages from graph theory.

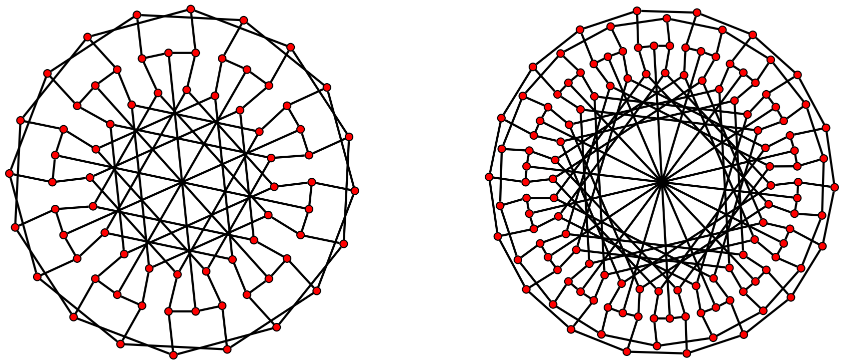
1.2 Cages

For an easy introduction to the subject the reader is referred to [24]. For a simple graph the girth is at least 3. The smallest trivalent graph of girth g is called a *g-cage*. Obviously K_4 is the unique 3-cage and $K_{3,3}$ is the only 4-cage. The Petersen graph $\mathbb{P}(5,2)$ is the only 5-cage. The 6-cage is known as the Heawood graph; see Figure 3(b). The unique 7-cage has 24 vertices. The 8-cage is known as the Cremona-Richmond graph or the Tutte 8-cage. There are 18 non-isomorphic 9-cages which are computed in [8].

Balaban found one of the three 10-cages which is shown in Figure 1(a), see [1]. The other two 10-cages are shown in Figure 2.

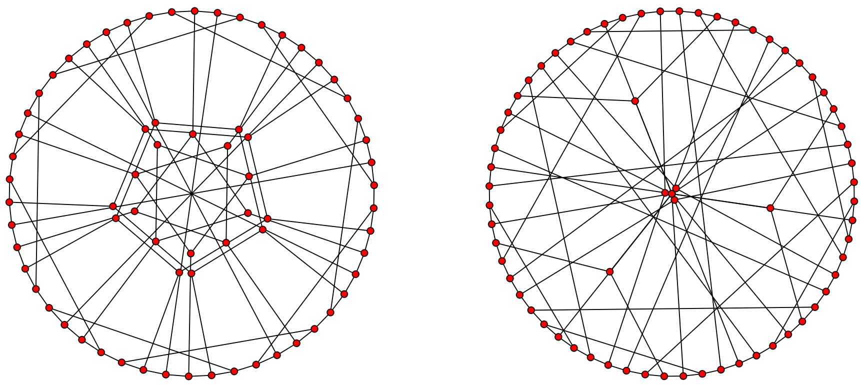
Up to $g \leq 10$ the g-cages are classified completely. For $g = 12$ there is a unique 12-cage. All other cases ($g = 11$, $g > 12$) are still open. For a survey on cages see also [23, 26, 28, 29].

Cages and related graphs have found their applications in chemistry [7], e.g. in modelling chemical reactions [2] and degenerate rearrangements [18].



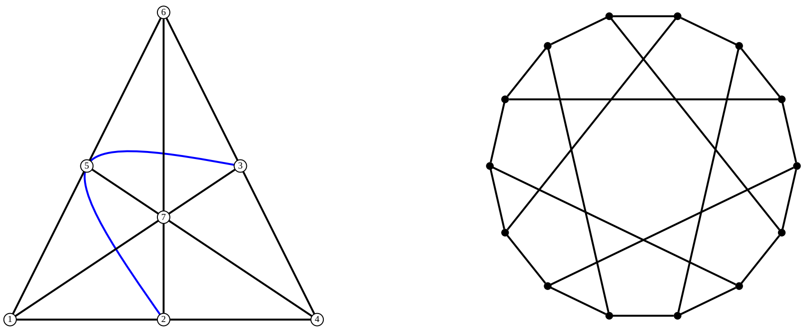
(a) (b)

Figure 1: The Balaban 10-cage or $B(5)$ (a). The two outer 10-gons can be interchanged with the two inner 10-gons. Its generalization $B(9)$ is shown in the right.



(a) (b)

Figure 2: The other two 10-cages are also bipartite and give rise to the configurations that are triangle- and quadrangle-free.



(a) (b)

Figure 3: Fano configuration (a). Its Levi graph is the Heawood graph or the 6-cage (b).

2 Cages and configurations

2.1 Overview

We have mentioned already that the size of the smallest n-gonal symmetric (v_3) configurations is related to the existence of cages. By Proposition 1, each bipartite $2n$-cage represents a Levi graph of the smallest m-gon-free (v_3) configuration for $m < n$, i.e. n-gonal configuration (possibly a pair of dual configurations).

The cages with an odd girth are not bipartite graphs and will not attract our attention. But, interestingly, all known n-cages where n is an even integer are bipartite graphs and hence generate $\frac{n}{2}$-gonal configurations for $n \geq 6$.

The only 6-cage, the Heawood graph, has 14 vertices. It gives the smallest configuration, the Fano (7_3) configuration or the projective plane of order 2, see Figure 3. The Tutte 8-cage is a bipartite graph on 30 vertices. It represents a Levi graph of the smallest (v_3) configuration without triangles, the Cremona-Richmond (15_3) configuration [11], see Figure 4. The Heawood graph and the Tutte 8-cage contain a Hamilton cycle and are highly symmetric. It is well known that the Fano configuration is not linear while the Cremona-Richmond configuration is. Both are self-dual configurations.

The remaining part of the article is devoted to the investigation of these properties in the case of 10-cages and the corresponding configurations.

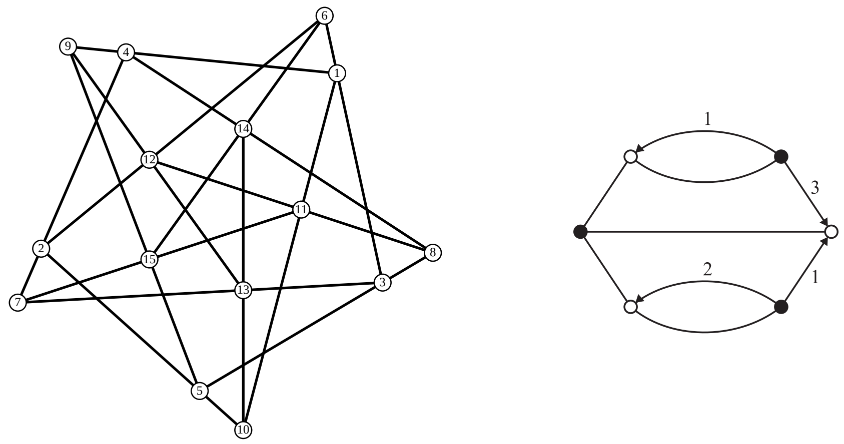
2.2 10-cages

As mentioned in the introduction, there are three non-isomorphic 10-cages. They are all bipartite graphs on 70 vertices.

Let us first investigate the existence of a Hamilton cycle. For cubic graphs with a given Hamilton cycle there is a useful notation due to Frucht et al. [12]. All we have to do is to list the lengths of chords measured in jumps when we traverse the vertices along the Hamilton cycle. Such a list is called the *LCF notation*. For instance, K_4 can be described by the sequence $[2, 2, 2, 2]$ which can be shortened to $[2^4]$. The LCF notation for the Tutte 8-cage is $[(-13, -9, 7, -7, 9, 13)^5]$.

The LCF codes for the 10-cages are, respectively,

$$
\begin{aligned}
[&-9, -25, -19, 29, 13, 35, -13, -29, 19, 25, 9, -29, 29, 17, 33, 21, 9,\\
&-13, -31, -9, 25, 17, 9, -31, 27, -9, 17, -19, -29, 27, -17, -9, -29, 33, -25,\\
&25, -21, 17, -17, 29, 35, -29, 17, -17, 21, -25, 25, -33, 29, 9, 17, -27,
\end{aligned}
$$



(a) (b)

Figure 4: Cremona-Richmond (15_3) configuration, the smallest (v_3) configuration without triangles (a) is linear. Its Levi graph is the Tutte 8-cage which is a $\mathbb{Z}_5$ covering graph over the voltage graph (b).

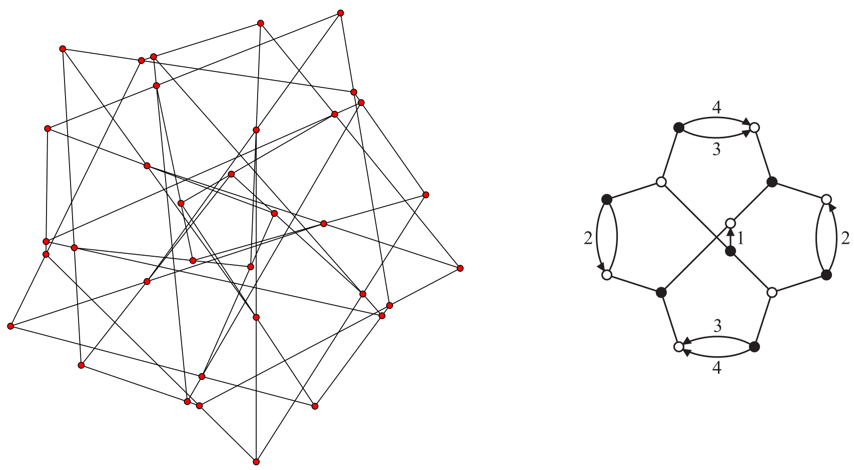
$$29, 19, -17, 9, -27, 31, -9, -17, -25, 9, 31, 13, -9, -21, -33, -17, -29, 29],$$
$$[(-29, -19, -13, 13, 21, -27, 27, 33, -13, 13, 19, -21, -33, 29)^5],$$
$$[9, 25, 31, -17, 17, 33, 9, -29, -15, -9, 9, 25, -25, 29, 17, -9, 9,$$
$$-27, 35, -9, 9, -17, 21, 27, -29, -9, -25, 13, 19, -9, -33, -17, 19, -31,$$
$$27, 11, -25, 29, -33, 13, -13, 21, -29, -21, 25, 9, -11, -19, 29, 9, -27, -19,$$
$$-13, -35, -9, 9, 17, 25, -9, 9, 27, -27, -21, 15, -9, 29, -29, 33, -9, -25].$$

This proves the following proposition.

Proposition 2. *Each* 10*-cage contains a Hamilton cycle.* □

An important part of interest when speaking about graph properties is an existence of symmetry. If the automorphism group of a graph G contains a semiregular element (fixing the two sets of a bipartition), then G is called a *polycirculant.* Part of the motivation for studying this class of graphs is a problem proposed in 1981 by the third author who asked if there exists a vertex-transitive graph without a nonidentity semiregular automorphism [19, Problem 2.4], that is, a vertex-transitive graph which is not a polycirculant. This problem attracted a wider interest in the mathematical community when it was reproposed in a slightly more general group-theoretic setting. The more general form, due to Klin [9, Problem BCC15.12], asks whether every transitive 2-closed permutation group contains a fixed-point-free element of prime order. The term *elusive* has recently been adopted for a transitive finite permutation group without a non-trivial semiregular subgroup. As was shown in [10], there are infinitely many elusive transitive permutation groups, but none of them is 2-closed. It is believed that no 2-closed transitive group is elusive. This conjecture is usually referred to as the *polycirculant conjecture.* For further results and recent advances with regards to this conjecture see [10, 13, 19, 20, 22]. The concept of polycirculants extends naturally to configurations (see the next subsection).

By computing the automorphism groups of all 10-cages we get the following result.



(a) (b)

Figure 5: The smallest 5-gonal (v_3) configuration (a) resulting from the Balaban 10-cage. It is a covering graph over (b) with group $\mathbb{Z}_5$.

Proposition 3. *The automorphism groups of the* 10*-cages have orders* 80*,* 120*,* 24*, respectively. The Balaban cage and the second* 10*-cage are polycirculants.* □

The concept of *voltage graphs* is generally used to simplify the description of large graphs. We refer the reader who is not familiar with this topic to the book by Gross and Tucker [14]. For example, the Tutte 8-cage is a $\mathbb{Z}_5$ covering graph over the voltage graph in Figure 4(b). The first and the second 10-cage are $\mathbb{Z}_5$ covering graphs over the voltage graphs in Figures 5(b) and 6(c).

2.3 The smallest 5-gonal (v_3) configurations

The smallest 5-gonal configurations, triangle- and quadrangle-free configurations, result from 10-cages.

The 5-gonal (35_3) configuration given by the Balaban cage is self-polar and is presented in Figure 5(a). We call it the *Balaban configuration.* The second cage gives rise to a pair of dual 5-gonal (35_3) configurations, see Figure 6(a), (b). The third cage also gives rise to a pair of dual configurations. Hence we have the following result.

Proposition 4. *There are* 5 *non-isomorphic* 5*-gonal* (35_3) *configurations. Also, these are the smallest* 5*-gonal* (v_3) *configurations.* □

Another important concept which has been extensively studied for over a century is the existence of a realization of configurations in the Euclidean plane, that is, the question whether a configuration is linear or not.

Proposition 5. *All five* 5*-gonal* (35_3) *configurations are linear.*

Proof. The coordinates can be found in [4]. □

The problem of finding a realization can be further extended to the search for *nice* realizations. There is no general theory, but the use of symmetry of their Levi graphs proves to be helpful.

A configuration $\mathcal{C}$ is *polycyclic* if there exists an automorphism $\alpha \in \mathrm{Aut}_0\,\mathcal{C}$ which is semiregular, meaning that all orbits on points and lines are of the same size. If α

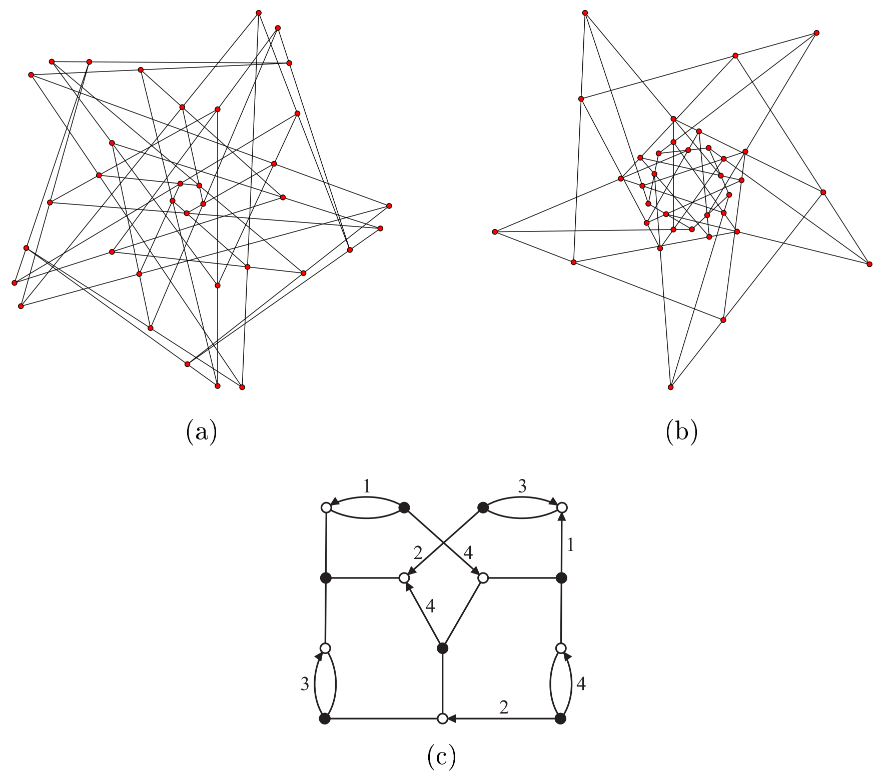


Figure 6: The second 10-cage is a $\mathbb{Z}_5$ covering graph over voltage graph (c) and gives rise to a pair of dual linear (35_3) configurations (a) and (b).

has order k then $\mathcal{C}$ is called a *k-cyclic* configuration. The corresponding Levi graph of a polycyclic configuration is a polycirculant.

There is a theory developed in [5] that enables one to construct the so-called *rotational drawings* of many configurations admitting a polycyclic structure. In this way the realizations in Figures 4, 5, 6 were obtained. See also [15, 16, 21] for related concepts regarding configurations and graphs which possess certain symmetries.

The Balaban configuration is polycyclic. Its rotational realization is shown in Figure 5(a). It results from the symmetry given by the description of its Levi graph using the voltage graph in Figure 5(b).

The same is true for the two dual configurations given by the second 10-cage. Their rotational realizations are depicted in Figure 6(a), (b).

The two 5-gonal (35_3) configurations given by the third cage are also linear, but they are not polycyclic and we cannot give rotational realizations as we did in the previous cases. Nevertheless, their realizability can be easily proved using the algebraic methods described in [6]. The coordinates $\mathbf{x}_i$, $1 \leq i \leq 35$, for one of them can be obtained from the construction sequence given in Table 1. Here we use the *bracket* notation $[i, j, k]$ for $\det(\mathbf{x}_i, \mathbf{x}_j, \mathbf{x}_k)$. See [6] for more details. The values for parameters α_i, $0 \leq i \leq 27$, can be determined in such way that the vectors $\mathbf{x}_i$ represent a realization over rational numbers, see Table 2.

2.4 Generalizations

Another advantage of using voltage graphs for a description of graphs is a possibility of constructing a sequence of graphs and configurations having the same properties.

Theorem 1. *There exists an infinite sequence of* 5*-gonal linear* (v_3) *configurations. Hence,* 5 *is a normal number.*

Proof. Let $B(n)$ be a covering graph of the base graph in Figure 5(b) with group $\mathbb{Z}_n$. All graphs $B(n)$ are bipartite. If $n \geq 5$ is odd then it is easy to see that the girth of the covering graph is at least 10. Thus, they determine a sequence of 5-gonal $((7n)_3)$ configurations $\mathcal{C}_n$ for $n = 5, 7, 9, \ldots$. The first configuration in the series is the Balaban configuration, the third one is shown in Figure 7.

To see that all configurations $\mathcal{C}_n$ in the series are linear we must first check whether the necesary condition (for rotational realization) from the algorithm given in [5] can be satisfied by some set of real parameters for any $\mathcal{C}_n$. After, for example, choosing the values $1, 2, \ldots, 8$ for the first eight of nine parameters, we get in our case the equation

$$\begin{aligned}
&x^2 \Big(1476 + 8120 \cos \tfrac{2\pi}{n} + 2936 \cos \tfrac{4\pi}{n} + 1460 \cos \tfrac{6\pi}{n} + 1952 \cos \tfrac{8\pi}{n} + \\
&\quad 312 \cos \tfrac{12\pi}{n} + 1435 \sin \tfrac{2\pi}{n} + 5515 \sin \tfrac{4\pi}{n} + 5515 \sin \tfrac{6\pi}{n} - 1004 \sin \tfrac{8\pi}{n} - \\
&\quad 2439 \sin \tfrac{12\pi}{n}\Big) + \\
&x \Big(-20616 - 229098 \cos \tfrac{2\pi}{n} - 6462 \cos \tfrac{4\pi}{n} - 19746 \cos \tfrac{6\pi}{n} - 37542 \cos \tfrac{8\pi}{n} \\
&\quad - 46200 \cos \tfrac{12\pi}{n} - 15561 \sin \tfrac{2\pi}{n} - 125673 \sin \tfrac{4\pi}{n} - 125673 \sin \tfrac{6\pi}{n} + \\
&\quad 35544 \sin \tfrac{8\pi}{n} + 51105 \sin \tfrac{12\pi}{n}\Big) - \\
&305100 - 360018 \cos \tfrac{2\pi}{n} + 1344474 \cos \tfrac{4\pi}{n} + 1344474 \cos \tfrac{6\pi}{n} - \\
&\quad 54918 \cos \tfrac{8\pi}{n} - 256284 \sin \tfrac{2\pi}{n} + 443412 \sin \tfrac{4\pi}{n} + 443412 \sin \tfrac{6\pi}{n} - \\
&\quad 256284 \sin \tfrac{8\pi}{n} = 0.
\end{aligned}$$

The discriminant of this quadratic equation is greater than zero for all $n \geq 5$. This ensures the existence of real solutions for $n \geq 5$ and the fulfillment of the necessary

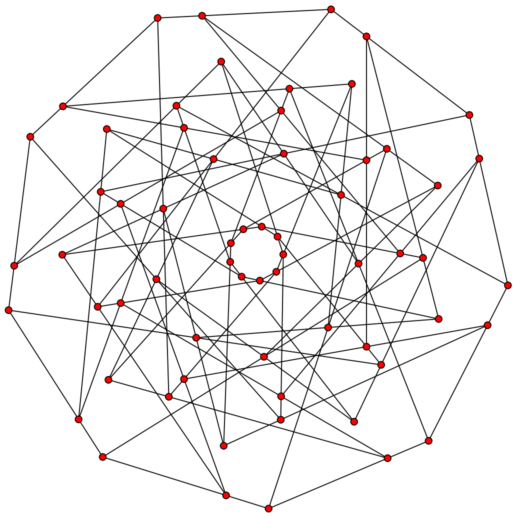


Figure 7: The third configuration in the infinite series of 5-gonal self-polar (v_3) configurations having $B(9)$ as its Levi graph.

condition for rotational realization. The fact that this condition is sufficient (that is, that no two points coincide) can be proved by computer calculations. □

3 Conclusion

In the text we already mentioned an interesting fact that all known even cages are bipartite graphs. Therefore it makes sense to set the following conjecture.

Conjecture. *All g-cages where g is an even integer are bipartite graphs.*

This conjecture is known to be true for even g up to 12. This is the largest number for which the cage problem is solved. The unique 12-cage has 126 points. It is a Levi graph of two dual (63_3) configurations which are also known as the generalized hexagons, see [27].

At this point we should also mention the enumeration results for (v_3) configurations, $v \leq 18$, in [3] obtained by computer calculations. The authors report that the results were extended to $v = 19$ after long parallel computer runs. The number of all non-isomorphic (19_3) configurations is 7 640 941 062. Among them there are 1 992 044 self-dual configurations, 1 991 320 self-polar configurations, 3 cyclic configurations, 14 triangle-free configurations. The number of trangle-free configurations is known up to $v = 21$. The gap between the computer approach and theoretical considerations seems to be unsurmountable which makes the conjecture attractive.

$$\mathbf{x}_{32} = (1,0,0)^{\mathrm{T}}, \qquad \mathbf{x}_{33} = (0,1,0)^{\mathrm{T}},$$
$$\mathbf{x}_{34} = (0,0,1)^{\mathrm{T}}, \qquad \mathbf{x}_{35} = (1,-1,1)^{\mathrm{T}},$$
$$\mathbf{x}_{31} = \mathbf{x}_{32} + \mathbf{x}_{33}\alpha_{26} + \mathbf{x}_{34}\alpha_{27}, \qquad \mathbf{x}_{30} = \mathbf{x}_{32} + \mathbf{x}_{33}\alpha_{24} + \mathbf{x}_{34}\alpha_{25},$$
$$\mathbf{x}_{29} = \mathbf{x}_{32} + \mathbf{x}_{33}\alpha_{22} + \mathbf{x}_{34}\alpha_{23}, \qquad \mathbf{x}_{27} = \mathbf{x}_{32} + \mathbf{x}_{33}\alpha_{20} + \mathbf{x}_{34}\alpha_{21},$$
$$\mathbf{x}_{25} = \mathbf{x}_{32} + \mathbf{x}_{33}\alpha_{18} + \mathbf{x}_{34}\alpha_{19}, \qquad \mathbf{x}_{23} = \mathbf{x}_{32} + \mathbf{x}_{33}\alpha_{16} + \mathbf{x}_{34}\alpha_{17},$$
$$\mathbf{x}_{22} = \mathbf{x}_{32} + \mathbf{x}_{33}\alpha_{14} + \mathbf{x}_{34}\alpha_{15}, \qquad \mathbf{x}_{20} = \mathbf{x}_{32} + \mathbf{x}_{33}\alpha_{12} + \mathbf{x}_{34}\alpha_{13},$$
$$\mathbf{x}_{19} = \mathbf{x}_{32} + \mathbf{x}_{33}\alpha_{10} + \mathbf{x}_{34}\alpha_{11}, \qquad \mathbf{x}_{16} = \mathbf{x}_{32} + \mathbf{x}_{33}\alpha_{8} + \mathbf{x}_{34}\alpha_{9},$$
$$\mathbf{x}_{15} = \mathbf{x}_{32} + \mathbf{x}_{33}\alpha_{6} + \mathbf{x}_{34}\alpha_{7}, \qquad \mathbf{x}_{12} = \mathbf{x}_{32} + \mathbf{x}_{33}\alpha_{4} + \mathbf{x}_{34}\alpha_{5},$$
$$\mathbf{x}_{8} = \mathbf{x}_{32} + \mathbf{x}_{33}\alpha_{2} + \mathbf{x}_{34}\alpha_{3},$$
$$\mathbf{x}_{26} = \mathbf{x}_{27} + \mathbf{x}_{33}\alpha_{1}, \qquad \mathbf{x}_{5} = \mathbf{x}_{25} + \mathbf{x}_{15}\alpha_{0},$$
$$\mathbf{x}_{28} = [29,35,32]\mathbf{x}_{22} + [35,29,22]\mathbf{x}_{32}, \qquad \mathbf{x}_{24} = [31,23,29]\mathbf{x}_{25} + [23,31,25]\mathbf{x}_{29},$$
$$\mathbf{x}_{21} = [34,22,26]\mathbf{x}_{25} + [22,34,25]\mathbf{x}_{26}, \qquad \mathbf{x}_{18} = [31,12,35]\mathbf{x}_{19} + [12,31,19]\mathbf{x}_{35},$$
$$\mathbf{x}_{17} = [32,16,26]\mathbf{x}_{18} + [16,32,18]\mathbf{x}_{26}, \qquad \mathbf{x}_{14} = [19,15,23]\mathbf{x}_{22} + [15,19,22]\mathbf{x}_{23},$$
$$\mathbf{x}_{13} = [33,14,30]\mathbf{x}_{29} + [14,33,29]\mathbf{x}_{30}, \qquad \mathbf{x}_{11} = [34,12,16]\mathbf{x}_{15} + [12,34,15]\mathbf{x}_{16},$$
$$\mathbf{x}_{10} = [27,11,30]\mathbf{x}_{20} + [11,27,20]\mathbf{x}_{30}, \qquad \mathbf{x}_{9} = [35,8,23]\mathbf{x}_{10} + [8,35,10]\mathbf{x}_{23},$$
$$\mathbf{x}_{7} = [16,8,21]\mathbf{x}_{20} + [8,16,20]\mathbf{x}_{21}, \qquad \mathbf{x}_{6} = [31,7,28]\mathbf{x}_{27} + [7,31,27]\mathbf{x}_{28},$$
$$\mathbf{x}_{4} = [9,5,13]\mathbf{x}_{12} + [5,9,12]\mathbf{x}_{13}, \qquad \mathbf{x}_{3} = [32,4,20]\mathbf{x}_{19} + [4,32,19]\mathbf{x}_{20},$$
$$\mathbf{x}_{2} = [3,24,33]\mathbf{x}_{8} + [24,3,8]\mathbf{x}_{33}, \qquad \mathbf{x}_{1} = [5,6,30]\mathbf{x}_{17} + [6,5,17]\mathbf{x}_{30}$$

Table 1: The construction sequence for the coordinates of a configuration arising from the third 10-cage. In order to obtain a realization, the parameters α_i, $0 \leq i \leq 27$, must satisfy the equation $[1,2,34] = 0$.

$$
\begin{aligned}
\alpha_0 = {} & -2574943097082595345544340065044256416763150642661777434707789 \\
& 5966787620877820717701948249681775720540264218915670672534689 0 \\
& 2262214036811204053240197500279284758605624718618526709560921 0 \\
& 7222402390830552317307325883192770706039020969269254417506252 1 \\
& 0086756733485633027070188539977944384763049494404519721011383 3 \\
& 4568658197903124433965927164994808784611798091030112338841372 0 \\
& 5705181599294151350015577603197295653753564338229903599253504 2 \\
& 67371715388260255670916165559834276819672066 0/ \\
& 8566420783355848130713300240718041905354631227190761890940109 6 \\
& 8054495792129924944592304113050662713949771614175390924889990 2 \\
& 7802989999659884175618477590524672264485722140355660555517585 9 \\
& 4636932276527270771611514269479749808029375781160354210906779 1 \\
& 0728946023635713593782486945482017303048700013013696487920825 7 \\
& 8030813277521107797734918913526366788592524678609237815999978 0 \\
& 6341401887391560949040192709388429849894474757815211590010788 9 \\
& 05944267184124981378187073576156229949974150 3, \\
\alpha_1 = {} & -7123834754731184528402693911971923451330058149875518104914623 \\
& 4805695343009450054707485348724711965433064698266226659315841 4 \\
& 4902740172982366442273109727826202634020431026333350033278964 1 \\
& 80373821353618055737905424 3/ \\
& 1278822422266327636135604207316558542996580734918505779610615 7 \\
& 0727973121193468282801294088265722180442809839467907552922478 7 \\
& 9844506260846116172181093714136482635976934334136626327123629 5 \\
& 44400986601488712671339 74, \\
\alpha_2 = {} & -3050903110258722487817260103307920379457268495239756349868330 \\
& 0662673654682267899245749398142509676398012593684685043158264 9 \\
& 0179113374985805683/ \\
& 8049963523141414447790910983227616154911214235378750516305452 9 \\
& 9657353118180049747271575312127788785462621281388768234743347 0 \\
& 5127358277359656, \\
\alpha_3 = {} & -99717684685911157566102515583604819 07/ \\
& 15146372744901918031042432812704839 2, \\
\alpha_4 = {} & 81, \alpha_5 = -73, \alpha_6 = 56, \alpha_7 = 98, \alpha_8 = 33, \alpha_9 = 71, \alpha_{10} = -62, \\
\alpha_{11} = {} & -87, \alpha_{12} = -35, \alpha_{13} = -55, \alpha_{14} = 68, \alpha_{15} = 69, \alpha_{16} = 73, \alpha_{17} = -29, \\
\alpha_{18} = {} & -73, \alpha_{19} = 87, \alpha_{20} = 91, \alpha_{21} = 73, \alpha_{22} = 49, \alpha_{23} = 25, \alpha_{24} = 28, \\
\alpha_{25} = {} & 100, \alpha_{26} = 24, \alpha_{27} = -13
\end{aligned}
$$

Table 2: The set of rational parameters $\alpha_0, \alpha_1, \ldots, \alpha_{27}$ for the construction of coordinates presented in the Table 1 which gives a rational realization of the configuration determined by the third 10-cage, see [4].

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