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Preprint series, Vol. 39 (2001), 770

POLYCYCLIC CONFIGURATIONS

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ISSN 1318-4865

August 14, 2001

Ljubljana, August 14, 2001

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Abstract

Polycyclic configurations constitute a generalization of the well-known class of cyclic configurations. They admit a concise description via voltage graphs over cyclic groups. Polycyclic (v_k) configurations are considered for $k = 2, 3, 4$. The structure of polycyclic configurations can be used in some cases to generate a rotational straight-line drawing of such configurations in the Euclidean plane.

1 Introduction

An *incidence structure* $\mathcal{C}$ is a triple $\mathcal{C} = (P, \mathcal{B}, I)$ where P is the set of *points*, $\mathcal{B}$ is the set of *lines* (or *blocks*), $I \subseteq P \times \mathcal{B}$ is the *incidence relation*. The elements of I are called *flags*. The bipartite incidence graph $L(\mathcal{C})$ with black vertices P, white vertices $\mathcal{B}$ and edges I is known as the *Levi graph* of $\mathcal{C}$, see [5]. An incidence structure is *connected* if its Levi graph is connected. Following the spirit of [8] we will call an incidence structure *lineal* if the girth of its Levi graph is at least 6 and we call it *triangle-free* if the girth is at least 8. Similarly we say that $\mathcal{C}$ is *triangle- and quadrangle-free* if $L(\mathcal{C})$ has girth at least 10, etc. Also, a configuration $\mathcal{C}$ is called *d-gonal* if the girth of $L(\mathcal{C})$ equals $2d$.

Each incidence structure $\mathcal{C} = (P, \mathcal{B}, I)$ gives rise to the so-called dual structure $\mathcal{C}^d = (\mathcal{B}, P, I^d)$ in which the roles of points and lines are reversed. Both $\mathcal{C}$ and $\mathcal{C}^d$ share the same Levi graph, only the black-white coloring of its vertices is reversed.

If $\mathcal{C}$ is isomorphic to its dual $\mathcal{C}^d$ we say that $\mathcal{C}$ is *self-dual* and the isomorphism is called a *duality*. A duality of order 2 is called a *polarity*. An isomorphism of $\mathcal{C}$ to itself is called an *automorphism* or *colinearity*. Automorphisms of $\mathcal{C}$ form a group denoted by $\mathrm{Aut}_0\,\mathcal{C}$. We may consider automorphisms and dualities (anti-automorphisms) as acting on the disjoint union $P \cup \mathcal{B}$. They together form the *extended group of automorphisms* $\mathrm{Aut}\,\mathcal{C}$.

A (v_r, b_s) configuration is an incidence structure $\mathcal{C} = (P, \mathcal{B}, I)$ such that $v = |P|$, $b = |\mathcal{B}|$, there are r lines through a point, there are s points on a line,

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and two lines meet in at most one point. The Levi graph of a configuration is semiregular of girth ≥ 6. Hence all our configurations are lineal.

A (v_r, b_s) configuration is symmetric if $v = b$ (which is equivalent to saying that $r = s$). A (v_r, v_r) configuration is shortened to a (v_r) configuration.

A (v_r, b_s) configuration $\mathcal{C}$ is *polycyclic* if there exists an automorphism $\alpha \in \mathrm{Aut}_0\,\mathcal{C}$ which is semi-regular, meaning that all orbits on points and lines are of the same size. If α has the order k then $\mathcal{C}$ is called a *k-cyclic* configuration. If the configuration $\mathcal{C}$ is k-cyclic then k divides $\gcd(v, b)$. Symmetric v-cyclic configurations are known as *cyclic* configurations. Since each configuration is 1-cyclic we will exclude this trivial case.

We can view α on the Levi graph $G = L(\mathcal{C})$ of the configuration $\mathcal{C}$. For $a, b \in V(G)$ let $a \approx_\alpha b$ denote that a and b are in the same orbit of α. Let $G/_{\approx_\alpha}$ denote the (multi-)graph obtained from G by identifying the vertices a and b if and only if $a \approx_\alpha b$ and identifying the edges aa' and bb' if and only if $a \approx_\alpha b$ and $a' \approx_\alpha b'$ and all four vertices a, a', b, b' are distinct. The projection $\psi : G \to G/_{\approx_\alpha}$ is a regular covering projection. For covering and voltage graphs see for instance [10].

If the automorphism group of a graph G contains a semiregular element, then G is called a *polycirculant*. In 1981 D. Marušič asked if there exists a vertex-transitive graph without a nonidentity semiregular automorphism [13, Problem 2.4], that is, a vertex-transitive graph which is not a polycirculant. The concept of polycirculants extends naturally to configurations via their Levi graphs.

Remark. A configuration is polycyclic if and only if its Levi graph is semiregular bipartite polycirculant graph with girth at least 6 and there exists a semiregular automorphism which fixes the two sets of the graph bipartition.

A polycyclic configuration $\mathcal{C}$ has a *transitive quotient* with respect to α if $G/_{\approx_\alpha}$ is vertex transitive. For graphs and groups see [2]. A configuration with a transitive quotient is obviously symmetric. Although the relation $\approx_\alpha$ and the graph $G/_{\approx_\alpha}$ depends on α we will drop it from the subscripts to simplify the notation.

If $\mathcal{C}$ is connected then $G/_{\approx}$ is connected, too. However, in general the converse is not necessarily true. See [10] for examples.

Any (v_r, b_s) configuration admits a description by a matrix

$$\begin{array}{cccccc}
a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & \dots & a_{1,b} \\
a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & \dots & a_{2,b} \\
\dots & \dots & \dots & \dots & \dots & \dots \\
a_{s,1} & a_{s,2} & a_{s,3} & a_{s,4} & \dots & a_{s,b}
\end{array}$$

with the entries from the set of points P such that each element from P occurs exactly r times, each column contains distinct entries, and no pair of elements from P occurs more than once in any column. Furthermore, when $v \geq b$ we can also achieve that each row contains distinct entries. The set P represents points and each column represents a line of the configuration. A k-cyclic configuration admits a shorter description.

Let $\alpha = (a_1^{(0)}\, a_1^{(1)}\, \dots\, a_1^{(k-1)})\,(a_2^{(0)}\, a_2^{(1)}\, \dots\, a_2^{(k-1)})\, \dots\, (a_n^{(0)}\, a_n^{(1)}\, \dots\, a_n^{(k-1)})$ be an automorphism of $\mathcal{C}$ (restricted to the set of points), where $n = v/k$. Let us define $a_i = a_i^{(0)}$ and $P_0 = \{a_1, a_2, \dots, a_n\}$, the set of point-orbit representatives.

Since $a_i^{(j)} = \alpha^j(a_i)$, the points are completely described by a_i, $i = 1, 2, \ldots, n$, k, and α. Since lines fall into $m = b/k$ orbits, one can describe them by m columnwise s-tuples:

$$\begin{array}{lll} \alpha^{j_{1,1}}(a'_{1,1}) & \ldots & \alpha^{j_{1,m}}(a'_{1,m}) \\ \alpha^{j_{2,1}}(a'_{2,1}) & \ldots & \alpha^{j_{2,m}}(a'_{2,m}) \\ \ldots & \ldots & \ldots \\ \alpha^{j_{s,1}}(a'_{s,1}) & \ldots & \alpha^{j_{s,m}}(a'_{s,m}). \end{array}$$

This representation is equivalent to the voltage graph description in which all edges are directed from lines to points. The points (from the set P_0) are labeled by a'-s while the voltages are labeled by j-s. We will shorten and simplify notation to

$$\begin{array}{lll} {a'_{1,1}}^{j_{1,1}} & \ldots & {a'_{1,m}}^{j_{1,m}} \\ {a'_{2,1}}^{j_{2,1}} & \ldots & {a'_{2,m}}^{j_{2,m}} \\ \ldots & \ldots & \ldots \\ {a'_{s,1}}^{j_{s,1}} & \ldots & {a'_{s,m}}^{j_{s,m}}. \end{array} \tag{1.1}$$

The following lemma summarizes some basic relations between $\mathcal{C}$ and $G/_{\approx}$.

Lemma 1.1. *If a (v_r, b_s) configuration $\mathcal{C}$ is k-cyclic then $G/_{\approx}$ is bipartite (r, s)-semiregular voltage graph on $(v + b)/k$ vertices with voltages in $\mathbb{Z}_k$ such that its covering graph is isomorphic to $G = L(\mathcal{C})$.*
Any semi-regular bipartite $\mathbb{Z}_k$ voltage graph gives rise to a Levi graph G of some configuration $\mathcal{C}$ precisely when no multiple edges, no adjacent multiple edges, and no 4-gons lift to a 2-gon or 4-gon.

Proof. Since $L(\mathcal{C})$ is bipartite and (r, s)-semi-regular and this property is inherited during the process of identifying the vertices, $G/_{\approx}$ is bipartite and (r, s)-semi-regular (multigraph), too. It has v/k black vertices and b/k white vertices, the first half representing point-orbits and the other half line-orbits, respectively. The way how voltages are attached to the edges of $G/_{\approx}$ was described above.

A bipartite (r, s)-semi-regular graph G is a Levi graph of some configuration if and only if its girth (the length of the shortest cycle in G) is at least 6. Since G is bipartite only the cycles of length 2 and 4 must be absent from G. One can obtain a 2-cycle only by lifting a multiple edge and a 4-cycle by lifting a multiple edge or two adjacent multiple edges or a 4-cycle itself. □

Let us stress again that throughout the paper we assume the edges of quotient graphs directed from vertices representing lines to the vertices representing points, in figures from black to white vertices.

Example 1.1. 1. The Cremona-Richmond configuration admits a description via the voltage graph in Figure 6(a) in the group $\mathbb{Z}_5$. Here $v = 15, r = 3, k = 5, n = 3$, and (1.1) is

$$\begin{array}{lll} a^3 & a^0 & b^0 \\ b^1 & a^1 & b^2 \\ c^0 & c^0 & c^0. \end{array}$$

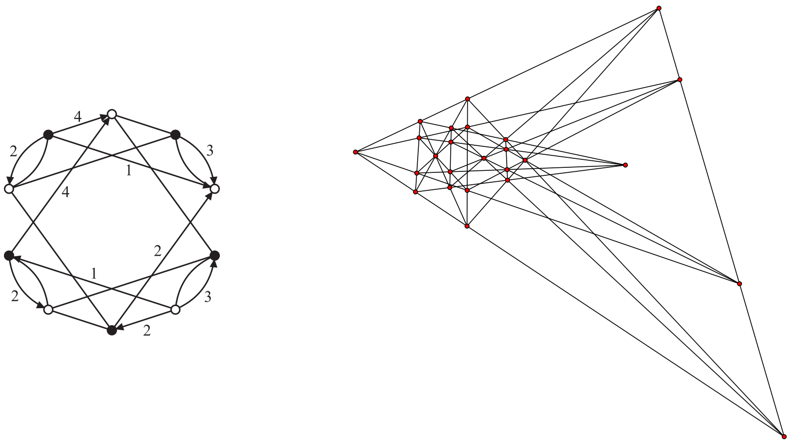

Figure 1: A linear (25_4) configuration that admits a 5-cyclic structure as shown by the voltage graph on the left.

2. In a similar way, the Pappus configuration admits a description via the voltage graph of Figure 3(a) in the group $\mathbb{Z}_3$. Here $v = 9, r = 3, k = 3, n = 3$, the underlying graph is $K_{3,3}$, and (1.1) is

$$\begin{array}{ccc} a^0 & a^0 & a^1 \\ b^0 & b^1 & b^0 \\ c^1 & c^0 & c^0. \end{array}$$

3. A recently discovered linear (25_4) configuration admits the following 5-cyclic structure, see Figure 1:

$$\begin{array}{ccccc} a^0 & a^1 & c^1 & c^0 & a^0 \\ b^0 & b^0 & d^0 & c^1 & a^1 \\ c^0 & b^4 & d^4 & d^2 & b^2 \\ d^0 & e^1 & e^1 & e^2 & e^2. \end{array}$$

4. Grünbaum [8] gives an example of a remarkable $(30_3, 15_6)$ configuration. It admits a rotational realization using $\mathbb{Z}_3$ and the following voltages:

$$\begin{array}{ccccc} a^0 & a^0 & a^0 & b^0 & b^0 \\ c^0_+ & c^0_- & d^0 & e^0_+ & e^0_- \\ g^2 & g^1 & f^0 & f^2 & f^1 \\ e^2_- & e^1_+ & h^0 & h^2 & h^1 \\ d^2 & d^1 & g^0 & e^2_+ & e^1_- \\ c^2_+ & c^1_- & b^0 & c^0_+ & c^0_-. \end{array}$$

5. And finally, the renowned Reye $(12_4, 16_3)$-configuration [12] has a 4-cyclic structure, see also Figure 2:

$$\begin{array}{cccc} a^0 & a^2 & a^1 & b^1 \\ a^1 & b^2 & b^0 & c^0 \\ b^2 & c^0 & c^1 & c^1. \end{array}$$

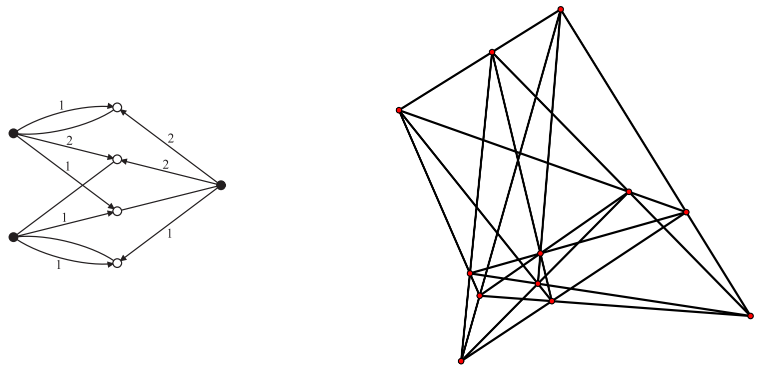

Figure 2: Reye $(12_4, 16_3)$ configuration. Although it is 4-cyclic with quotient graph on the left, it does not have a rotational realization. We show its non-rotational realization.

2 Rotational drawings of polycyclic configurations

The existence of a polycyclic structure can help us produce *nice* drawings of polycyclic configurations with a given automorphism α.

First we present some terminology from the general theory. Let $\mathcal{C} = (P, \mathcal{B}, I)$ and $\mathcal{D} = (P', \mathcal{B}', I')$ be two incidence structures and let us define the maps

$$\phi_P : P \to P' \qquad \text{and} \qquad \phi_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}'.$$

Together they define a map $\phi : P \cup \mathcal{B} \to P' \cup \mathcal{B}'$. If

$$(p, B) \in I \Rightarrow (\phi_P(p), \phi_{\mathcal{B}}(B)) \in I' \tag{2.1}$$

holds for every $p \in P$ and $B \in \mathcal{B}$ then we call ϕ a *representation* of $\mathcal{C}$ in $\mathcal{D}$. If, in addition to (2.1), both ϕ_P and $\phi_{\mathcal{B}}$ are 1-to-1 maps then we say that ϕ is a *weak realization* of $\mathcal{C}$ in $\mathcal{D}$. A map ϕ is called a *realization* (or *strong realization*) if ϕ_P and $\phi_{\mathcal{B}}$ are 1-to-1 maps and

$$(p, B) \in I \Leftrightarrow (\phi_P(p), \phi_{\mathcal{B}}(B)) \in I'$$

holds for every $p \in P$ and $B \in \mathcal{B}$. Another term which is used in this context is coordinatization.

A *coordinatization* $X_{\mathcal{C}}$ of an incidence structure $\mathcal{C} = (P, \mathcal{B}, I)$ over a field K is a 1-to-1 map

$$X_{\mathcal{C}} : P \to K^3, \qquad p \mapsto \mathbf{x}_p$$

such that

$$p, q, r \text{ are collinear in } \mathcal{C} \quad \Leftrightarrow \quad \det(\mathbf{x}_p, \mathbf{x}_q, \mathbf{x}_r) = 0. \tag{2.2}$$

for all distinct $p, q, r \in P$. A coordinatization is called *weak* if $X_{\mathcal{C}}$ is 1-to-1 and the equivalence relation in (2.2) is replaced by an implication. Any coordinatization over reals gives a strong realization in the projective plane (and hence in the Euclidean plane). The converse in not necessarily true, see [6] for an example. The problem of coordinatization is extensively discussed in [3, 15]. However, we can exploit a special structure of polycyclic configurations to obtain *nice* realizations in the Euclidean plane.

A representation (weak, strong realization) of a k-cyclic configuration $\mathcal{C}$ in the Euclidean plane $\mathcal{E}$ is called *rotational* if the automorphism α of $\mathcal{C}$ is realized as a rotation for $\frac{2\pi}{k}$, that is, the points from the same orbit are equally spaced on a circle.

Let α be an automorphism of $\mathcal{C}$ of order k as above. The corresponding rotational representation ϕ in $\mathcal{E}$ can be completely described by a restriction of ϕ_P to P_0. We will denote $\mathbf{x}_a := \phi_P(a)$, $\mathbf{x}_a \in \mathbb{R}^2$. If $\mathbf{x}_a$ is a coordinate vector of a point orbit representative a then the coordinates for other points in the orbit can be obtained by rotating $\mathbf{x}_a$ around the origin. To be more precise, $\mathbf{x}_b = R_k^j \mathbf{x}_a$ if $b = \alpha^j(a)$ and $a \in P_0$ where R_k is the rotation matrix for $\frac{2\pi}{k}$ around the origin,

$$R_k = \begin{bmatrix} \cos\frac{2\pi}{k} & -\sin\frac{2\pi}{k} \\ \sin\frac{2\pi}{k} & \cos\frac{2\pi}{k} \end{bmatrix}.$$

In a similar way we define a rotational coordinatization. A *rotational coordinatization* $X_{\mathcal{C},\alpha}$ of a k-cyclic configuration $\mathcal{C}$ for an automorphism α with the orbit representatives P_0 is a coordinatization over $K = \mathbb{R}$ such that $\mathbf{x}_b = R_k^j \mathbf{x}_a$ and $b = \alpha^j(a)$ for some $a \in P_0$ where

$$R_k = \begin{bmatrix} \cos\frac{2\pi}{k} & -\sin\frac{2\pi}{k} & 0 \\ \sin\frac{2\pi}{k} & \cos\frac{2\pi}{k} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Example 2.1. We continue with the polycyclic form of the Pappus configuration from Example 1. Figure 3 shows a rotational realization of the Pappus configuration. Here, the realization is given by

$$\mathbf{x}_1 = (1,0), \qquad \mathbf{x}_2 = (1,1), \qquad \mathbf{x}_3 = (1,-3)$$

where $a_1 = 3$, $a_2 = 6$, and $a_3 = 9$.

Of course, in order to obtain a realization of $\mathcal{C}$ we must suitably choose the values for $\mathbf{x}_a$, $a \in P_0$. These coordinates must satisfy the conditions imposed by the set of lines. Again, because of the polycyclic structure of $\mathcal{C}$, we only need to take into account the line representatives of each orbit, that is, the columns in (1.1).

In [3] there is a description of a simple algorithm which enables us to find a solution of v equations arising from v lines of a (v_3) configuration. A necessary condition for an existence of a coordinatization over reals can be reduced to an existence of a certain number of suitable real parameters to be zeroes of one polynomial which is called a *final polynomial*. With this method we can easily obtain a realization of a configuration or prove its non-realizability.

But what is an advantage of looking for rotational realization? The first motivation is for those who are concerned about aesthetics in a representation of configurations, rotational realizations are usually “nice” and are obtained by an equal effort than “ordinary” realizations given by the algorithm mentioned

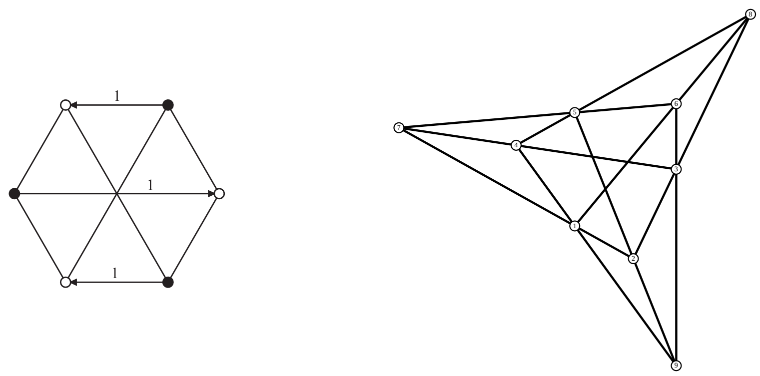

Figure 3: Rotational realization of Pappus configuration and its quotient graph $G/_{\approx}$ with the voltage group $\mathbb{Z}_3$; compare [6], Figure 14.

above. Examples are the pictures in this article. Secondly, the use of symmetry of polycyclic configuration reduces the number of line conditions and consequently simplifies its final polynomial (in this case, a final polynomial for rotational coordinatization). An important fact is also that rotational realizations (final polynomials for rotational realizations) are similar for certain families of configurations, for example, for configurations sharing the same quotient graph. The examples for the last two observations are given in the fifth section.

The problem of rotational realization is far from being understood. Many situations can occur. Although a Reye configuration from Example 1.1 is 4-cyclic and 2-cyclic, no rotational realizations exist. Its non-rotational realization is shown in Figure 2. Some configurations admit rotational realizations over different quotient graphs. Figures 13 and 10 show two rotational realizations of the same (10_3) configuration.

However, one may ask more topological question which is implicit in the work of Grünbaum [11, 8, 7]. An *arrangement of pseudolines* is a finite family of simple closed curves in the projective plane $P(\mathbb{R}^3)$ such that any two curves have exactly one point in common, each crossing the other at this point, while no point is common to all curves.

Question 2.1. Which polycyclic configurations admit rotational realizations with pseudolines.

The above question has several implicit subqestions. For instance, it may happen that the rotational realization is weak, that is, we have false incidences. These may be avoided by using pseudolines. It is not clear if in all instances where false incidence would occur one may replace lines by pseudolines that still preserve rotational character of a realization.

3 Polycyclic (v_2) configurations

In this short section we discuss the simplest case. Obviously connected (v_2) configurations are, speaking in graph theoretic language, v-cycles. They are all

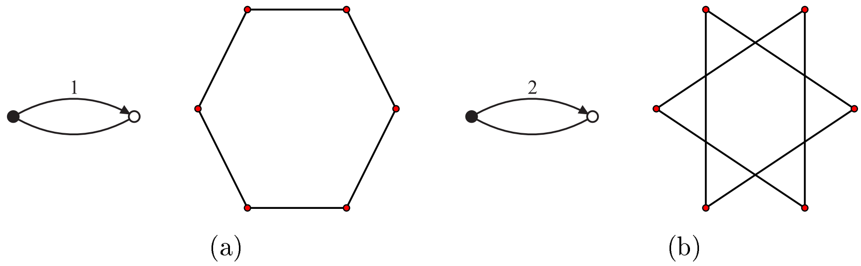

Figure 4: Configurations $\mathcal{C}_2(6,1)$, (a), and $\mathcal{C}_2(6,2)$, (b), and their quotient graphs.

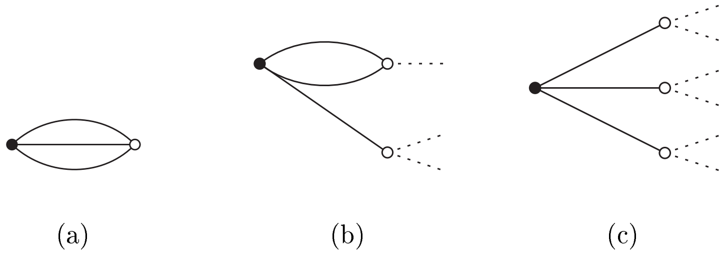

Figure 5: Possible local structures of $G/_{\approx}$ for (v_3) configurations.

cyclic configurations and therefore polycyclic. If we do not restrict to connected configurations, polycyclic (v_2) configurations are unions of equal-length cycles.

Let $\mathcal{C}_2(k,p)$, $0 < p < k/2$, be a configuration with k points on a circle and a line connects the i-th point to the $(i+p)$-th point (addition is mod k). There are $\gcd(k,p)$ connected components, each one is a multilateral [6]. The special case $\mathcal{C}_2(k,1)$ will be abbreviated to $\mathcal{C}_2(k)$ and is called a simple multilateral.

Polycyclic (v_2) configurations $\mathcal{C}_2(k,p)$ can be described by voltage graphs over a digon. The voltages on the two edges of a digon are 0 and p while the voltage group is $\mathbb{Z}_k$. Two examples are in Figure 4.

4 Polycyclic (v_3) configurations

The first cases of polycyclic (v_3) configurations were studied by Dorwart and Grünbaum in [6] and were explored in a much more general way by Grünbaum in the short paper [7] that we obtained quite recently.

Observing the local structure of the quotient graph $G/_{\approx}$ we distinguish three cases, see Figure 5. Case (a) is possible only for $k = v$, when the corresponding configuration is cyclic. In general, the graph $G/_{\approx}$ can have points only of form (a), (b) or (c), or it can consist of a mixture of them, see Figure 6. In the latter case the quotient graph is not vertex-transitive.

Example 4.1. Here are some examples of rotational realizations.

1. $G/_{\approx} = K_{3,3}$. The $\mathbb{Z}_3$ covering graph on $K_{3,3}$ with voltages 1 on one 1-factor and 0 on the other edges is the Levi graph of Pappus configuration. Its rotational realization is depicted in Figure 3.

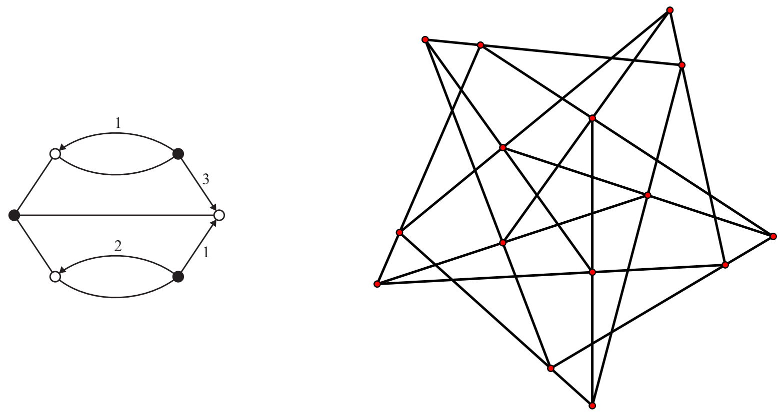

Figure 6: Rotational realization of Cremona-Richmond (15_3) configuration and its quotient graph $G/_{\approx}$; the voltage group is $\mathbb{Z}_5$. The drawing does not capture very high symmetry of the configuration as the quotient graph is not vertex transitive.

2. Figure 11 represents rotational realization of the configuration $(9_3)_2$. Its quotient graph is vertex transitive of type (b).
3. The smallest triangle-free configuration is called the *Cremona-Richmond configuration*. Its Levi graph is the Tutte 8-cage. Figure 6 shows its rotational realization together with the corresponding quotient graph. Even though the configuration is highly symmetric, its quotient is not vertex transitive for any automorphism.

Any cubic bipartite connected graph on $2n$ vertices can serve as the underlying graph of a voltage graph of a polycyclic (v_3) configuration. We will call it *realizable* if there exists at least one voltage graph whose configuration is rotationally realizable.

The construction procedure for these graphs is relatively simple. The graphs with given n consist of all simple connected bipartite graphs with $2n$ vertices and all connected bipartite graphs with $2n-2$ vertices in which a digon is inserted at every possible edge.

If we restrict our attention to the vertex-transitive case there is a unique non-simple family of graphs that is very useful for constructing polycyclic configurations.

Let Θ denote the graph consisting of two vertices and three parallel edges joining them. Let $\Theta_1 = \Theta$. In general, Θ_n consists of vertices $0, 1, \ldots, 2n-1$ and edges $(2i, 2i+1)$ and double edges at $(2i-1, 2i)$ (addition is mod $2n$), see Figure 9.

Proposition 4.1. *A connected bipartite cubic vertex-transitive graph is either simple or isomorphic to some $\Theta_n, n \geq 1$.*

Proof. All three possibilities for the local structure around a vertex of a cubic graph are shown in Figure 5. For a connected vertex-transitive graph G this

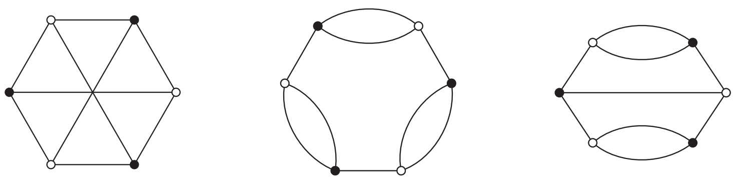

Figure 7: Underlying graphs for the voltage graphs for polycyclic (v_3) configurations with three orbits.

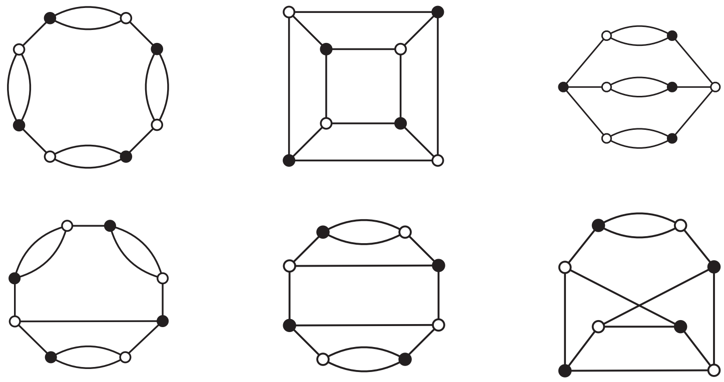

Figure 8: All possible underlying graphs for the voltage graphs for polycyclic (v_3) configurations with four orbits.

structure must be the same for each vertex. Case (c) in Figure 5 implies that graph G is simple, case (b) implies that G is isomorphic to some Θ_i, $i \geq 2$, and case (a) shows Θ_1. □

Case $n = 1$**.** For $n = 1$ the only graph is the Θ graph. The derived configurations are cyclic configurations; none of them is rotationally realizable over Θ.

Case $n = 2$**.** The case $n = 2$ consists of a single graph Θ_2. It is obtained from Θ by insertion of a digon at some edge. For a rotational realization, see Figure 10.

Case $n = 3$**.** There are three graphs available. From Θ_2 we obtain Θ_3 by inserting a digon in a single edge and we get a graph of Figure 6, the underlying graph of a Cremona-Richmond configuration. Finally, $K_{3,3}$ is the smallest simple bipartite graph having 6 vertices. It is the underlying graph of the Pappus configuration. See Figure 7.

Case $n = 4$**.** There are 6 graphs in this group. The cube Q_3 is the only simple cubic graph on 6 vertices. The other five graphs are obtained from the 3 graphs of case $n = 3$ via the indicated procedure. All 6 of them are depicted in Figure 8. They are all realizable.

Conjecture 4.1. Every connected trivalent bipartite graph except for Θ is realizable.

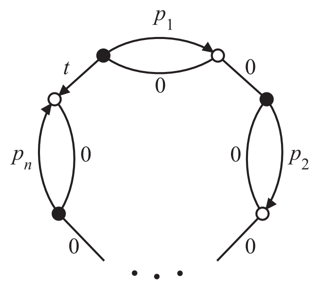

Figure 9: Transitive quotient graph $G/_\approx$.

5 Configurations $\mathcal{C}_3(k,p,t)$

Now, let us focus on the transitive case where all vertices are of type (b) according to the Figure 5, that is, $G/_\approx$ is an even cycle with alternating single and double edges.

We can assume that the voltages on double edges are 0 and p_i, $0<p_i<k/2$, $i=1,2,\ldots,n$, and 0 on all single edges but one which we denote by t, see Figure 9. This is true since we can add the same element from $\mathbb{Z}_k$ to the voltages on all edges which are incident with the same vertex — this means that we only change the numbering of the vertices in the same vertex-fiber. We will refer to this procedure by “rotating voltages around a vertex”. It follows that a configuration given by such graph can be described by

$$\begin{array}{ccccc} a_1^{p_1} & a_2^{p_2} & \ldots & a_{n-1}^{p_{n-1}} & a_n^{p_n} \\ a_1^0 & a_2^0 & \ldots & a_{n-1}^0 & a_n^0 \\ a_2^0 & a_3^0 & \ldots & a_n^0 & a_1^t, \end{array} \tag{5.1}$$

where $n=v/k$. Therefore we can denote these k-cyclic (v_3) configurations by $\mathcal{C}_3(k,p,t)$ where $p=(p_1,p_2,\ldots,p_n)$. In case $p_1=p_2=\cdots=p_n=:p_0$ we shall write $\mathcal{C}_3(k,p_0^n,t)$.

Theorem 5.1. *For $n>2$ any sequence $p=(p_1,p_2,\ldots,p_n)$, $0<p_i<k/2$, and number t, $0\le t<k$ define a (v_3) configuration $\mathcal{C}_3(k,p,t)$. For $n=2$ the following conditions are required:*

$$p_1+p_2\neq t,\quad p_1\neq t,\quad p_2\neq t,\quad t\neq 0.$$

Proof. Let us first check the conditions of Lemma 1.1 for $n>2$. Since we have voltages 0 and p_i, $0<p_i<k/2$, on all double edges, no double edge lifts to a 2- or 4-cycle. There are also no 4-cycles and adjacent multiple edges in $G/_\approx$. Now, it follows from Lemma 1.1 that a $\mathbb{Z}_k$ covering graph over $G/_\approx$ represents a Levi graph of some (v_3) configuration. For $n=2$ we must prevent that some 4-cycle is lifted to a 4-cycle. It is easy to check that the conditions above include all possibilities. □

The following results show some basic properties of $\mathcal{C}_3(k,p,t)$ configurations.

Proposition 5.2. *The dual configuration of $\mathcal{C}_3(k,(p_1,p_2,\ldots,p_n),t)$ is the configuration $\mathcal{C}_3(k,(p_n,p_{n-1},\ldots,p_1),p_1+p_2+\cdots+p_n-t)$.*

Proof. Configurations $\mathcal{C}$ and $\mathcal{C}^d$ share the same Levi graph and consequently the same quotient graph $G/_{\approx}$, only the role of white and black vertices is reversed. To obtain the same description for the dual of $\mathcal{D} = \mathcal{C}_3(k,(p_1,p_2,\ldots,p_n),t)$ we have to reverse the arrows on edges of $G/_{\approx}$ to point from white to black vertices. We do this by substituting voltage v with $k-v$ on each edge. This voltages can again be rotated around vertices and set to 0 and p_i on double edges and to 0 on all single edges but the last one. On the last single edge we get $p_1+p_2+\cdots+p_n-t$. When we read the voltages along the cycle (in the reverse order) we obtain $\mathcal{C}_3(k,(p_n,p_{n-1},\ldots,p_1),p_1+p_2+\cdots+p_n-t)$ for the dual of $\mathcal{D}$. □

It is also easy to check whether a $\mathcal{C}_3$ configuration is connected or not.

Proposition 5.3. *The configuration $\mathcal{C}_3(k,(p_1,p_2,\ldots,p_n),t)$ is connected if and only if* $\gcd(k,p_1,\ldots,p_n,t)=1$.

Proof. Levi graph of $\mathcal{C}_3$ is a $\mathbb{Z}_k$ covering graph on $G/_{\approx}$. It is easy to see that it is connected precisely when gcd of all voltages and k is 1, see [10]. □

Proposition 5.4. *The configuration $\mathcal{C}_3(k,(p_1,p_2,\ldots,p_n),t)$ is isomorphic to the configurations $\mathcal{C}_3(k,(p_i,p_{i+1},\ldots,p_n,p_1,\ldots,p_{i-1}),t)$, $i=1,2,\ldots,n$.*

Proof. The result is true, since the voltage t can be moved to any single edge. □

This type of configurations admits a simple algorithm which gives a solution of the system of n equations given by the lines in (5.1) and is necessary to obtain a rotational coordinatization.

We can assume that $\mathbf{x}_1=[1,0,1]$. Here we write $\mathbf{x}_i$ for the coordinates of the orbit-representative a_i. The collinearity of $a_1^{p_1}$, a_1, and a_2 implies

$$\mathbf{x}_2=(\alpha_1 I+R^{p_1})\,\mathbf{x}_1,$$

where α_1 is a real parameter, I identity matrix, and R rotation matrix R_k from the definition of the rotational coordinatization. The construction of vectors $\mathbf{x}_i$ can be continued in the following way:

$$\begin{aligned}\mathbf{x}_3&=(\alpha_2 I+R^{p_2})\,\mathbf{x}_2=(\alpha_2 I+R^{p_2})(\alpha_1 I+R^{p_1})\,\mathbf{x}_1,\\ &\cdots\\ \mathbf{x}_{i+1}&=\Big(\prod_{j=i}^{1}(\alpha_j I+R^{p_j})\Big)\mathbf{x}_1=:P_i\,\mathbf{x}_1,\quad i=1,2,\ldots,n-1.\end{aligned}$$

where $\alpha_2,\ldots,\alpha_{n-1}$ are real parameters. Finally, the condition

$$\begin{aligned}&f(\alpha_1,\alpha_2,\ldots,\alpha_{n-1}):=\\ &\quad\det(\mathbf{x}_n,R^{p_n}\mathbf{x}_n,R^q\mathbf{x}_1)=\det(P_{n-1}\,\mathbf{x}_1,R^{p_n}P_{n-1}\,\mathbf{x}_1,R^t\mathbf{x}_1)=0\end{aligned}\tag{5.2}$$

imposed by the last line ensures that the construction closes. This is a necessary condition for $\alpha_1,\ldots,\alpha_{n-1}$ to determine $\mathbf{x}_1,\ldots,\mathbf{x}_n$ in the way that they represent a rotational coordinatization. Note that the choice 0 for some α_i which results in point coincidence must be avoided.

Proposition 5.5. *If there exists a rotational coordinatization of* $\mathcal{C}_3(k,p,t)$ *configuration then the equation* $f(\alpha_1, \alpha_2, \ldots, \alpha_{n-1}) = 0$ *where* f *is defined in* (5.2) *has real solutions with* $\alpha_i \neq 0$ *for all* $i = 1, 2, \ldots, n-1$. □

Here are some examples showing the applications of Proposition 5.5.

Example 5.1. 1. Let us take configurations $\mathcal{C}_3(k, 2^2, 1)$, $k \geq 5$. The equation (5.2) in this case is

$$\alpha_1{}^2 + \alpha_1\left(2 + 2\cos\tfrac{2\pi}{k} + 2\cos\tfrac{4\pi}{k}\right) - 1 - 2\cos\tfrac{2\pi}{k} = 0.$$

This quadratic equation has real solutions for any $k \geq 5$. They determine a rotational realization of the family members. Example for $k = 5$ is in Figure 10, see also Figure 6 in [6].

2. In the next example we take $\mathcal{C}_3(k, 1^3, 1)$. This are cyclic (v_3) configurations with symbol $(1, 2, 4)$ where $v = 3k$. The condition (5.2) in this case is

$$1 - \alpha_1\,\alpha_2 - \alpha_1{}^2\,\alpha_2 - \alpha_1\,\alpha_2{}^2 - 2\,\alpha_1\,\alpha_2\,\cos\tfrac{2\pi}{k} = 0.$$

It is not difficult to see that this equation has real solutions for $k \geq 3$ which actually give rotational realization of these configurations. Examples for $v = 9, 12, 15$ are in Figure 11. The realizations constructed here are typical for these configurations, see for example Figures 3(a), 4(a), 5(a) in [6].

3. Figures 3(b), 4(b), 5(b) in the same paper represent rotational drawings of $\mathcal{C}_3(k, 1^4, 1)$ for $k = 3, 4, 5$. Here we show $\mathcal{C}_3(5, 1^4, 1)$ in Figure 12.

4. Here is a negative example. Configurations $\mathcal{C}_3(k, p^3, 3p)$, $p \geq 1$, do not admit rotational realization for any $k > 3p$. We can see this by showing that the equation (5.2)

$$\alpha_1{}^2 + \alpha_1\,\alpha_2 + \alpha_1{}^2\,\alpha_2 + \alpha_2{}^2 + \alpha_1\,\alpha_2{}^2 + \alpha_1{}^2\,\alpha_2{}^2 +$$
$$2\,\alpha_1{}^2\,\alpha_2\,\cos\tfrac{2p\pi}{k} + 2\,\alpha_1\,\alpha_2{}^2\,\cos\tfrac{2p\pi}{k} + 2\,\alpha_1{}^2\,\alpha_2{}^2\,\cos\tfrac{2p\pi}{k} = 0$$

has no real solutions other than $\alpha_1 = \alpha_2 = 0$.

The $\mathcal{C}_3$ family is defined over even cycles with alternating single and double edges. Similarly, we can define a configuration family over odd Möbius ladder graphs M_{2n+1}. Graph M_m can be defined as a $2m$ cycle with additional edges between antipodal vertices. For example, graphs in Figures 3 and 13 are M_3 and M_5. Only odd ladders are bipartite and can serve as underlying voltage graphs for (v_3) configurations. By rearranging voltages on M_{2n+1} we can achieve that only those on the outer cycle are non-zero; we denote them successively by $p_1, p_2, \ldots p_{2(2n+1)}$. A $\mathbb{Z}_k$ covering graph over such graph can give us a $((k(2n+1))_3)$ k-cyclic configuration which we denote by $\mathcal{M}(k, (p_1, p_2, \ldots, p_{2(2n+1)}))$, $0 \leq p_i < k$. For $n > 3$ every sequence defines some configuration, some care is needed only when $k = 3$. The precise conditions can be derived by the reader himself. Examples of $\mathcal{M}(3, (1, 2, 2, 1, 2, 2))$ and $\mathcal{M}(2, (1, 0, 1, 0, 1, 0, 1, 0, 1, 0))$ are in Figures 3 and 12, respectively.

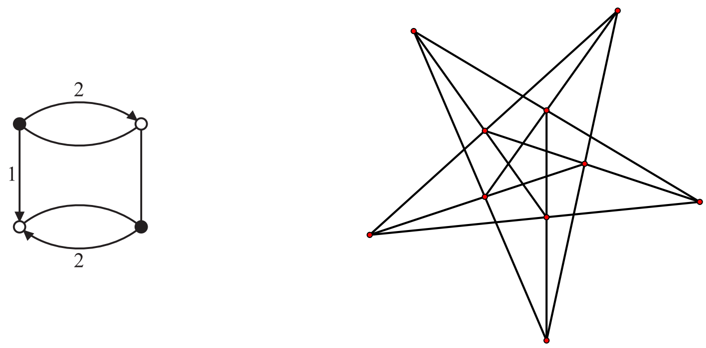

Figure 10: Rotational realization of $\mathcal{C}_3(5, 2^2, 1)$ and its quotient graph; compare [6], Figure 6(a).

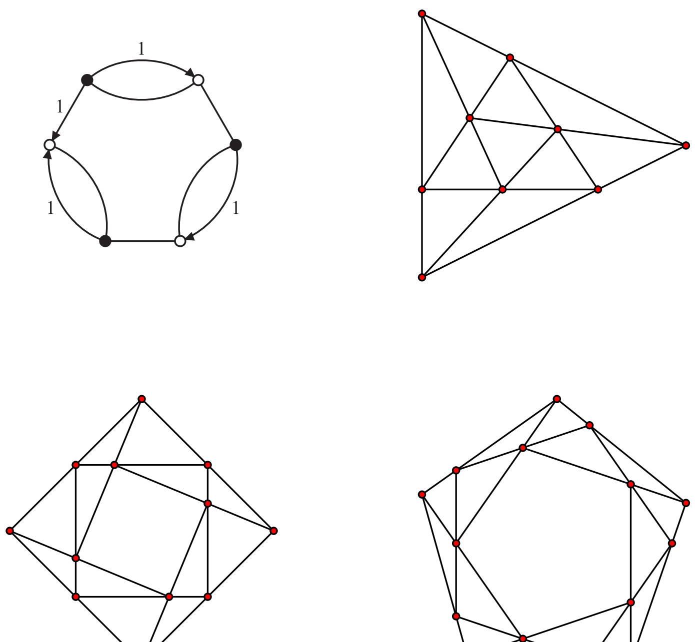

Figure 11: Rotational realization of (9_3), (12_3), and (15_3) configurations $\mathcal{C}_3(k, 1^3, 1)$, $k = 3, 4, 5$, and their quotient graph with the respective voltage groups $\mathbb{Z}_3, \mathbb{Z}_4$, and $\mathbb{Z}_5$; compare [6], Figures 2, 4(a) and 5(a).

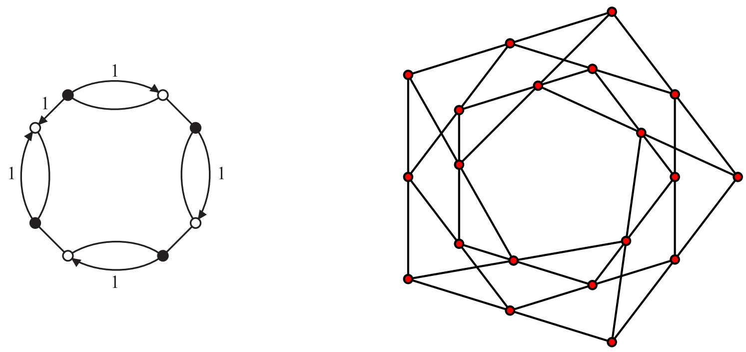

Figure 12: Rotational realization of $\mathcal{C}_3(5, 1^4, 1)$ and its quotient graph; compare [6], Figure 5(b).

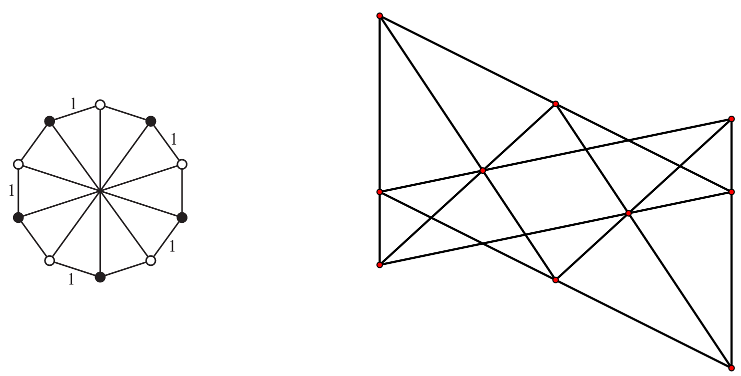

Figure 13: Another rotational realization of $\mathcal{C}_3(5, 2^2, 1)$ configuration already given in Figure 10. This time over Möbius ladder M_5 with the voltage group $\mathbb{Z}_2$.

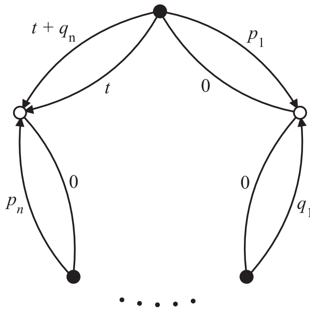

Figure 14: Transitive $G/_{\approx}$ of a (v_4) configuration.

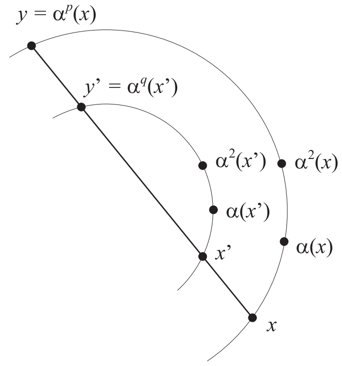

Figure 15: Description of a line in case of $\mathcal{C}_4$ configurations.

6 Polycyclic (v_4) configurations

Here, generalizing the idea from the previous section, observing the local structure of the quotient graph $G/_{\approx}$ we distinguish four cases. Let us restrict our attention to the configurations with transitive quotient graphs as shown in Figure 14, that is, to even cycles with double edges.

Let us first give a description of these configurations. Again we do it by describing the way of attaching the voltages to $G/_{\approx}$. The voltages on the edges can be set to 0 and r_1 to the first (arbitrary) double edge, 0 and r_2 to the adjacent one, etc. This is true since the voltages can be rotated around the vertex as we showed in the description of $\mathcal{C}_3$ configurations in the previous section. In this way we can achieve that one voltage is 0 and the other is $0 \leq r_i \leq k/2$. Of course, this is not true for the last edge. We denote the voltages on it by t and $t + r_{2n}$, $0 \leq r_{2n} \leq k/2$. Therefore, this type of k-cyclic (v_4) configurations can be fully described by specifying k, the sequence $r = (r_1, r_2, \ldots, r_{2n})$, and t. For the reasons which will become clear soon we split the sequence r into two sequences $p = (p_1, p_2, \ldots, p_n)$ and $q = (q_1, q_2, \ldots, q_n)$, where $p_i = r_{2i-1}$ and $q_i = r_{2i}$, $1 \leq i \leq n$ (the sequences of odd and even entries of r). See Figure 14.

Having a voltage graph, we can naturally ask which p, q, t actually give a description of some configuration and when these configurations can be rotationally represented with straight lines or rotationally realized. Answers to these questions gives the following theorem.

Theorem 6.1. *For given $n \geq 2$, $k \geq 7$ the sequences $p = (p_1, p_2, \ldots, p_n)$, $q = (q_1, q_2, \ldots, q_n)$, $1 \leq p_i, q_i < k/2$, and the number t determine a $((nk)_4)$ configuration $\mathcal{C}_4(k, p, q, t)$ if and only if*

$$p_i \neq q_i, \qquad p_i \neq q_{i-1}, \qquad i = 1, 2, \ldots, n. \tag{6.1}$$

For $n = 2$, in addition to (6.1), *there are conditions*

$$a - b + c - d \not\equiv 0 \pmod{k}. \tag{6.2}$$

for any possible choice of a, b, c, d, where $a \in \{0, p_1\}$, $b \in \{0, q_1\}$, $c \in \{0, p_2\}$, $d \in \{t, t + q_2\}$.

Furthermore, this configuration can be rotationally represented with straight lines if and only if

$$\cos\frac{p_1\pi}{k}\cos\frac{p_2\pi}{k}\cdots\cos\frac{p_n\pi}{k}=\cos\frac{q_1\pi}{k}\cos\frac{q_2\pi}{k}\cdots\cos\frac{q_n\pi}{k}, \tag{6.3}$$

$$\#\textit{odd elements in } p+\#\textit{odd elements in } q\equiv 0 \pmod 2. \tag{6.4}$$

In addition to this two conditions, it admits a rotational realization if and only if there exist no i and j, $0<|j-i|<n-1$, and r, $0<r<k/2$, such that the sequences $p'=(r,p_{i+1},\dots,p_j)$ and $q'=(q_i,q_{i+1},\dots,q_j)$ or $p''=(p_i,\dots,p_{j-1},p_j)$ and $q''=(q_i,\dots,q_{j-1},r)$ satisfy the equations (6.3) *and* (6.4).

Remark. Everywhere in this section the addition of elements of p in q is taken mod k and indices denoting the elements of sequences p and q are taken mod_1 n, for example $p_{n+1}=p_1$.

Most of this section will be devoted to the proof of this theorem. We will do this step by step. Let us start with the first part.

Lemma 6.2. *For given $n\geq 2$, $k\geq 7$ the sequences $p=(p_1,p_2,\dots,p_n)$, $q=(q_1,q_2,\dots,q_n)$, $1\leq p_i,q_i<k/2$, and the number t determine a $((nk)_4)$ configuration $\mathcal{C}_4(k,p,q,t)$ if and only if*

$$p_i\neq q_i,\qquad p_i\neq q_{i-1},\qquad i=1,2,\dots,n. \tag{6.5}$$

For $n=2$, in addition to (6.5), *there are conditions*

$$a-b+c-d\not\equiv 0 \pmod k \tag{6.6}$$

for any possible choice of a,b,c,d, where $a\in\{0,p_1\}$, $b\in\{0,q_1\}$, $c\in\{0,p_2\}$, $d\in\{t,t+q_2\}$.

Proof. We use Lemma 1.1. For $n>2$ only the adjacent double edges can be lifted to a 4-cycle. This is possible only when the difference between voltages on them is the same. This is stated in (6.5).

For $n=4$ we must also prevent that none of the 4-cycles lifts to some 4-cycle. All possibilities are listed in (6.6). □

Example 6.1. Figure 16 shows (21_4) configuration of this type. It was discovered by Felix Klein, studied in the complex plane by Coxeter [4] and later shown to exist in the real plane by Grünbaum and Rigby [11]. The corresponding sequences are $p=(1,2,3)$, $q=(3,1,2)$ and $t=0$.

Following the way of observing $\mathcal{C}_3$ configurations let us also add a characterization of the dual configuration to $\mathcal{C}_4(k,p,q,t)$.

Proposition 6.3. *The dual of $\mathcal{C}_4(k,(p_1,p_2,\dots,p_n),(q_1,p_2,\dots,q_n),t)$ is the configuration*

$$\mathcal{C}_4(k,(q_n,\dots,q_1),(p_n,\dots,p_1),p_1+\cdots+p_n-q_1-\cdots-q_n-t).$$

Proof. We simply imitate the proof of Lemma 5.2. First we reverse the direction of edges by substituting voltages v with $-v$, then we return them back to their original values by rotating them around vertices. The values which help us rotate the voltages accumulate on the last edge giving us the new value for t. □

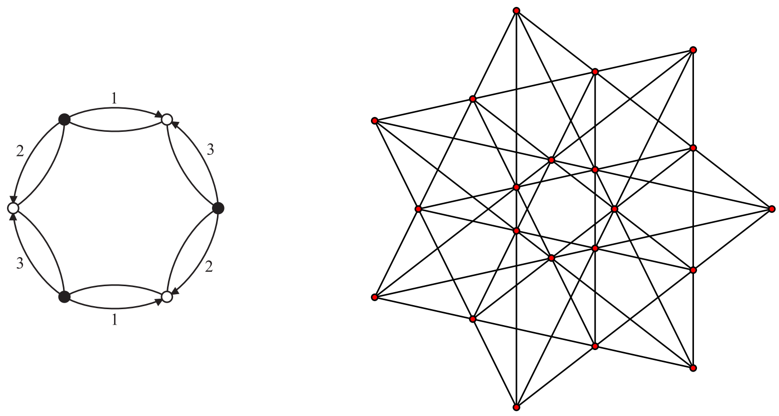

Figure 16: (21_4) configuration $\mathcal{C}_4(7,(1,2,3),(3,1,2),0)$ along its quotient graph.

We add another two simple observations.

Proposition 6.4. *The configuration* $\mathcal{C}_4(k,(p_1,p_2,\ldots,p_n),(q_1,q_2,\ldots,q_n),t)$ *is connected if and only if* $\gcd(k,p_1,\ldots,p_n,q_1,\ldots,q_n,t)=1$.

Proof. We tell once more what we did in the case of $\mathcal{C}_3$ configurations. The Levi graph of $\mathcal{C}_4$ which is a $\mathbb{Z}_k$ covering graph on $G/_\approx$ is connected precisely when gcd of all voltages and k is 1. □

Proposition 6.5. *The configuration* $\mathcal{C}_4(k,(p_1,p_2,\ldots,p_n),(q_1,q_2,\ldots,q_n),t)$ *is isomorphic to*

$$\mathcal{C}_4(k,(p_i,p_{i+1},\ldots,p_n,p_1,\ldots,p_{i-1}),(q_i,q_{i+1},\ldots,q_n,q_1,\ldots,p_{i-1}),t) \quad \textit{and}$$
$$\mathcal{C}_4(k,(p_i,p_{i-1},\ldots,p_1,p_n,\ldots,p_{i+1}),(q_i,q_{i-1},\ldots,q_1,q_n,\ldots,p_{i+1}),t)$$

for each $i=1,2,\ldots,n$.

Proof. This is true since the voltage t on $G/_\approx$ can be moved around the cycle to any pair of double edges. □

Now we reveal the geometric meaning of p, q, and t. Taking a line ℓ from line-orbit $\mathcal{P}_i$ of $\mathcal{C}_4(k,p,q,t)$ it has two points x and y in one point-orbit $\mathcal{O}_i$ and the other two points x' and y' in orbit $\mathcal{O}_{i+1}$. The difference of the voltages on a double edge between $\mathcal{P}_i$ and $\mathcal{O}_i$ is p_i. This means that $y=\alpha^{p_i}(x)$. Similarly we find out that $y'=\alpha^{q_i}(x')$. This is schematically shown in Figure 15. Now, we can understand p_i and q_i as "jumps" the line makes on the points of the orbit $\mathcal{O}_i$ and $\mathcal{O}_{i+1}$, respectively.

Since we are interested in rotational realizations we further investigate this topic in case of $\mathcal{C}_4(k,p,q,t)$ configurations.

Lemma 6.6. *If a rotational realization of* $\mathcal{C}_4(k,p,q,w)$ *exists then the equation*

$$\cos\frac{p_1\pi}{k}\,\cos\frac{p_2\pi}{k}\cdots\cos\frac{p_n\pi}{k}=\cos\frac{q_1\pi}{k}\,\cos\frac{q_2\pi}{k}\cdots\cos\frac{q_n\pi}{k}. \tag{6.7}$$

holds and

$$w = \frac{1}{2}\sum_{i=1}^{n}(p_i - q_i) \tag{6.8}$$

is an integer. Hence the rotational realization of $\mathcal{C}(k,p,q,w)$ uniquely determines w.

Remark. We will refer to w defined in (6.8) as *twist.* It is an integer precisely when

$$\#\text{odd elements in } p + \#\text{odd elements in } q \equiv 0 \pmod 2. \tag{6.9}$$

We will also write $\mathcal{C}_4(k,p,q)$ instead of $\mathcal{C}_4(k,p,q,t)$ when we take $t = \lfloor w \rfloor$.

Proof. Let R_i be the radius of the circle representing orbit $\mathcal{O}_i$. If we want the line ℓ to be "straight" then we have to set the radius R_{i+1} of the next circle (orbit $\mathcal{O}_{i+1}$) to

$$R_{i+1} = R_i \frac{\cos\frac{p_i\pi}{k}}{\cos\frac{q_i\pi}{k}}, \qquad i = 1, 2, \ldots, n. \tag{6.10}$$

In order to "close" the construction there must be $R_1 = R_{n+1}$. After recursively substituting R_{i+1} with $R_i \cos\frac{p_i\pi}{k} / \cos\frac{q_i\pi}{k}$ and finally canceling R_1 we get (6.7).

But the condition (6.7) is not sufficient for a configuration $\mathcal{C}_4(k,p,q,t)$ to be rotationally drawn with straight lines. The crucial is also the choice of the parameter t. We will show that t is uniquely determined if it is possible to obtain a straight-line drawing of $\mathcal{C}_4(k,p,q,t)$ by this method.

In the process of drawing a configuration which we described, we notice the following. The first point on the orbit $\mathcal{O}_{i+1}$ is shifted for $\frac{1}{2}(p_i - q_i)$ with regard to the first one on the orbit $\mathcal{O}_i$. When the construction closes, we must take into account the sum of this shifts (w) over all orbits if we want the lines between $\mathcal{O}_n$ and $\mathcal{O}_1$ to be straight. In the language of voltages this shift is exactly t. If we take $t = w$ and w is an integer, then the line through points in $\mathcal{O}_n$ hits $\mathcal{O}_1$ in some point of the configuration. Of course, equation (6.7), which handles the sizes of the radii of the orbits, must also hold. □

Example 6.2. When q is a permutation of p then the condition (6.7) is trivially satisfied. This situation occurred in example 6.1. But the equality holds also for $k = 17$, $p = (1,8,4,2)$, $q = (3,7,5,6)$ and $k = 12$, $p = (1,5)$, $q = (4,4)$. The corresponding configurations are shown in Figures 18 and 17. Table 1 contains sequences p and q for small k for which (6.7) holds and $p \cap q = \emptyset$.

To obtain a (strong) rotational realization we must avoid two situations. The first is a coincidence of points.

It is possible that two subsequences $p' = (p_i, \ldots, p_j)$ and $q' = (q_i, \ldots, q_j)$ of p and q already satisfy condition (6.7). In this case the orbits i and j lie on the same circle. Furthermore, if (6.9) holds for p' and q' then the points on these two orbits coincide, otherwise they are shifted for $\frac{\pi}{k}$.

Example 6.3. The twists w in examples in Figures 16, 17, and 18, are 0, -1, and -3 respectively. Two examples where the cosine equation (6.7) holds, but w is not an integer are $\mathcal{C}_4(15,(1,4),(3,3))$ and $\mathcal{C}_4(15,(5,5),(3,6))$ with $w = -\frac{1}{2}$ and $w = \frac{1}{2}$. Combining them we get configuration $\mathcal{C}_4(15,(1,4,5,5),(3,3,3,6))$ with $w = 0$. Here two orbits coincide, but the points on them are shifted. See Figure 19.

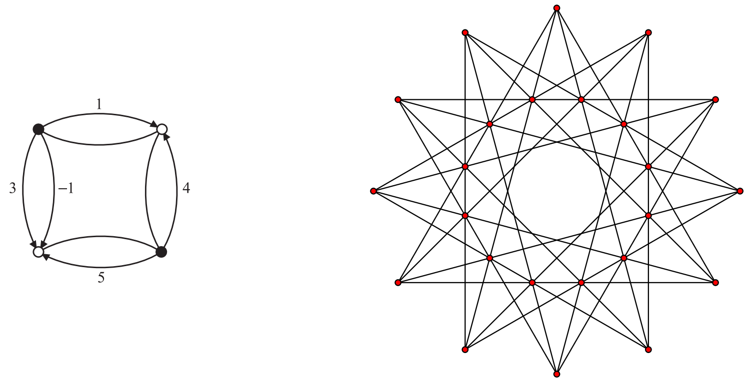

Figure 17: Configuration $\mathcal{C}_4(12, (1,5), (4,4), -1)$.

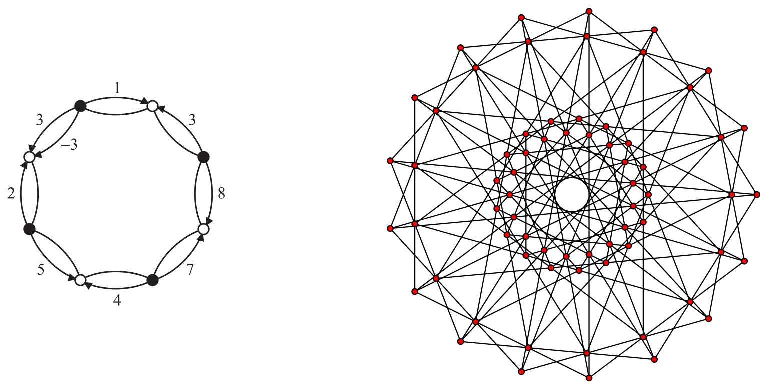

Figure 18: Configuration $\mathcal{C}_4(17, (1,8,4,2), (3,7,5,6), -3)$.

k	p, q
12	$(1,5), (4,4)$
15	$(\mathbf{1}, \mathit{4}), (\mathbf{3}, \mathbf{3})$; $(\mathit{2}, \mathbf{7}), (\mathit{6}, \mathit{6})$; $(\mathbf{3}, \mathit{6}), (\mathbf{5}, \mathbf{5})$
18	$(1,6), (4,5)$; $(1,8), (7,6)$; $(2,7), (5,6)$
24	$(1,11), (8,10)$; $(2,8), (5,7)$; $(2,10), (8,8)$; $(3,9), (6,8)$
9	$(1,2,4), (3,3,3)$
10	$(1,1,4), (2,3,3)$
15	$(1,4,6), (3,5,5)$; $(2,3,7), (5,5,6)$
14	$(1,1,6), (3,4,5)$; $(1,2,5), (3,3,4)$; $(1,3,6), (2,5,5)$
16	$(1,2,7), (3,5,6)$; $(1,4,7), (2,6,6)$; $(2,2,6), (3,4,5)$
17	$(1,2,4,8), (3,5,6,7)$

Table 1: Sequences p and q satisfying conditions (6.7) and $p \cap q = \emptyset$ for small k.

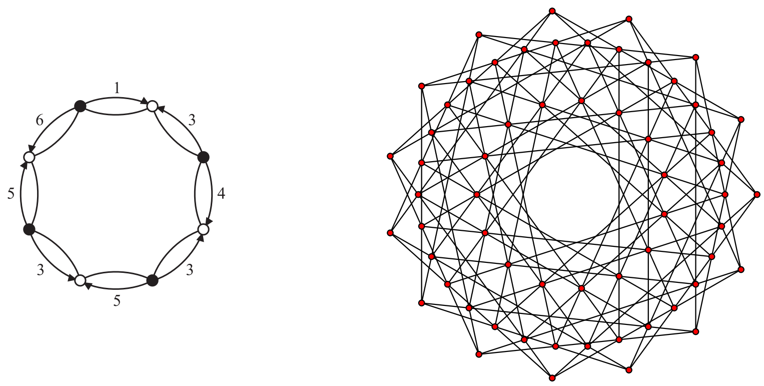

Figure 19: Configuration $\mathcal{C}_4(15,(1,4,5,5),(3,3,3,6))$ has the property that there are four vertex orbits but only three radii.

In the other situation we try to avoid additional incidences. This means that no line is (geometrically) incident with more than four points.

Lemma 6.7. *For a given k let us have the sequences $p = (p_1, p_2, \ldots, p_n)$, $q = (q_1, q_2, \ldots, q_n)$, $0 < p_i, q_i < k/2$, satisfying* (6.7) *and* (6.9)*. Then there are lines in a rotational representation of $\mathcal{C}_4(k,p,q)$ which are geometrically incident (i.e. in the Euclidean plane) with more than four points if and only if there exist two indices i and j, $0 < |j-i| < n-1$, and r, $0 < r < k/2$, such that the sequences $p' = (r, p_{i+1}, \ldots, p_j)$ and $q' = (q_i, q_{i+1}, \ldots, q_j)$ or $p'' = (p_i, \ldots, p_{j-1}, p_j)$ and $q'' = (q_i, \ldots, q_{j-1}, r)$ satisfy the equations* (6.7) *and* (6.9)*.*

Proof. Additional incidences can occur precisely when there exists a $\mathcal{C}_4$ "subconfiguration" satisfying (6.7) and (6.9). By (6.10), the radii of this configuration are

$$R_i, R_{i+1} = R_i \frac{c_k(p_i)}{c_k(q_i)}, \ldots, R_i = R_{j-1} \frac{c_k(p_j)}{c_k(r)}, \tag{6.11}$$

$c_k(x) = \cos(x\pi/k)$, for indices i, j and some r, $0 < r < k/2$. Here, the "jump" r uses the additional incidence to obtain a subconfiguration. If we want it to be rotationally realizable then the equations (6.7) and (6.9) must hold for the sequences $(p_i, \ldots, p_{j-1}, p_j)$ and $(q_i, \ldots, q_{j-1}, r)$ by Lemma 6.6 and (6.11). The other possibility is obtained by a similar argument or observing that $\mathcal{C}_4(k,p,q) = \mathcal{C}_4(k, (q_n, q_{n-1}, \ldots, q_1), (p_n, p_{n-1}, \ldots, p_1))$. □

Example 6.4. For $p = (1,2,5,3)$, $q = (3,4,4,2)$. According to Table 1 sequences $p' = (1,5)$ and $q' = (4,4)$ cancel for $k = 12$ and satisfy (6.9). By the previous Lemma, the rotational drawing of $\mathcal{C}_4(12,(1,2,5,3),(3,4,4,2))$ contains lines with more than four points on them, see Figure 20. Here $i = 2$, $j = 3$, $r = 1$. Another example is $\mathcal{C}_4(18,(1,6,4),(4,1,6))$. The sequences from Lemma 6.7 are $p'' = (1,6)$ and $q'' = (4,5)$ which satisfy (6.7) and (6.9).

Finally we list some references to the papers with the related contents. In [1, 7, 8, 9] our rotationally realizable $\mathcal{C}_4(k,(p_1,p_2),(q_1,q_2))$ configurations are called *astral* while the other $\mathcal{C}_4$ configurations fall into a family of *stelar configurations*.

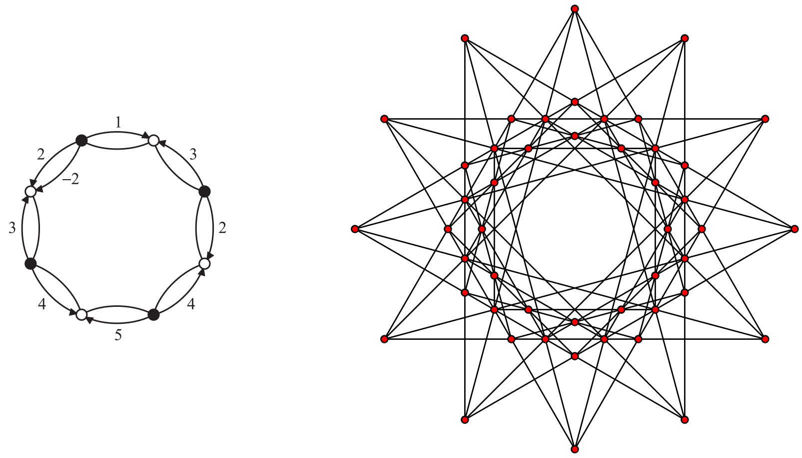

Figure 20: The weak rotational realization of $\mathcal{C}_4(12,(1,2,5,3),(3,4,4,2))$ contains additional incidences, lines with six points on them.

In [1] there is also a list of infinite families and sporadic astral configurations. They correspond to the solutions of the cosine equation (6.7).

Reference [14] gives a construction of weakly flag-transitive (v_4) configurations. All of the constructed configurations belong to the clall of $\mathcal{C}_4(k,p,q,t)$ configurations. It transpires that the smallest known weakly flag-transitive configuration is $\mathcal{C}_4(9,(1,4,2),\ (2,1,4))$ and that the smallest known weakly flag-transitive non-self polar configuration is $\mathcal{C}_4(17,(1,8,4,2),(3,7,5,6))$.

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