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# GENERALIZATION OF CARATHEODORY'S INEQUALITY AND THE BOHR RADIUS FOR MULTIDIMENSIONAL POWER SERIES

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# Generalization of Caratheodory's inequality and the Bohr radius for multidimensional power series *

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**Abstract**

In the present paper we generalize the Caratheodory's inequality for functions in one or several complex variables. New results concerning some multidimensional analogues of Bohr's theorem on power series are obtained.

## 1 Caratheodory's inequality in Several Complex variables

Let $\mathcal{D} \subset \mathbf{C}^n$ be a circular domain, that is a Cartan domain, characterized by the fact that if $z \in \mathcal{D}$ then $ze^{i\phi} \in \mathcal{D}$, where $z = (z_1, \ldots, z_n)$ and $0 \leq \phi \leq 2\pi$.

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Furtermore, we assume also that the domain $\mathcal{D}$ is a strongly starlike domain, that is, for every homothetic transformation $\lambda\overline{\mathcal{D}} \subset \mathcal{D}$, where $0 < \lambda < 1$.
It is a well known fact that in this type of domains every holomorphic function can be expanded into a series of homogeneous polynomials. Now we are ready to formulate our first lemma.

**Lemma 1.1** *Let $\mathcal{D}$ be a Cartan, strongly starlike domain. Let $f$ be a holomorphic function in $\mathcal{D}$, with expansion into homogeneous polynomials given by*

$$f(z) = \sum_{k=0}^{\infty} P_k(z), \ z \in \mathcal{D}, \tag{1.1}$$

*where $P_k(z)$ is a homogeneous polynomial of degree $k$. If $\Re f(z) > 0$ for every $z \in \mathcal{D}$ then for every $k \geq 1$ the following inequality holds*

$$|P_k(z)| \leq 2\Re P_0(z), \ \forall z \in \mathcal{D} \tag{1.2}$$

**Proof:** It is known that for every $k = 1, \ldots,$the polynomials $P_k(z)$ in the expansion (1.1) are given by

$$\begin{aligned} P_k(z) &= \lim_{r \to 1+0} \frac{1}{2\pi r^k} \int_0^{2\pi} f(zre^{i\phi}) e^{-ik\phi} d\phi \\ &= \lim_{r \to 1+0} \frac{1}{2\pi r^k} \int_0^{2\pi} \Big( f(zre^{i\phi}) + \overline{f(zre^{i\phi})} \Big) e^{-ik\phi} d\phi \end{aligned}$$

Therefore

$$\begin{aligned} |P_k(z)| &\leq \lim_{r \to 1+0} \frac{1}{2\pi r^k} \int_0^{2\pi} 2\Re f(zre^{i\phi}) d\phi = \lim_{r \to 1+0} \frac{1}{r^k} 2\Re f(0) \\ &= \lim_{r \to 1+0} \frac{1}{r^k} 2\Re P_0(z) = 2\Re P_0(z) \end{aligned}$$

This completes the proof of the lemma. $\diamondsuit$

Lemma 1.1 and the simple fact that every convex domain in the complex plane **C** is intersection of half-planes allows us to deduce the following theorem.

**Theorem 1.1** *Let $\mathcal{D}$ be a Cartan domain and $f$ be a function holomorphic in $\mathcal{D}$ . If in the domain $\mathcal{D}$ the expansion* (1.1) *for the function $f$ is valid and $f(\mathcal{D}) \subset G \subset \mathbf{C}$, then for every $z \in \mathcal{D}$ the following inequality holds for every $k \geq 1$*

$$|P_k(z)| \leq 2dist(P_0(z), \partial\widetilde{G}),$$

*where $\widetilde{G}$ is the convex hull of $G$.*

**Remark 1.1** *We point out here that in the above theorem $\widetilde{G}$ cannot be replaced by $G$.*

Theorem 1.1 has a number of interesting corollaries. We state them without proofs.

**Corollary 1.1** *Let $\mathcal{D}$ be a Cartan domain and $f$ be a function holomorphic in $\mathcal{D}$. If $|f(z)| < 1$ for every $z \in \mathcal{D}$, then for every $k \geq 1$ the following holds*

$$|P_k(z)| \leq 2(1 - |P_0(z)|), \ \forall z \in \mathcal{D} \tag{1.3}$$

**Corollary 1.2** *Let $\mathcal{D}$ be a complete, bounded, Reinhardt domain. Let $f$ be a function holomorphic in $\mathcal{D}$ with the corresponding multidimensional power series*

$$f(z) = \sum_{|\alpha| \geq 0} c_\alpha z^\alpha, \ z \in \mathcal{D}, \tag{1.4}$$

*where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$, $z^\alpha = z_1^{\alpha_1} \ldots z_n^{\alpha_n}$ and all $\alpha_i$ are nonegative integers. If $f(\mathcal{D}) \subset G \subset \mathbf{C}$, then for every $\alpha$ such that $|\alpha| \geq 1$ the following holds*

$$|c_\alpha| \leq \frac{2dist(c_0, \partial\widetilde{G})}{d_\alpha},$$

*where $d_\alpha(\mathcal{D}) = max_{\overline{\mathcal{D}}}|z^\alpha|$.*

We point out here that the proof of the Corollary 1.2 follows from Theorem 1.1 and the Cauchy estimates in the case of power series taken from [5].
In one complex variable Theorem 1.1 leads to

**Corollary 1.3** *If in the unit disk* $K = \{z_1 \in \mathbf{C} : |z_1| < 1\}$ *the function* $f$ *is a power series, that is*

$$f(z_1) = \sum_{k=0}^{\infty} c_k z_1^k \tag{1.5}$$

*and* $f(K) \subset G$, *then for every* $k \geq 1$ *we have*

$$|c_k| \leq 2dist(c_0, \partial\widetilde{G})$$

The following result is the known Caratheodory's inequality [10]

**Corollary 1.4** *If in the unit disk* $K = \{z_1 \in \mathbf{C} : |z_1| < 1\}$ *the equality* (1.5) *holds and if* $\Re f(z_1) > 0$ *for every* $z_1 \in K$ *then*

$$|c_k| \leq 2\Re c_0, \ \forall k \geq 1$$

Another classical inequality, known as Landau's inequality [13], is also deduced from Theorem 1.1

**Corollary 1.5** *If in the unit disk* $K = \{z_1 \in \mathbf{C} : |z_1| < 1\}$ *the equality* (1.5) *holds and if* $|f(z_1)| < 1$ *for every* $z_1 \in K$ *then*

$$|c_k| \leq 2(1 - |c_0|), \ \forall k \geq 1$$

# 2 Bohr's theorem in several complex variables

Let us recall the theorem of H.Bohr [9], proven at the beginning of the 20th century.

**Theorem 2.1** *If a power series* (1.5) *converges in the unit disk* $K$ *and its sum has modulus less than* 1, *then*

$$\sum_{k=0}^{\infty} |c_k z_1^k| < 1$$

*in the disk* $\{z_1 \in \mathbf{C} : \ |z_1| < \frac{1}{3}\}$. *Moreover the constant* $\frac{1}{3}$ *cannot be improved.*

Formulations of Bohr's theorem in several complex variables appeared very recently. We recall some of them.
Given a complete Reinhardt domain $\mathcal{D}$, we denote by $R(\mathcal{D})$ the largest non-negative number $r$ witht he property that if the power series (1.4) converges in $\mathcal{D}$ and its sum has modulus less than 1, then

$$\sum_{|\alpha|\geq 0} |c_\alpha z^\alpha| < 1, \tag{2.1}$$

in the homothety $r\mathcal{D}$. In [8] the following result is proved in case $\mathcal{D}$ is the unit polydisk

$$U^n = \{z \in \mathbf{C}^n : \ |z_j| < 1, \ j = 1, \ldots, n\}$$

**Theorem 2.2** *For $n > 1$ one has*

$$\frac{1}{3\sqrt{n}} < R(U^n) < \frac{2\sqrt{logn}}{\sqrt{n}}$$

We see from Theorem 2.2 that $R(U^n) \to 0$ when $n \to \infty$.
If $\mathcal{D}$ is the hypercone

$$\mathcal{D}_1^n = \{z \in \mathbf{C}^n : \ |z_1| + \ldots + |z_n| < 1\},$$

then the situation is quite different as the following theorem, taken from [1], shows.

**Theorem 2.3** *For $n > 1$ one has*

$$\frac{1}{3e^{\frac{1}{3}}} < R(\mathcal{D}_1^n) < \frac{1}{3}$$

For further estimates of $R(\mathcal{D})$ in the domains

$$\mathcal{D}_p^n = \{z \in \mathbf{C}^n : \ |z_1|^p + \ldots + |z_n|^p < 1\},$$

where $1 \leq p < \infty$, we refer the reader to [7]. For other generalizations of the Bohr's theorem see [2], [3],[4], [6], [12].
As it was pointed out already in [1], it seems more natural to consider not a single number in the Bohr problem in $\mathbf{C}^n$, but the largest subdomain $\mathcal{D}_B$ of $\mathcal{D}$, such that (2.1) holds. At this stage, we state the following new result in this direction.

**Theorem 2.4** *If the power series* (1.4) *converges in the unit ball*

$$\mathcal{D}_2^n = \{z \in \mathbf{C}^n : \ |z_1|^2 + \ldots + |z_n|^2 < 1\}$$

*and the modulus of its sum there is less than* 1*, then* (2.1) *holds in the hypercone*

$$\frac{1}{3}\mathcal{D}_1^n = \{z \in \mathbf{C}^n : \ |z_1| + \ldots + |z_n| < 1\}$$

*and the constant* $\frac{1}{3}$ *cannot be improved.*

The proof of the Theorem 2.4 is based on the following lemma.

**Lemma 2.1** *If* $P_k(z)$ *is the homogeneous polynomial*

$$P_k(z) = \sum_{|\alpha|=k} c_\alpha z^\alpha$$

*and* $|P_k(z)| < 1$ *in the ball* $\mathcal{D}_2^n$*, then*

$$\sum_{|\alpha|=k} |c_\alpha z^\alpha| < 1$$

*for every* $z$ *in the hypercone* $\mathcal{D}_1^n$*.*

**Proof:** Using induction, one can prove the inequality

$$\sqrt{\frac{|\alpha|^{|\alpha|}}{\alpha_1^{\alpha_1} \ldots \alpha_n^{\alpha_n}}} \leq \frac{|\alpha|!}{\alpha_1! \ldots \alpha_n!}, \tag{2.2}$$

From Cor.1.2 follows that

$$|c_\alpha| \leq \frac{1}{d_\alpha(\mathcal{D}_2^n)} = \sqrt{\frac{|\alpha|^{|\alpha|}}{\alpha_1^{\alpha_1} \ldots \alpha_n^{\alpha_n}}}$$

From this, the relation (2.2) implies that for every $z \in \mathcal{D}_1^n$ we have

$$\begin{aligned} \sum_{|\alpha|=k} |c_\alpha z^\alpha| &\leq \sum_{|\alpha|=k} \frac{|\alpha|!}{\alpha_1! \ldots \alpha_n!} |z_1|^{\alpha_1} \ldots |\alpha_n|^{\alpha_n} \\ &= (|z_1| + \ldots + |z_n|)^k < 1 \end{aligned}$$

The proof of the lemma is now complete. $\diamondsuit$
We remark here that the condition $|P_k(z)| < 1$ in the ball $\mathcal{D}_2^n$ cannot be replaced by the analogous inequality in the hypercone $\mathcal{D}_1^n$ as the example of second order homogeneous polynomial

$$Q(z) = \frac{\sqrt{3}}{2}(z_1^2 + z_2^2) + 3iz_1z_2$$

shows. Actually, for $z \in \overline{\mathcal{D}}_1^2$, we have that $|Q(z)| \leq 1$, but

$$Max_{\overline{\mathcal{D}}_1^2}\Big[\frac{\sqrt{3}}{2}(|z_1|^2 + |z_2|^2) + 3|z_1z_2|\Big] = \frac{3+\sqrt{3}}{4} > 1$$

Now, we are ready to to return to the proof of the Theorem 2.4.
**Proof of the Theorem 2.4:** From the assumption $|f(z)| < 1$ in the ball $\mathcal{D}_2^n$ and the Corollary 1.1 we get that for every $k \geq 1$ the estimate

$$|\sum_{|\alpha|=k} c_\alpha z^\alpha| \leq 2(1-|c_0|)$$

holds for every $z \in \mathcal{D}_2^n$. From this and the lemma 2.1 follows that for every $z \in \mathcal{D}_1^n$ one has that

$$\sum_{|\alpha|=k} |c_\alpha z^\alpha| \leq 2(1-|c_0|)$$

Now, if $z \in \frac{1}{3}\mathcal{D}_1^n$, then

$$\sum_{|\alpha|=k} |c_\alpha z^\alpha| \leq |c_0| + 2(1-|c_0|)\sum_{k=1}^{\infty}\frac{1}{3^k} = 1$$

The fact that the constant $\frac{1}{3}$ cannot be improved can be seen if we consider the function of the type $f(z_1, 0, \ldots, 0)$. This completes the proof of the theorem.$\diamondsuit$
In order to improve the estimate $R(\mathcal{D}_1^n) > \frac{1}{e^{\frac{1}{3}}} = 0.238844$ taken from the Theorem 2.3 in the case $n = 2$, we use the following lemma taken from [11], [14].

**Lemma 2.2** *Let*

$$F(t) = \sum_{j=-n}^{n} a_j e^{ijt}$$

*be a real trigonometric polynomial. Then*

$$|a_o| + |a_{-k}| + |a_k| \leq max_t |F(t)|$$

The final theorem of the paper is the following one.

**Theorem 2.5**

$$R(\mathcal{D}_1^2) > 0.304236$$

**Proof:** If for every $z \in \overline{\mathcal{D}}_1^2$ one has that

$$|P_k(z)| = |\sum_{|\alpha|=k} c_\alpha z^\alpha| \leq 1,$$

then in particular this holds for the points of the form $(\frac{1}{2}, \frac{1}{2}e^{it})$ and we obtain

$$|\sum_{\substack{\alpha_2=0 \\ \alpha_1=k-\alpha_2}}^{k} c_{\alpha_1,\alpha_2} e^{it\alpha_2}| \leq 2^k$$

From this we deduce that

$$\begin{aligned} |P_k(\frac{1}{2}, \frac{e^{it}}{2})|^2 &= \sum_{\alpha_1+\alpha_2=k} |c_{\alpha_1,\alpha_2}|^2 + c_{0,k}\bar{c}_{k,0} e^{ik\phi} + \\ &+ \bar{c}_{0,k} c_{k,0} e^{-ik\phi} + \ldots \leq 2^{2k} \end{aligned}$$

Lemma 2.2 implies

$$\sum_{\alpha_1+\alpha_2=k} |c_{\alpha_1,\alpha_2}|^2 + 2|c_{0,k}||c_{k,0}| \leq 2^{2k}$$

or

$$\sum_{\alpha_1=1}^{k=1} |c_{\alpha_1,k-\alpha_1}|^2 + (|c_{0,k}| + |c_{k,0}|)^2 \leq 2^{2k} \qquad (2.3)$$

Now, we find that under the condition (2.3) the following is valid

$$max_{\substack{x_1+x_2\leq 1\\ x_1,x_2\geq 0}} \sum_{\alpha_1+\alpha_2=k} |c_{\alpha_1,\alpha_2}|x_1|^{\alpha_1}|x_2|^{\alpha_2} = A_k$$

It is not very difficult to show that

$$A_k = \frac{1+\sqrt{(k-1)(2^{2(k-1)}-1)}}{2^{k-1}}$$

Furthermore, we consider now the equation

$$\sum_{k=1}^{\infty} A_k x^k = \sum_{k=1}^{\infty} \frac{1+\sqrt{(k-1)(2^{2(k-1)}-1)}}{2^{k-1}} x^k = \frac{1}{2} \tag{2.4}$$

Using the program Mathematica 3.0 [15], we estimated a root of (2.4): the equation (2.4) has a root greater than $x_0 = 0.304236$. In addition, if $z \in x_0\mathcal{D}_1^2$ then

$$\sum_{|\alpha|=0}^{\infty} |c_\alpha||z^\alpha| < |c_0| + (\sum_{k=1}^{\infty} A_k x^k) 2(1-|c_0|) < 1$$

This completes the proof of the theorem. $\diamondsuit$

**Remark 2.1** *The paper by Boas and Khavinson [8] contains essentially the following result (Remark 4 in [1]): If the power series* (1.4) *converges in the unit polydisk* $U^n$ *and the modulus of its sum is less than* 1*, then* (2.1) *holds in the ball* $\frac{1}{3}\mathcal{D}_2^n$ *and the constant* $\frac{1}{3}$ *cannot be improved.*

**Remark 2.2** *The estimates of Bohr radius from the Theorem* 2.3 *hold also for every Reinhardt domain of the type*

$$\mathcal{D} = \{z \in \mathbf{C}^n : \ \phi(|z_1|, \ldots, |z_n|) < 0\},$$

*where* $\phi$ *is a convex function, that is,* $\mathcal{D}$ *is the union of hypercones*

$$\{z \in \mathbf{C}^n : \ a_1|z_1| + \ldots + a_n|z_n|) < 1\}$$

*The same holds about the estimates from the Theorem* 2.5*.*

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