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CIRCULAR COLORINGS OF EDGE-WEIGHTED GRAPHS

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Abstract

The notion of (circular) colorings of edge-weighted graphs is introduced. This notion generalizes the notion of (circular) colorings of graphs, the channel assignment problem, and several other optimization problems. For instance, its restriction to colorings of weighted complete graphs corresponds to the traveling salesman problem (metric case). It also gives rise to a new definition of the chromatic number of directed graphs. Several basic results about the circular chromatic number of edge-weighted graphs are derived.

1 Introduction

The theory of circular colorings of graphs has become an important branch of chromatic graph theory with many interesting results, leading to new methods and exciting new results. We refer to the survey article by Zhu [9].

In this paper, circular colorings of edge-weighted graphs are introduced. This notion contains, as special cases, several other optimization problems, e.g., the channel assignment problem. When restricted to complete graphs, it generalizes the traveling salesman problem and the hamiltonicity problem. Edge-weights need not be symmetric. This possibility leads to a new definition of colorings of directed graphs.

A *weighted graph* is a pair $G = (V, A)$, where V is the vertex set and $A : V \times V \to \mathbb{R}^+ \cup \{0\}$ are the edge-weights. For $u, v \in V$, we shall write

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$a_{uv} = A(u,v)$. The set $\vec{E}$ of ordered pairs, $\vec{E} = \{(u,v) \mid a_{uv} > 0\}$ is called the set of *directed edges* of G. The unordered pair $\{u,v\}$, shortly written as uv or vu, is an *edge* of G if $a_{uv} > 0$ or $a_{vu} > 0$. The set of edges is denoted by $E = E(G)$. For $(u,v) \in \vec{E}$, a_{uv} is called the *weight* of the directed edge (u,v). It is allowed that $a_{uv} \neq a_{vu}$, and if this happens for at least one edge, then we say that the weights are *nonsymmetric*. Throughout this paper, it will be assumed that G has no loops, i.e., $a_{vv} = 0$ for every $v \in V$.

For a positive real number p, denote by $S_p \subset \mathbb{R}^2$ the circle with radius $\frac{p}{2\pi}$ (hence with perimeter p) centered at the origin of $\mathbb{R}^2$. In the obvious way, we can identify the circle S_p with the set $\mathbb{R}/p\mathbb{Z}$. For $x,y \in S_p$, let us denote by $S_p(x,y)$ the arc on S_p from x to y in the clockwise direction, and let $\mathrm{d}(x,y)$ denote the length of this arc.

Let $G = (V,A)$ be a weighted graph with at least one edge. A *circular p-coloring* of G is a function $c : V \to S_p$ such that for every directed edge $(u,v) \in \vec{E}$, $\mathrm{d}(c(u),c(v)) \geq a_{uv}$. Since $\mathrm{d}(c(u),c(v)) + \mathrm{d}(c(v),c(u)) = p$, a necessary condition for existence of a circular p-coloring is that

$$p \geq \max\{a_{uv} + a_{vu} \mid u,v \in V\}. \tag{1}$$

The *circular chromatic number* $\chi_c(G)$ of the edge-weighted graph G is the infimum of all real numbers p for which there exists a circular p-coloring of G. It will be shown later that the infimum is indeed attained, i.e., there exists a circular $\chi_c(G)$-coloring of G.

The circular chromatic number of weighted graphs introduced above generalizes some other graph invariants and can be used as a model for several well-known optimization problems.

(a) If all edge-weights are equal to 1, then $\chi_c(G)$ is the usual circular chromatic number of G (cf., e.g., [9]).

(b) If there is a function $f : V \to \mathbb{R}^+$, and weights of edges are defined as $a_{uv} = f(u)+f(v)$, then we get the notion of weighted circular colorings that were studied by Deuber and Zhu [1].

(c) Let G be an arbitrary (unweighted) graph with vertex set V. Let K_G be the complete graph with the same vertex set as G and edge-weights 1 (for edges of G) and 2 (for nonedges of G). Then $\chi_c(K_G) = |V|$ if and only if G has a hamiltonian cycle. If G has no hamiltonian cycle, then $\chi(K_G) = |V| + \mathrm{la}(G)$, where $\mathrm{la}(G)$ is the *linear arboricity* of G, i.e., the minimum number of paths whose vertex sets partition $V(G)$. This example shows that computation of the weighted circular chromatic number is NP-hard even for complete graphs with edge-weights 1 and 2 only.

(d) Let $D = [d_{uv}]_{u,v\in V}$ be the cost matrix for a metric traveling salesman problem (TSP), i.e., D satisfies the triangular inequality. Then every circular p-coloring of the weighted complete graph K_V (with edge-weights D) determines a tour of the traveling salesman of cost $\le p$, and vice versa. Therefore, $\chi_c(K_V)$ is the optimum for the considered TSP.

Another closely related area is the channel assignment problem for which we refer to the recent survey article by McDiarmid [6].

The notion of the circular chromatic number thus generalizes several well-known optimization problems and hence introduces the possibility to apply tools from one area into another one. As the edge-weights are not discrete integer values, one may also get use of some tools from continuous optimization. The author of this paper is quite optimistic about such possibilities which may yield better understanding of graph coloring theory. As an example we refer to an extension of Hajós theorem to circular colorings of edge-weighted graphs [8] which sheds some new light to why no nontrivial applications of this celebrated theorem are known.

2 Tight edges

First, we shall show that the infimum in the definition of the circular chromatic number is attained. It will be shown that for every weighted graph G of order n, there exists a circular p-coloring for $p = \chi_c(G)$, and that $\chi_c(G)$ can be expressed as an integer fraction with denominator smaller than n of a sum of at most n edge-weights. This implies, in particular, that $\chi_c(G)$ is a rational number if all edge-weights are rational.

Let c be a circular p-coloring of G. A directed edge (u, v) is said to be *tight* if $\mathrm{d}(c(u), c(v)) = a_{uv}$. A cycle $C = v_1v_2 \dots v_kv_1$ is *tight* if the directed edges $(v_1, v_2), \dots, (v_{k-1}, v_k)$, and (v_k, v_1) are all tight. If $k = 2$ and the edges (v_1, v_2) and (v_2, v_1) are both tight, then we also consider the 2-cycle $v_1v_2v_1$ to be a tight cycle. If C is a tight cycle, then the weight of C,

$$a(C) := a_{v_1v_2} + \cdots + a_{v_{k-1}v_k} + a_{v_kv_1} \tag{2}$$

is an integer multiple of p, and the number $w(C) = a(C)/p$ is called the *winding number* of C.

Lemma 2.1 *If $p_0 = \chi_c(G)$, then there is a circular p_0-coloring of G which has a tight cycle.*

Proof. We may assume that G is connected. Suppose that $p_1 \geq p_0$ and that there is a circular p_1-coloring of G. Let $p \leq p_1$ be a real number such that G has a circular p-coloring c with maximum number of tight edges.

Let v_0 be a vertex of G and let $V_0 = \{v_0\}$. For $i = 1, 2, \ldots$, let $V_i = \{v \in V \mid \exists u \in V_{i-1} \text{ such that } (u,v) \text{ is tight}\}$. If $V_0 \cup \cdots \cup V_{i-1} \neq V$, and $V_i = \emptyset$, we can shift the colors of $V \setminus (V_0 \cup \cdots \cup V_{i-1})$ counterclockwise until a new tight edge occurs. By the maximality of c, this does not happen. Consequently, for each $v \in V$ there is a path from v_0 to v consisting only of tight edges.

Suppose that there is no tight cycle. For $v \in V$, let $l(v)$ be the maximum of $a(P)$ taken over all directed walks P from v_0 to v which consist of tight edges only. Since there are no tight cycles, the values $l(v)$ are finite. By the definition of $l(v)$, if $l(u) > l(v)$, then the edge uv is not tight. This implies that for $p' = p - \varepsilon$, where $\varepsilon > 0$ is small enough, the mapping $c'(v) = l(v) \bmod p' \in \mathbb{R}/p'\mathbb{Z}$ determines a circular p'-coloring of G. By increasing the value of ε as much as possible, a new tight edge occurs. This contradiction to the maximality of c shows that there exists a tight cycle.

Clearly, if the cycle C is tight, then $p = a(C)/w(C)$. The winding number of C is bounded by the number of edges in C. Therefore, the same cycle can be tight for at most n distinct values of p. Since there are only finitely many cycles of G, this easily implies the statement of the lemma. □

Corollary 2.2 *For every forest F with at least one edge,*

$$\chi_c(F) = \max\{a_{uv} + a_{vu} \mid u, v \in V\}.$$

Let C be a tight cycle with respect to a circular p-coloring. Then $p = a(C)/w(C)$. If the edge-weights are symmetric, then each edge-weight is at most $p/2$. Therefore, $a(C) \leq np/2$, so the winding number is at most $n/2$. In the nonsymmetric case, the winding number is at most $n-1$.

Corollary 2.3 *The circular chromatic number of G is of the form $\frac{a(C)}{k}$ where C is a cycle and k is an integer which is smaller than $n = |V|$ (and is smaller or equal to $n/2$ if the edge-weights are symmetric). In particular, the infimum in the definition of the circular chromatic number is attained.*

3 Upper bounds

Let $G = (V, A)$ be a weighted graph. For $v \in V$, let $d_G^+(v) = \sum_{u \in V} a_{uv}$ and $d_G^-(v) = \sum_{u \in V} a_{vu}$, and let $D_G(v) = d_G^+(v) + d_G^-(v)$. The graph G is said to be *weakly p-degenerate* if every subgraph H of G contains a vertex v with $D_H(v) \leq p$. The following result is obvious:

Proposition 3.1 *If G is a weakly p-degenerate graph, then $\chi_c(G) \leq p$.*

The inequality of Proposition 3.1 has an analogy in the theory of usual graph colorings of unweighted graphs, except that Proposition 3.1 gives a bound which is (roughly) for the factor 2 worse. The following example shows that the loss of factor 2 in comparison with the usual chromatic number cannot be improved.

Let G_n be the graph obtained from the complete graph K_n with unit edge-weights by adding a new vertex t_{uv} for each edge $uv \in E(K_n)$, and joining t_{uv} with the vertices u and v. The weight of each new edge joining t_{uv} with u or v is equal to $\kappa = \frac{n-1}{2}$.

Proposition 3.2 *The graph G_n is $(n-1)$-degenerate. In n is odd, then $\chi_c(G_n) = 2n - 4 + \frac{4}{n+1}$.*

Proof. It is clear by construction that G_n is $(n-1)$-degenerate.

Suppose now that n is odd. Let $p_0 = 2n - 4 + \frac{4}{n+1} = nr$, where $r = 2 - \frac{4}{n+1}$. Let $v_1, \ldots, v_n$ be the vertices of $K_n \subset G_n$. By setting $c(v_i) = (i-1)r \in \mathbb{R}/p_0\mathbb{Z}$, a circular p_0-coloring of K_n is obtained which can be extended to a circular p_0-coloring of G_n. Therefore, $\chi_c(G_n) \leq p_0$.

Suppose now that $p < p_0$, and suppose that there is a circular p-coloring c of G_n. Let $x_i = c(v_i)$. We may assume that the cyclic order of these colors on S_p is $x_1, \ldots, x_n$. For $x \in S_p$, let $\bar{x}$ be the point of S_p which lies diametrically opposite x on the circle S_p. Let $r_i = \mathrm{d}(x_i, x_{i+1})$, $i = 1, \ldots, n-1$, and let $r_n = \mathrm{d}(x_n, x_1)$.

Let α be the minimum distance of a point x_j from some $\bar{x}_i$, $i, j \in \{1, \ldots, n\}$. Since the color $c(t_{v_i v_j})$ has distance at least κ from x_i and from x_j, it is necessary that $p/2 + \alpha \geq 2\kappa$. This implies:

$$\alpha \geq 2\kappa - \frac{p}{2} > 1 - \frac{2}{n+1}. \tag{3}$$

If the opposite segment of $S_p(x_i, x_{i+1})$ contains some point x_j $(1 \leq j \leq n)$, then we say that i is *normal*. Otherwise, i is said to be *abnormal*. If i

is normal, then (3) implies that

$$r_i > 2 - \frac{4}{n+1}. \tag{4}$$

If i is abnormal, let j be the index such that $S_p(x_j, x_{j+1})$ contains $\bar{x}_i$. Then j is normal. This shows that there exists a normal index, and we shall assume that n is normal.

Let $i, i+1, \ldots, i+k$ $(k \geq 0)$ be a maximal subsequence of $1, \ldots, n$ such that $i, \ldots, i+k$ are all abnormal. Let j $(1 \leq j \leq n)$ be the index such that the segment $S_p(x_j, x_{j+1})$ (index $j+1$ modulo n) contains $\bar{x}_i$. Then $S_p(x_j, x_{j+1})$ also contains $\bar{x}_{i+1}, \ldots, \bar{x}_{i+k+1}$. This fact and (3) imply that

$$r_j > 2(1 - \frac{2}{n+1}) + r_i + \cdots + r_{i+k}.$$

Recall that $r_l \geq 1$ $(1 \leq l \leq n)$. Consequently, $r_i + \cdots + r_{i+k} + r_j > 2(k+2) - \frac{4}{n+1}$. Since every j appears at most once opposite some maximal abnormal sequence $i, \ldots, i+k$, and since every normal i satisfies (4), we get

$$p = r_1 + r_2 + \cdots + r_n > 2n - \frac{4n}{n+1} = p_0,$$

a contradiction. This shows that $\chi_c(G_n) = p_0$. $\square$

On the other hand, the $\Delta(G) + 1$ upper bound for the usual chromatic number has a generalization to the weighted case. Such a result, derived for the setting of channel assignment problems, was recently obtained by McDiarmid [5]. His proof can be extended to work also in the case of circular colorings of edge-weighted graphs.

Theorem 3.3 *Let $\Delta^+(G) = \max\{d_G^+(v) + a_{vu} \mid u, v \in V\}$. Then $\chi_c(G) \leq \Delta^+(G)$.*

Proof. Let us first assume that all edge-weights are integers. Let $v_1, \ldots, v_n$ be the vertices of G. We assign them colors $c(v_i) \in \mathbb{N}$ $(i = 1, \ldots, n)$ by applying the following "greedy" algorithm. For consecutive values of $\alpha = 0, 1, 2, \ldots$, traverse all vertices $v_1, \ldots, v_n$ and assign color $c(v_i) = \alpha$ to every uncolored vertex v_i such that for every vertex v_j that already received a color $c(v_j)$, $\alpha - c(v_j) \geq a_{v_j v_i}$.

We claim that $c(v_i) \leq d_G^+(v_i)$, $i = 1, \ldots, n$. Suppose that v_i has not been colored for $\alpha = 0, 1, \ldots, d$. Then for each such α, there was a vertex $v_{j(\alpha)}$

such that $\alpha - c(v_{j(\alpha)}) < a_{v_{j(\alpha)}v_i}$. If v_j is a neighbor of v_i, then $|\{\alpha \mid j(\alpha) = j\}| \leq a_{v_j v_i}$. This implies that $d + 1 \leq d_G^+(v_i)$.

Using the above conclusion, it is easy to see that c determines a circular $\Delta^+(G)$-coloring of G.

If the edge-weights are not integers, we proceed as follows. Let N be a large positive real number, and let $a'_{uv} = \lceil N a_{uv} \rceil$. Denote by G' the weighted graph thus obtained. Clearly, $\chi_c(G') \geq N \cdot \chi_c(G)$. By the above, $\chi_c(G') \leq \Delta^+(G') = \max\{d_{G'}^+(v) + a'_{vu}\} \leq N\Delta^+(G) + n$. Therefore, $\chi_c(G) \leq \Delta^+(G) + \frac{n}{N}$. Since N is arbitrarily large, $\chi_c(G) \leq \Delta^+(G)$. $\square$

Let G^T be the weighted graph whose weight function a^T is the transpose of a, i.e., $a^T_{uv} = a_{vu}$. Every circular p-coloring of G determines a circular p-coloring of G^T obtained by the reflection of the circle S_p. Hence, $\chi_c(G^T) = \chi_c(G)$. Note that $\Delta^+(G^T) = \Delta^-(G) = \max\{d_G^-(v) + a_{uv} \mid u, v \in V\}$. Therefore, $\chi_c(G) \leq \Delta^-(G)$.

4 Local changes

The following transformation gives a new edge-weighting but preserves the chromatic number. Let t be a real number and let v be a vertex of G. For each neighbor u of v, define new edge-weights $a'_{vu} = a_{vu} + t$ and $a'_{uv} = a_{uv} - t$. If the absolute value of t is small enough so that all new edge-weights remain nonnegative, then the resulting weighted graph has the same circular chromatic number as G.

If u, v are nonadjacent vertices of G, let $G_{u,v}$ denote the graph obtained from G by identifying u and v into a new vertex w. The edge-weights in $G_{u,v}$ are the same as in G except that for each $z \in V(G_{u,v}) \setminus \{w\}$, the weight a'_{zw} of zw and a'_{wz} of wz are equal to $a'_{zw} = \max\{a_{zu}, a_{zv}\}$ and $a'_{wz} = \max\{a_{uz}, a_{vz}\}$, respectively. Then

$$\chi_c(G) \leq \chi_c(G_{u,v}) \tag{5}$$

since every circular p-coloring c of $G_{u,v}$ determines a coloring of G by setting $c(v) = c(u) := c(w)$.

If u, v are adjacent vertices of G, then we define $G_{u,v}$ in the same way except that the weights of edges incident with w are determined differently:

$$a'_{zw} = \max\{a_{zu}, a_{zv} - a_{uv}\} \qquad \text{and}$$

$$a'_{wz} = \max\{a_{uz}, a_{uv} + a_{vz}\}.$$

If c is a circular p-coloring of $G_{u,v}$, then setting $c(u) = c(w)$ and $c(v)$ to be the point on S_p such that $\mathrm{d}(c(u), c(v)) = a_{uv}$ yields a circular p-coloring of G. This shows that

$$\chi_c(G) \leq \chi_c(G_{u,v}). \tag{6}$$

5 Colorings and orientations

A mapping $T : \vec{E} \to \{-1, 1\}$ is an *(edge-)orientation* of G if for every $(u,v) \in \vec{E}$, $T(u,v) = -T(v,u)$. We say that the edge $uv \in E$ is *oriented* from u to v if $T(u,v) = 1$.

Let T be an orientation. A mapping $t : E \to \mathbb{R}$ is a *tension* if for every cycle $C = v_1 v_2 \dots v_k v_1$ of G, we have

$$\sum_{i=1}^{k} T(v_i, v_{i+1})\, t(v_i v_{i+1}) = 0. \tag{7}$$

The tension t is *p-admissible* if for every edge $uv \in E$ oriented from u to v, $a_{uv} \leq t(uv) \leq p - a_{vu}$.

Lemma 5.1 *A weighted graph G has a circular p-coloring if and only if there is an orientation of G for which there exists a p-admissible tension.*

Proof. If c is a circular p-coloring, the following determines an orientation T and a p-admissible tension t. Fix a point $o \in S_p \backslash c(V)$. Suppose that $uv \in E$. If $o \notin S_p(c(u), c(v))$, then we set $T(u,v) = 1$ and $t(uv) = \mathrm{d}(c(u), c(v))$. Otherwise, we set $T(u,v) = -1$ and $t(uv) = \mathrm{d}(c(v), c(u))$.

Conversely, let T be an orientation and t a p-admissible tension. We may assume that G is connected. Let D be a spanning tree of G, and let v_0 be a vertex of G. For $v \in V$, let $P = v_0 v_1 \dots v_{k-1} v_k$ be the path in D from v_0 to $v = v_k$. Set $l(v) = \sum_{i=0}^{k-1} T(v_i, v_{i+1}) t(v_i v_{i+1})$ and $c(v) = l(v) \bmod p \in \mathbb{R}/p\mathbb{Z} \approx S_p$. Consider an arbitrary directed edge $(u,v) \in \vec{E}$ such that $T(u,v) = 1$. If $uv \in E(D)$, then $l(v) = l(u) + t(uv)$. The same relation holds if $uv \notin E(D)$ by (7). Therefore, $a_{uv} \leq l(v) - l(u) \leq p - a_{vu}$. This implies that $\mathrm{d}(c(u), c(v)) = t(uv) \geq a_{uv}$ and that $\mathrm{d}(c(v), c(u)) = p - t(uv) \geq a_{vu}$. This shows that c is a circular p-coloring of G. $\square$

The following result of Hoffman [4] and Ghouila-Houri [2] gives a necessary and sufficient condition for existence of an admissible tension. Let us mention that a *directed cycle* of G is a cycle in the digraph with edge set $\vec{E}$. Each cycle of G determines two directed cycles (one in each direction), and each edge of G determines a directed cycle of length 2.

Theorem 5.2 *Let G be a weighted graph with a given orientation T, and let $l, u : E \to \mathbb{R}$ be nonnegative functions such that $0 \le l(e) \le u(e)$ for every $e \in E$. Then there exists a tension t such that $l(e) \le t(e) \le u(e)$ for every $e \in E$ if and only if for every directed cycle $C = v_1 v_2 \dots v_k v_1$ $(k \ge 2)$ of G, we have*

$$\sum_{T(v_i,v_{i+1})=-1} l(v_i v_{i+1}) \le \sum_{T(v_i,v_{i+1})=1} u(v_i v_{i+1}), \tag{8}$$

where i takes values $1, \dots, k$, and $v_{k+1} = v_1$. Moreover, if l and u are rational (integer) valued, then t can be chosen to be rational (integer) valued.

Observe that Hoffman's condition in Theorem 5.2 must also hold for the reverse directed cycle $C' = v_1 v_k \dots v_2 v_1$. This gives:

$$\sum_{T(v_i,v_{i+1})=1} l(v_i v_{i+1}) \le \sum_{T(v_i,v_{i+1})=-1} u(v_i v_{i+1}). \tag{9}$$

For a directed cycle $C = v_1 \dots v_k v_1$ of G, let C^+ (resp. C^-) be the set of edges of C whose T-orientation is the same (resp. opposite) as on C. For p-admissible tensions we have $l(v_i v_{i+1}) = a_{v_i v_{i+1}}$ if $T(v_i, v_{i+1}) = 1$, and we have $u(v_i v_{i+1}) = p - a_{v_i v_{i+1}}$ if $T(v_i, v_{i+1}) = -1$. Condition (9) is equivalent to the following requirement:

$$a(C) = \sum_{i=1}^{k} a_{v_i v_{i+1}} \le p|C^-|. \tag{10}$$

This shows that G has no circular p-coloring if and only if for every orientation T there exists a cycle C such that (10) is violated. This implies:

Theorem 5.3 *Let G be a weighted graph. Then*

$$\chi_c(G) = \min_T \max_C \frac{a(C)}{|C^-|} \tag{11}$$

where the minimum is taken over all (acyclic) orientations T of G, and the maximum is over all directed cycles of G.

A version of Theorem 5.3 for usual colorings of graphs was proved by Minty [7]. A version for circular colorings was proved by Goddyn, Tarsi, and Zhang [3] who also pointed out that the same result can be proved in the setting of matroids. Our extension to weighted graphs can also be extended to matroids with weighted elements.

A corollary of Theorem 5.3 is that the infimum in the definition of the cyclic chromatic number is attained. In fact, $\chi_c(G)$ is of the form $\frac{a}{k}$ where $a = a(C)$ is a sum of at most n edge-weights and k is a positive integer smaller or equal to n. In the case of symmetric edge-weights, $k \leq n/2$ since either C or its inverse C' has at most $n/2$ negatively oriented edges, so that $\max\{\frac{a(C)}{|C^-|}, \frac{a(C')}{|C'^-|}\} \geq \frac{a(C)}{n/2}$. Cf. also Corollary 2.3.

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