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2-LOCAL 4/3-COMPETITIVE ALGORITHM FOR MULTICOLORING HEXAGONAL GRAPHS

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2-Local 4/3-Competitive Algorithm For Multicoloring Hexagonal Graphs

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An important optimization problem in the design of cellular networks is to assign sets of frequencies to transmitters to avoid unacceptable interference. A cellular network is generally modeled as a subgraph of the infinite triangular lattice. The distributed frequency assignment problem can be abstracted as a multicoloring problem on a weighted hexagonal graph, where the weight vector represents the number of calls to be assigned at vertices. In this paper we present a distributed algorithm for multicoloring hexagonal graphs using only the local clique number $\omega(v)$ at each vertex v of the given hexagonal graph, which can be computed from local information available at the vertex. We prove the algorithm uses no more than $\left\lceil \frac{4\omega(G)}{3} \right\rceil$ colors for any hexagonal graph G, without explicitly computing the global clique number $\omega(G)$. Hence the competitive ratio of the algorithm is $4/3$. We also prove that our algorithm is 2-local.

keywords: graph coloring, frequency planning, cellular networks, 2-local distributed algorithm.

1. INTRODUCTION

A basic problem concerning cellular networks is to assign sets of frequencies (colors) to transmitters (vertices) to avoid unacceptable interference [1]. The number of frequencies demanded at a transmitter may vary between transmitters. In a usual cellular model, transmitters are centers of hexagonal cells and the corresponding adjacency graph is a subgraph of the infinite triangular lattice. An integer $d(v)$ is assigned to each vertex of the triangular lattice and will be called the *demand* of the vertex v. A *hexagonal graph* $G(V,E,d)$ is the vertex weighted graph induced on the set of vertices of positive demand. We will introduce a distributed algorithm which will produce a proper multicoloring of the hexagonal graph. A *proper multicoloring* of G is a mapping c from $V(G)$ to subsets of integers such that $c(v) \geq d(v)$ for any vertex $v \in G$ and $c(v) \cap c(u) = \emptyset$ for any pair of adjacent vertices u and v in the graph G. The minimal cardinality of a proper multicoloring of G, $\chi(G)$, is called the *multichromatic number*. Another invariant of interest in this context is the *(weighted) clique number*, $\omega(G)$, defined as follows: The weight of a clique of G is the sum of demands on its vertices and $\omega(G)$ is the maximal clique weight on G. Clearly, $\chi(G) \geq \omega(G)$. Recently, the bound

$$\chi(G) \leq (4/3)\omega(G) + c$$

was independently proved by several authors [5, 6, 9]. All proofs are constructive thus implying the existence of 4/3-approximation algorithms. McDiarmid and Reed [5] also show that it is NP-complete to decide whether $\chi(G) = \omega(G)$. A distributed algorithm which guarantees the $\left\lceil \frac{4\omega(G)}{3} \right\rceil$ bound is reported by Narayanan and Shende [6, 7]. A framework for studying distributed online assignment in cellular networks was developed in [4]. In particular, competitive ratios of distributed algorithms which utilize information about increasingly larger neighborhoods are addressed. The best competitive ratios for 0-,1-,2- and 4-local algorithms reported are 3, 3/2, 17/12 and 4/3, respectively. (An algorithm is k-local if the computation at a vertex v uses only information about the demands of vertices whose graph distance from v is less than or equal to k.)
The algorithm of [5] gives the best possible bound among the above mentioned early papers, which is:

$$\chi(G) \leq \left\lfloor \frac{4\omega(G)+1}{3} \right\rfloor.$$

Better bounds can be obtained for triangle-free hexagonal graphs: in [3] a distributed algorithm with competitive ratio 5/4 is given, while an inductive proof for 7/6 ratio is reported in [2]. McDiarmid and Reed conjectured that for triangle free hexagonal graphs the inequality $\chi(G) \leq (9/8)\omega(G)+c$ holds . A distributed algorithm which gives a proper multicoloring of a hexagonal graph with at most $\left\lfloor \frac{4\omega(G)+1}{3} \right\rfloor$ colors, which is less than or equal to $\left\lceil \frac{4\omega(G)}{3} \right\rceil$, is given in [8]. In the algorithm proposed in [8], each vertex of the graph is assumed to know its demand, its position (i.e. coordinates) and the global constant $\omega(G)$. A vertex can communicate to its neighbors to obtain some local information. It is important to note that the computation time does not depend on the size of the graph. An obvious drawback of the distributed algorithm in [8] is that each vertex needs one piece of global information, namely $\omega(G)$. Here we define the notion of local $\omega(v)$ at each vertex v of the graph, which can by definition be computed from local information available at the vertex. In this paper we develop a new version of the algorithm using the local clique number and give its correctness proof. Directly from the definition it will follow that

$$\omega(G) = \max_{v \in G} \omega(v)$$

and hence the algorithm presented here will use no more than $\left\lceil \frac{4\omega(G)}{3} \right\rceil$ colors, without explicitly computing the $\omega(G)$. We will also prove that the algorithm is 2-local. This improves on the results of [4], where a 2-local algorithm with performance ratio 17/12 and a 4-local algorithm with performance ratio 4/3 are given. More formally, we will prove that

THEOREM 1.1. *There is a 2-local distributed approximation algorithm for multicoloring a hexagonal graph which uses at most $\lceil 4\omega(G)/3 \rceil$ colors. Time complexity of the algorithm at each vertex is constant.*

The paper is organized as follows. In the next section we formally define some basic terminology. In Section 3 we present the overview of the distributed algorithm for multicoloring hexagonal graphs which uses only the local clique number. In Section 4 the precise definition of the actions taken in the algorithm and their proofs are given. We show the algorithm is 2-local and its performance ratio is 4/3.

2. PRELIMINARIES

Following [5] the vertices of triangular lattice, denoted by L, may be described as follows. The position of each vertex is an integer linear combination $x\vec{p} + y\vec{q}$ of the two vectors $\vec{p} = (1, 0)$ and $\vec{q} = (\frac{1}{2}, \frac{\sqrt{3}}{2})$. Thus, we may identify the vertices of triangular grid with the pairs (x, y) of integers. Two vertices are adjacent when the Euclidean distance between them is one. Therefore each vertex (x, y) has six neighbors $(x \pm 1, y)$, $(x, y \pm 1)$, $(x + 1, y - 1)$, $(x - 1, y + 1)$, see Fig. 1.

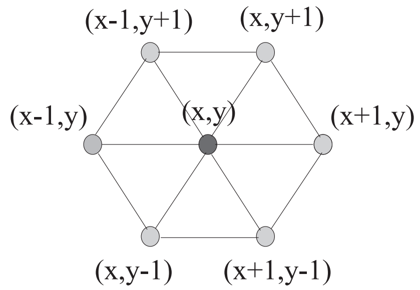

FIG. 1. The coordinates of neighbors of vertex (x, y).

For simplicity, we will refer to the neighbors as R (right), L (left), UR (up-right), DL (down-left), DR (down-right) and UL (up-left), respectively.
We will use coordinates when the distributed implementation of the algorithm will be discussed in more detail.
There is an obvious 3-coloring of the infinite triangular lattice which gives rise to partition of vertex set of any triangular lattice graph into three independent sets. This partition of the triangular grid into three independent sets I_0, I_1 and I_2 can be decided locally from the coordinates by the rule: a vertex with coordinates (x, y) is in the independent set I_i where $i = x + 2y \mod 3$. According to the partition I_0, I_1 and I_2 , vertices are assigned their *base colors* which are denoted by r (red), b (blue) and g (green) respectively. We will denote the base color of the vertex v by $C(v) \in \{r, b, g\}$.

In accordance with the coordinates as given above let us define a new system of coordinates, so that each vertex will have three coordinates (i, j, k). Note that a triangular lattice is composed of three sets of parallel straight lines. Although the third coordinate is redundant, the new coordinate system is introduced because all three directions are symmetric, which allows a simpler description of the steps that follow.
We define the new coordinates of a vertex v by

$$(i, j, k) := (x, y, x + y),$$

where (x, y) are the coordinates of v as defined above.
We omit a straightforward proof of the following proposition:

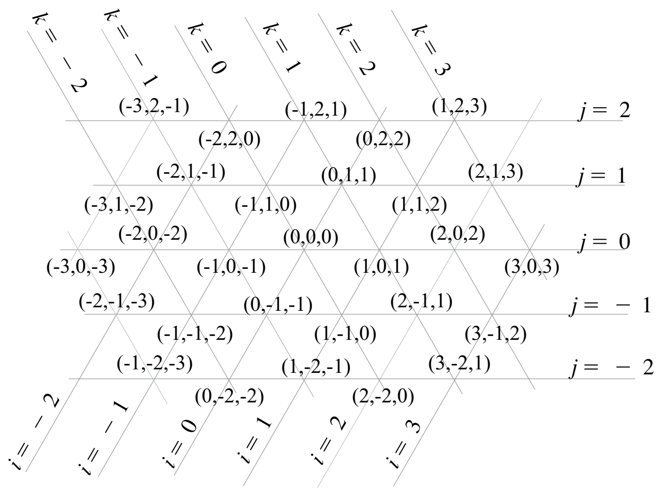

FIG. 2. The coordinates (i, j, k) of triangular lattice.

PROPOSITION 2.1.
- each line which goes from bottom-left to up-right has the first coordinate, i, constant,
- each horizontal line has the second coordinate, j, constant,
- lines of the third set have the third coordinate, k, constant.

The distributed algorithm we give is based on the algorithm of [8], with two essential improvements. First, instead of using the global clique number we are using the *local clique number* $\omega(v)$, which can be computed locally at each vertex and second, the coloring of the remaining configurations after Step 3 is done according to new rules which can be decided based on local information. We will prove correctness of the algorithm using the local clique number and show that all steps at vertex v can be computed using only information about demands of those vertices whose graph distance from v is less than or equal to 2. Hence showing that the algorithm is 2-local.

DEFINITION 2.1. Let c stand for red, blue or green. A vertex v is said to be *c-free* if it is not a c vertex and it has no c neighbors. A c-free vertex with at least two neighbors is called a *center*.

DEFINITION 2.2. If a center v has neither R, LD nor LU neighbor, it is called a *left center*. A left center can be either green-free blue center or blue-free red center or red-free green center. In all other cases a center v is said to be the *right center.*

Notation *left* or *right* center follows from the base coloring and the geometry of neighbors of the center. Thus, there are three different possibilities for left and three for right centers. For each possibility we have four different positions (Fig. 3).

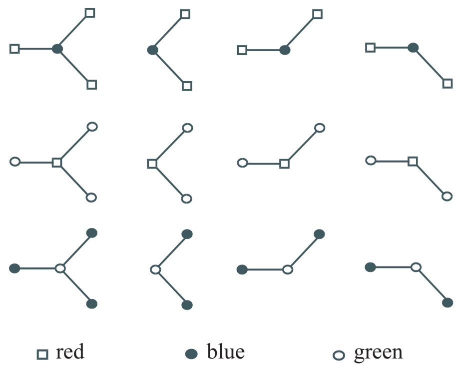

FIG. 3. All possible positions for left centers.

Let $G = (V, E, d)$ denote a weighted hexagonal graph. For each vertex $v \in G$ we will define so-called *local clique number* $\omega(v)$. It is the maximal demand of a clique containing v. Equivalently, we can define $\omega(v)$ on the triangular lattice L as:

Definition 2.3. For every vertex v in triangular lattice L let

$$\omega(v) = \max\{d(v) + d(u) + d(z) \mid uvz \text{ is triangle of } L \text{ containing } v\}$$

be the *local clique number* of a vertex v and $p(v) = \left\lceil \frac{\omega(v)}{3} \right\rceil$, $q(v) = \omega(v) - 2p(v)$.

A vertex $t \in L$ with demand $d(t) = 0$ will be called an *empty vertex.* We prefer the last definition, of the local clique number, because using the notion of empty vertices will allow simpler arguments later.
The following statements follow directly from the definition.

Lemma 2.1. *For every vertex $v \in G$ holds $\omega(v) \leq 3p(v)$.*

LEMMA 2.2. *For an arbitrary triangle uvz of G and its arbitrary vertex v holds* $\omega(v) \geq d(u) + d(v) + d(z)$.

LEMMA 2.3. *For any vertices u, v from G the inequality* $\omega(u) \leq \omega(v)$ *implies* $p(u) \leq p(v)$ *and* $q(u) \leq q(v) + 1$.

COROLLARY 2.1. *For an arbitrary triangle uvz of G we have* $d(u) + d(v) + d(z) \leq p(u) + p(v) + p(z)$.

Proof. Without loss of generality we may assume that $\omega(v) = \min\{\omega(u), \omega(v), \omega(z)\}$. This implies $p(v) \leq p(u)$ and $p(v) \leq p(z)$. As $\omega(v) \leq 3p(v)$ by Lemma 2.1 and using Lemma 2.2 we have $d(u)+d(v)+d(z) \leq p(u)+p(v)+p(z)$, as claimed. ∎

Our color palette at a vertex $v \in G$ will be composed of $3p(v)+q(v)$ distinct colors. Namely, at $v \in G$ we associate a class of $p(v)$ hues of each base color (red, blue and green) and $q(v)$ additional white colors. Hues of a base color c at a vertex v will be identified with the interval $[c, i]$, $1 \leq i \leq p(v)$ and white hues with the interval $[\mathrm{w}, j]$, $1 \leq j \leq q(v)$. We will sometimes refer to particular subsets of $[c, i]$, which will be called *low*, *high* or *middle c* colors. By the low c colors we mean the consecutive hues starting from $[c, 1]$ to $[c, l]$ and by the high c colors at a vertex v we mean the consecutive hues starting from $[c, p(v)]$ to $[c, p(v) - m + 1]$. When the middle c colors will be used, we will assume that a (possibly empty) subset of l low c colors was already assigned. The k middle c colors at a vertex v are then formally the subset of $[c, i]$, starting at $[c, l + 1]$ to $[c, l + k]$, where $l + k$ must be less than or equal to $p(v)$.

3. THE ALGORITHM

We will describe the algorithm in six steps. A vertex $v \in G$ is defined to be *light* if $d(v) \leq p(v)$ and to be *heavy* otherwise. Recall that $p(v) = \left\lceil \frac{\omega(v)}{3} \right\rceil$ and $q = \omega(v) - 2p(v)$.

With $d_i(v)$ we denote the demand of the vertex v after Step i (i.e. the difference between $d(v)$ and the number of assigned colors in previous steps). $G_i = (V_i, E_i, d_i)$ denotes the remaining graph after Step i and is formally the induced graph on the vertices of positive demand $d_i(v)$. The degree of a vertex v in the graph G_i will be denoted by $\deg_i(v)$.

OVERVIEW OF THE ALGORITHM

Step 0: For each vertex $v \in G$ compute $\omega(v)$, $p(v)$ and $q(v)$.

Step 1: For each vertex $v \in G$ assign $p(v)$ colors to the vertex v from its base color palette.

Step 2: Completely color all left centers in such a way that c-free left center v borrows $p(v) - \mathbf{c}(v)$ high c colors (i.e. c colors which were not used by its neighbors), where $\mathbf{c}(v)$ is the maximum among the weights of the light c neighbors of v in G.

Step 3: Color isolated vertices by using white colors and by borrowing from the neighbor with maximal demand.

Step 4: Color centers of quasistars and their neighbors in G_3. (using white colors for centers and no white colors for neighbors of centers, see more details below.)
Step 5: Color the remaining paths by white colors. Each odd vertex $v \in G_4$ gets $d_4(v)$ low white colors, each even vertex $u \in G_4$ gets $d_4(u)$ high white colors. (Definition of odd and even vertices on a straight path of G_4 is given in Section 4.)
Step 6: Color the remaining isolated vertices by low white colors.

Details of Step 4:
(a) If $v \in G_3$ is a center of a quasistar with neighbors u_1, u_2 and u_3 in G_3 then v replaces $M(v) = \max\{d_3(u_1), d_3(u_2), d_3(u_3)\}$ high $C(v)$ colors with $M(v)$ low white colors and takes further $[\mathrm{w}, i]$, $M(v)+1 \leq i \leq M(v)+d_3(v)$ white colors. (If $u_i \notin G_3$ then $d_3(u_i) = 0$, $i = 1, 2, 3$.)
(b) If $u \in G_3$ is a neighbor of a center of a quasistar $v \in G_3$ and $\deg_3(u) = 2$ then u gets $d_3(u)$ high $C(v)$ colors.
(c) If $u \in G_3$ is a neighbor of a center of a quasistar $v \in G_3$ and $\deg_3(u) = 1$ then
(c)-(i) if $\deg_1(u) = 1$ then u gets $d_3(u)$ high $C(*)$ colors, where $C(*)$ means the base color different from $C(u)$ and $C(v)$.
(c)-(ii) if u has another neighbor t in G_1 (different from v) then u gets $d_3(u)$ middle $C(v)$ colors.

In the sequel we will give a precise definition of the actions taken at a vertex showing explicitly that only 2-local information is needed. It will be proved that the resulting graphs after each of the steps are partially but properly multicolored and that after Step 6 the graph G is multicolored completely.

4. CORRECTNESS PROOF

Step 0. Nothing to prove.

Step 1. For each vertex $v \in G$ assign $p(v)$ colors to the vertex v from its base color palette.

Claim: The resulting graph G_1 is triangle free.
[[Assume the opposite, i.e. there exist a triangle $\{u, v, z\}$ in G_1. That means that vertices u, v and z were heavy in G. Thus

$$d(u) + d(v) + d(z) > p(u) + p(v) + p(z),$$

which contradicts Corollary 2.1.]]

Step 2. Completely color all left centers in such a way that a c-free left center v borrows $p(v) - \mathbf{c}(v)$ high c colors (i.e. c colors which were not used by its neighbors), where $\mathbf{c}(v)$ is the maximum among the weights of the light c neighbors of v in G.

Note that a vertex can decide whether it is a left center or not easily and using only information available at its neighbors. A vertex is a left center if and only if its DL, UL and R neighbors are light (i.e. have demand $d_1(.) = 0$ and at least two of the remaining three neighbors have positive demand $d_1(.) > 0$).

Claim: Without loss of generality we may suppose that v is a blue-free red center. Let $\mathbf{b}(v)$ be the maximum among the weights of the light blue neighbors of v. Then $p(v) - \mathbf{b}(v)$ high blue colors suffice to color the remaining demand of v.
[[Assume the opposite (i.e. $d_1(v) > p(b) - \mathbf{b}(v)$) and let g be the common heavy green neighbor of v and b, the blue neighbor of v, for which the equality $d(b) = \mathbf{b}(v)$ holds. (Note that such a green neighbor of v exists, since v has to be the center. Furthermore $d_1(g) \geq 1$ because g is heavy.) Therefore we have:

$$d(v) + d(g) + d(b) = p(v) + d_1(v) + p(g) + d_1(g) + \mathbf{b}(v) >$$

$$> p(v) + p(b) - \mathbf{b}(v) + p(g) + d_1(g) + \mathbf{b}(v) > p(v) + p(b) + p(g),$$

which contradicts Corollary 2.1.]]

Remark: The blue colors assigned to v in the Step 2 are $[b, i]$, $i = p(v), p(v) - 1, ..., p(v) - d_1(v) + 1$.

The remaining graph G_2 is bipartite. In particular, the connected components of G_2 are isolated vertices, paths or quasistars (i.e. trees with exactly one vertex of degree 3). This is obvious because all centers which remain in G_2 are right centers.

Step 3. Color isolated vertices by using white colors and by borrowing high colors from the neighbor with maximal demand.

It is trivial for a vertex to decide whether it is isolated or not by inspecting the demand of its neighbors.

Claim: No conflict can occur.
[[Without loss of generality we may suppose that u is a red vertex. There are two different possibilities.

(a) u was already isolated in G or u became isolated after Step 1.
Let v be a neighbor of u such that $d(v)$ is the maximal demand among all the neighbors of u in G (if u was already isolated in G, then $d(v) = 0$). Note that $d(v) \leq p(v)$ and $d(u) + d(v) \leq \min\{\omega(u), \omega(v)\}$.

If $\omega(u) \leq \omega(v)$ then

$$d_1(u) \leq \omega(u) - d(v) - p(u) = q(u) + p(u) - d(v) \leq q(u) + (p(v) - d(v)).$$

If $\omega(u) > \omega(v)$ then

$$d_1(u) \leq \omega(v) - d(v) - p(u) = 2p(v) + q(v) - d(v) - p(u) \leq (p(v) - d(v)) + q(v).$$

It follows that u may be assigned $\max\{q(u), q(v)\}$ low white colors and $p(v) - d(v)$ high blue colors, if v is a blue vertex or $p(v) - d(v)$ high green colors if v is green, without conflict.

(b) u became isolated after Step 2.
Vertex u was a neighbor of some left center $v \in G$ in this particular case. We claim that $q(u)$ white colors suffice to color u (i.e. $d_2(u) \leq q(u)$). Let a and b be the common neighbors of u and v. If they do not appear in G, then we take $d(.) = 0$. Let us define

$$\omega_{uv} = d(u) + d(v) + \max\{d(a), d(b)\}$$

and $p_{uv} = \lceil \frac{\omega_{uv}}{3} \rceil$, $q_{uv} = \omega_{uv} - 2p_{uv}$. Then we have

$$d_1(u) = \omega_{uv} - p(u) - d_1(v) - p(v) - \max\{d(a), d(b)\}.$$

From $\omega_{uv} \leq \omega(u)$ and $\omega_{uv} \leq \omega(v)$ it follows $p_{uv} \leq p(u)$, $q_{uv} \leq q(u) + 1$, $p_{uv} \leq p(v)$ and $q_{uv} \leq q(v) + 1$. Hence

$$d_2(u) = d_1(u) \leq \omega_{uv} - 2p_{uv} - d_1(v) - \max\{d(a), d(b)\}$$

$$\leq q_{uv} - \max\{d(a), d(b)\} - d_1(v) \leq q_{uv} - 1 \leq q(u),$$

using $d_1(v) \geq 1$. Because u has no neighbor which has received white colors, no conflicts can occur.]]

As the isolated vertices have been completely colored, the remaining components after Step 3 are only nontrivial straight paths and quasistars (i.e. trees with exactly one vertex of degree 3).
It is trivial that the paths and quasistars are bipartite graphs and if this bipartition is known it is easy to see that only white colors are sufficient (follows from Lemma 4.1). However, as we want to design a 2-local algorithm, we may not assume that this bipartition is known. In order to achieve 2-locality, we will have to use a more complicated method. First we will color the centers using white colors and the neighbors of centers without using white colors. This will assure that there will be no white colors used for neighbors of the remaining paths (and isolated vertices, which again may appear if a ray of a quasistar was of length 2). Paths will be colored by white colors and new isolated vertices will again have to be treated slightly different.

LEMMA 4.1. *Let u and v be neighbors in the graph G_3. Then*

$$d_3(u) + d_3(v) \leq \max\{q(u), q(v)\}.$$

Proof. Let t be a light common neighbor of vertices u and v in G. (If it doesn't appear in G, then it is an empty vertex of the lattice L and $d(t) = 0$.) Then

$$d(u) + d(v) + d(t) = p(u) + p(v) + d_3(u) + d_3(v) + d(t).$$

If we assume the opposite ($d_3(u) + d_3(v) > \max\{q(u), q(v)\}$) then we get:

$$d(u) + d(v) + d(t) > p(u) + p(v) + \max\{q(u), q(v)\} + d(t).$$

In the case $p(u) \leq p(v)$ we have

$$d(u) + d(v) + d(t) > 2p(u) + q(u) + d(t) \geq \omega(u)$$

and if $p(u) > p(v)$,

$$d(u) + d(v) + d(t) > 2p(v) + q(v) + d(t) \geq \omega(v).$$

Both cases contradict Lemma 2.2. ∎

Step 4a: If $v \in G_3$ is a center of a quasistar with neighbors u_1, u_2 and u_3 in G_3, then v replaces $M(v) = \max\{d_3(u_1), d_3(u_2), d_3(u_3)\}$ high $C(v)$ colors with $M(v)$ low white colors and takes further $[\mathrm{w}, i]$, $M(v) + 1 \leq i \leq M(v) + d_3(v)$ white colors.

This may be done by Lemma 4.1.

Step 4b: If $u \in G_3$ is a neighbor of a center of a quasistar $v \in G_3$ and $\deg_3(u) = 2$ then u gets $d_3(u)$ high $C(v)$ colors.

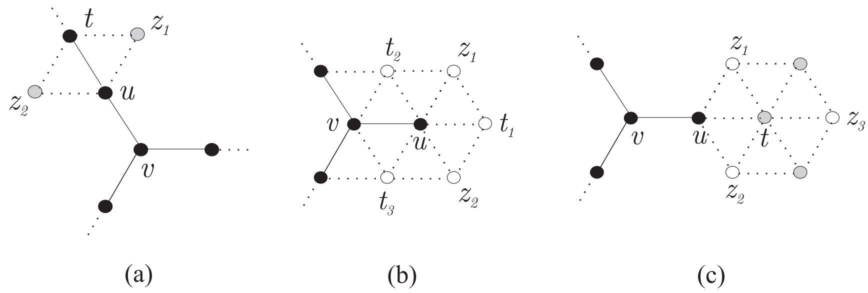

FIG. 4. Neighborhoods of a center of a quasistar v.

Claim: $\min\{p(z_1) - d(z_1), p(z_2) - d(z_2)\}$ $C(v)$ colors suffice to complete the remaining demand of u, where z_1 and z_2 are light neighbors of u in G of color $C(v)$.

[[Without loss of generality let say $p(z_1) - d(z_1) \leq p(z_2) - d(z_2)$. Let t be the neighbor of u in G_3 (different from v), see Fig. 4 (a). If we assume the opposite ($d_3(u) > \min\{p(z_1) - d(z_1), p(z_2) - d(z_2)\}$) then we have:

$$d(u) + d(t) + d(z_1) = d_3(u) + p(u) + d_3(t) + p(t) + d(z_1) >$$
$$> p(z_1) - d(z_1) + p(u) + d_3(t) + p(t) + d(z_1) > p(z_1) + p(u) + p(t),$$

which contradicts Corollary 2.1. So we can assign $d_3(u)$ high $C(v)$ colors to u. Because v already replaced its $M(v) \geq d_3(u)$ high $C(v)$ colors by low white colors in Step 4a no conflict can occur.]]

Step 4c(i): If $u \in G_3$ is a neighbor of a center of a quasistar $v \in G_3$ and $\deg_3(u) = \deg_1(u) = 1$ then u gets high $C(*) = \{r, b, g\} \backslash \{C(u), C(v)\}$ colors and high $C(v)$ colors if needed.

Note that all neighbors of u except v were light in G and color $C(*)$ is the base color different from the colors $C(v)$ and $C(u)$.

Claim: Let t_1, t_2 and t_3 be the light neighbors of u in G, all of the same color $C(t_1)$, different from the colors $C(u)$ and $C(v)$ (see Fig. 4 (b)).

Case 1: If $p(t_1) - d(t_1) = \min\{p(t_1) - d(t_1), p(t_2) - d(t_2), p(t_3) - d(t_3)\} =$ MIN then MIN high $C(t_1)$ colors and $\min\{p(z_1) - d(z_1), p(z_2) - d(z_2)\}$ high $C(v)$ colors suffice to complete the remaining demand of u, where z_1 and z_2 are the light neighbors of u and t_1 in G with color $C(v)$.

Case 2: If $p(t_1) - d(t_1) \geq$ MIN then only high $C(t_1)$ colors suffice to complete the remaining demand of u.

[[**Case 1:** Without loss of generality let say $p(z_1) - d(z_1) \leq p(z_2) - d(z_2)$. If we assume that the colors as claimed are not sufficient (i.e. $d_3(u) > p(t_1) - d(t_1) + p(z_1) - d(z_1)$) then we get:

$$d(u) + d(z_1) + d(t_1) = d_3(u) + p(u) + d(z_1) + d(t_1) >$$
$$> p(t_1) - d(t_1) + p(z_1) - d(z_1) + p(u) + d(z_1) + d(t_1) = p(t_1) + p(z_1) + p(u),$$

which contradicts Corollary 2.1.

Case 2: Without loss of generality let say $\text{MIN} = p(t_2) - d(t_2)$. If we assume opposite to the claim ($d_3(u) > \text{MIN}$) then

$$d(u) + d(v) + d(t_2) = d_3(u) + p(u) + d_3(v) + p(v) + d(t_2) >$$
$$> p(t_2) - d(t_2) + p(u) + d_3(v) + p(v) + d(t_2) > p(t_2) + p(u) + p(v),$$

which contradicts Corollary 2.1.]]

Step 4c(ii): If $u \in G_3$ is a neighbor of a center of a quasistar $v \in G_3$ and $\deg_3(u) = 1$ and u had another neighbor t in G_1 (different from v) then u gets $d_3(u)$ middle $C(v)$ colors.

Note that t was a left center in G_1 and it must be in the same straight line as vertices u and v in G (see Fig. 4(c)). Namely, if t would be DR or UR neighbor of u then u would be a left center in G_1 and hence completely multicolored in Step 2. In all other cases there would be a triangle in G_1 which contradicts Claim of the Step 1.

Claim: $\min\{p(z_1) - d(z_1), p(z_2) - d(z_2)\} - d_1(t)$ middle $C(v)$ colors suffice to complete the remaining demand of u, where t is heavy neighbor of u (different from v) and z_1, z_2 are light common neighbors of u and t in G, see Fig. 4 (c).

[[Without loss of generality let us say $p(z_1) - d(z_1) \leq p(z_2) - d(z_2)$. If we assume the opposite ($d_3(u) > \min\{p(z_1) - d(z_1), p(z_2) - d(z_2)\} - d_1(t)$) then

$$d(u) + d(t) + d(z_1) = d_3(u) + p(u) + p(t) + d_1(t) + d(z_1) >$$
$$> p(z_1) - d(z_1) - d_1(t) + p(u) + p(t) + d_1(t) + d(z_1) = p(z_1) + p(u) + p(t),$$

which contradicts Corollary 2.1. Hence although the vertex t received high $C(v)$ colors in Step 2 there is enough middle $C(v)$ colors which can be used by u.]]

The last claim assures that there is enough colors. We have to show that the set of middle colors can be determined at v using only 2-local information. The middle $C(v)$ colors used at u are $[C(v), i]$; $\max\{d(z_1), d(z_2), d(z_3)\} + 1 \leq i \leq \max\{d(z_1), d(z_2), d(z_3)\} + d_3(u)$, where z_1, z_2 and z_3 are light neighbors of t in G, see Fig. 4 (c). Hence, the information needed to

compute the set of middle $C(v)$ colors for u are $\max\{d(z_1), d(z_2), d(z_3)\}$ and $d_3(u)$. Therefore the computation of the set of the middle $C(v)$ colors at u is 2-local.
The remaining components after Step 4 are only isolated vertices and straight paths. Vertices of a straight path will be called *odd* and *even* respectively depending on the parity of the coordinate which changes along the path. More formal definition is given below.

DEFINITION 4.1. A vertex $v \in G_4$ with coordinates (i, j, k) is
• *odd*, if one of the following conditions is fulfilled:
- v has neighbors $(i-1, j, k-1)$ or $(i+1, j, k+1)$ in G_3 with coordinate $i \equiv 1 \pmod 2$,
- v has neighbors $(i, j-1, k-1)$ or $(i, j+1, k+1)$ in G_3 with coordinate $k \equiv 1 (\bmod 2)$,
- v has neighbors $(i-1, j+1, k)$ or $(i+1, j-1, k)$ in G_3 with coordinate $j \equiv 1 \pmod 2$.

• *even*, if one of the following conditions is fulfilled:
- v has neighbors $(i-1, j, k-1)$ or $(i+1, j, k+1)$ in G_3 with coordinate $i \equiv 0 \pmod 2$,
- v has neighbors $(i, j-1, k-1)$ or $(i, j+1, k+1)$ in G_3 with coordinate $k \equiv 0 \pmod 2$,
- v has neighbors $(i-1, j+1, k)$ or $(i+1, j-1, k)$ in G_3 with coordinate $j \equiv 0 \pmod 2$.

Step 5: Each odd vertex $v \in G_4$ gets $d_4(v)$ low white colors, each even vertex $u \in G_4$ gets $d_4(u)$ high white colors.
Claim: There is enough white colors which will not cause any conflicts with already used colors.
[[There is enough white colors by Lemma 4.1. No color conflict can occur because white colors have only been used for isolated vertices in Step 2 and for centers of quasistars in Step 4a. Those vertices were not neighbors of vertices participating in this step.]]

After Step 5 only isolated vertices remain.

Step 6: Each $v \in G_5$ gets $d_5(v)$ low white colors.

Claim: No conflict can occur.
[[Note that the only possibility for a vertex $v \in G$ is that it was the terminal vertex of a quasistar in G_3 at distance 2 from the center of a quasistar $u \in G_3$. Let t be the common neighbor of u and v in G_3 (such a vertex must exist otherwise v would be completely colored in Step 3). By Lemma 4.1 we have $d_3(v) + d_3(t) \leq \max\{q(v), q(u)\}$ and hence $d_5(v) = d_3(v) \leq \max\{q(v), q(u)\}$.

Since white colors were used only for isolated vertices in Step 3, for centers of quasistars in Step 4a and for straight paths in Step 5, there is enough white colors to color v completely without any conflicts.]]

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