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A REMARK ON LORENTZ ALGEBRAS

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Abstract. An interesting topic in the theory of Jordan Banach algebras is that of ultraprimeness. In this note we prove that for Lorentz algebras, equipped with the spectral norm, the constant of ultraprimeness is $\frac{1}{3}$ regardless of dimension.

1. Introduction

Many problems that arise from quantum physics and theories of interaction of elementary particles give rise to special algebraic structures. Jordan algebras were introduced as a possible algebraic foundation of calculus of observables. In general, a vector space $(\mathcal{A}, \circ)$ over field $\mathbf{F}$, equipped with the bilinear mapping $(a,b) \longmapsto a \circ b$ is a Jordan algebra if

$$a \circ b = b \circ a$$

and

$$\left(a^2 \circ b\right) \circ a = a^2 \circ (a \circ b)$$

hold for all $a, b \in \mathcal{A}$. A considerable supply of Jordan algebras is given by

Example 1. *Let $\mathcal{B}$ be a linear subspace of an associative algebra $\mathcal{A}$ which is square stable, i.e. for any $a \in \mathcal{A}$, a^2 belongs to $\mathcal{B}$. Then $\mathcal{B}$ equipped with the Jordan product*

$$a \circ b = \tfrac{1}{2}(ab + ba)$$

is a Jordan algebra.

Jordan algebras are not used only in quantum mechanics, they also have applications in various parts of mathematics such as geometry of bounded domains, analysis on symmetric cones, statistics, etc. Some recommended books about them are [4, 6, 9, 10] and [13].

Let $\mathcal{A}$ be an associative Banach algebra. Given any $a, b \in \mathcal{A}$ we define the algebraic operator

$$M_{a,b} : \mathcal{A} \longrightarrow \mathcal{A}$$

by

$$M_{a,b}(x) = axb.$$

Linear combinations of such operators are called elementary operators and have been widely studied in recent years. As part of the study of elementary operators and related topics, M. Mathieu (see [9]) introduced the idea of an ultraprime Banach algebra. An algebra $\mathcal{A}$ is defined to be ultraprime if there is a positive number κ such that the estimate

$$\|M_{a,b}\| \geq \kappa \|a\| \|b\|$$

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is valid for all $a, b \in \mathcal{A}$. Obviously every ultraprime algebra is prime, while the converse is not true. One of the counterexamples is the algebra of the Hilbert-Schmidt operators. For Jordan algebras this theory was first considered by M. Cabrera and P. Rodriguez in [2] (see also [11]). In Jordan algebras the operator which represents their primeness is the Jacobson-McCrimmon operator, defined by

$$U_{a,b}(x) = (a \circ x) \circ b + (b \circ x) \circ a - (a \circ b) \circ x.$$

In the case of the Jordan algebras from Example 1 this reduces to

$$U_{a,b}(x) = \tfrac{1}{2}(axb + bxa).$$

The definition of ultraprimeness can be given in the form that there should exist a positive constant κ such that

$$\|U_{a,b}\| \geq \kappa \|a\| \|b\|,$$

for all $a, b \in \mathcal{A}$. In [12] L.L. Stacho and B. Zalar considered the ultraprimeness of the Jordan algebras of hermitian operators on a real or complex Hilbert space. These algebras belong to a wider class, which is related to the analysis on cones in $\mathbf{R}^n$ (see [4] for details). These algebras are called Euclidean algebras and are completely classified. Beside hermitian operators on a real, complex, quaternionic or octonionic space, there is one more class of Euclidean algebras. They are called Lorentz algebras and are associated to Minkowski light cones (see [4] for details).

From the results of [1, 2] and [3] it is already known that in the spectral norm the estimate

$$\|U_{a,b}\| \geq \tfrac{1}{2}\left(\sqrt{2}-1\right) \|a\| \|b\|,$$

is valid for all elements a, b from a Lorentz algebra, regardless of its dimension. It is the purpose of the present note to sharpen this result and improve the upper estimate.

2. Definitions

Let $\mathcal{H}$ be a finite dimensional real Hilbert space and let $\langle a, b\rangle$ be the inner product defined on $\mathcal{H}$. On the vector space $\mathcal{L} = \mathbf{R} \oplus \mathcal{H}$ we define the product

$$(\lambda + a) \circ (\mu + b) = \lambda\mu + \langle a, b\rangle + \mu a + \lambda b.$$

It is not difficult to verify that $\mathcal{L}$ is a Jordan algebra with the identity element $e = 1 + 0$. It is called Lorentz algebra and belongs to the class of Euclidean algebras associated to symmetric cones in real vector spaces (see [4]).

Remark 1. *Lorentz algebra also arises in the framework of Example 1 as a subspace of the Clifford algebra constructed from inner product $\langle .,. \rangle$. There exist Jordan algebras that cannot be obtained from such a construction.*

In the monograph of J.Faraut and A. Koranyi there are two important norms on Euclidean Jordan algebras which are considered. The first one is the obvious Hilbert norm $\|x\| = \sqrt{\langle x, x\rangle}$. The second one is the so called spectral norm and is defined as

$$\|u\|_\infty = \max\{|\lambda_i|\}, \tag{1}$$

where λ_i are eigenvalues of the spectral decomposition of $u \in \mathcal{L}$. We recall that by spectral decomposition ([4], page 43), for each $u \in \mathcal{L}$ there exist unique real numbers $\lambda_1, \ldots, \lambda_k$ all distinct, and a unique complete system of orthogonal idempotents $\{c_1, \ldots, c_k\}$ such that

$$u = \lambda_1 c_1 + \ldots + \lambda_k c_k.$$

We also recall that $\{c_1, \ldots, c_k\}$ is a complete system of orthogonal idempotents if

$$c_i^2 = c_i,$$

$$c_i \circ c_j = 0 \quad \text{for} \quad i \neq j$$

$$c_1 + c_2 + \ldots + c_k = e.$$

It is not difficult to verify that the only possible nonzero idempotents in Lorentz algebra $\mathcal{L}$ are e and $\frac{1}{2} + \frac{1}{2}\frac{a}{\|a\|}$, for all $a \in \mathcal{H}$. The spectral norm is the analogue of the operator norm which is the framework of most existing results on ultraprimeness, so it is natural to use it in our work as well. In this way we obtain a result which is independent of the dimension of $\mathcal{L}$.

Let $u = \lambda + a \in \mathcal{L}$ be arbitrary and let

$$u = \lambda_1 c_1 + \lambda_2 c_2 = \lambda_1 \left(\tfrac{1}{2} + \tfrac{1}{2}\tfrac{u}{\|u\|} \right) + \lambda_2 \left(\tfrac{1}{2} + \tfrac{1}{2}\tfrac{v}{\|v\|} \right)$$

be its spectral decomposition. Obviously $2\lambda = \lambda_1 + \lambda_2$. Since the idempotents c_1 and c_2 are orthogonal, we have $u = -v$ and so $2\|a\| = |\lambda_1 - \lambda_2|$. Hence we have $\lambda_1 = \lambda + \|a\|$ and $\lambda_2 = \lambda - \|a\|$. Considering both identities, the norm (1) in the case of Lorentz algebra can be rewritten as

$$\|u\|_\infty = |\lambda| + \|a\|.$$

In the context of a general study of ultraprime Jordan Banach algebras, the authors of [3] consider in particular the case of JB^*-algebras, and prove that for Lorentz algebras the constant of ultraprimeness is $\frac{1}{2}\left(\sqrt{2} - 1\right)$. It is our purpose in the sequel to improve the existing estimate. More precisely we shall prove

Theorem 1. *Let $\mathcal{L}$ be a Lorentz algebra. Then we have*

$$\|U_{a,b}\|_\infty \geq \tfrac{1}{3}\|a\|_\infty \|b\|_\infty,$$

for all $a, b \in \mathcal{L}$.

Note that $\|U_{a,b}\|_\infty$ means

$$\sup_{\|x\|_\infty \leq 1} \|U_{a,b}(x)\|_\infty.$$

3. Proof of the Theorem

In the case of Lorentz algebra the Jacobson-McCrimmon operator can be described as

$$U_{a,b}(x) = \lambda\mu\rho + \lambda\langle v, w\rangle + \mu\langle u, w\rangle + \rho\langle u, v\rangle +$$

$$+\lambda\mu w + \mu\rho u + \lambda\rho v + \langle v, w\rangle u + \langle u, w\rangle v - \langle u, v\rangle w,$$

where $a = \lambda + u$, $b = \mu + v$ and $x = \rho + w \in \mathcal{L}$.

Lemma 1. *Let $\mathcal{L}$ be a Lorentz algebra and let $a = \lambda + u$, $b = \mu + v \in \mathcal{L}$ be ∞-norm one elements. Then we have*

$$\|U_{a,b}\|_\infty \geq \max\{|2\|u\| - 1|, |2\|v\| - 1|\}.$$

Proof. From the definitions of norm and $U_{a,b}$ we can easily calculate that

$$\left\|U_{a,b}(x)\right\|_{\infty} = \left|\lambda\mu\rho + \lambda\left\langle v,w\right\rangle + \mu\left\langle u,w\right\rangle + \rho\left\langle u,v\right\rangle\right| + \left\|\lambda\mu w + \mu\rho u + \lambda\rho v + \left\langle v,w\right\rangle u + \left\langle u,w\right\rangle v - \left\langle u,v\right\rangle w\right\|.$$

Choose $x = \rho + w \in \mathcal{L}$ such that $\rho = -\lambda$ and $w = u$ and compute expression $\left\|U_{a,b}x\right\|$. We have

$$\begin{aligned}\left\|U_{a,b}\right\|_{\infty} &\geq \left|-\lambda^2\mu + \lambda\left\langle u,v\right\rangle + \mu\left\langle u,u\right\rangle - \lambda\left\langle u,v\right\rangle\right| + \\ &+ \left\|\lambda\mu u - \lambda\mu u - \lambda^2 v + \left\langle u,v\right\rangle u + \left\langle u,u\right\rangle v - \left\langle u,v\right\rangle u\right\| = \\ &= \left|-\lambda^2\mu + \left\|u\right\|^2\mu\right| + \left\|-\lambda^2 v + \left\|u\right\|^2 v\right\| \geq \left|-\lambda^2 + \left\|u\right\|^2\right|\left(\left|\mu\right| + \left\|v\right\|\right).\end{aligned}$$

Since $\left|\lambda\right| + \left\|u\right\| = \left|\mu\right| + \left\|v\right\| = 1$ we have

$$\left\|U_{a,b}\right\|_{\infty} \geq \left|2\left\|u\right\| - 1\right|.$$

If we replace ρ by $-\mu$ and w by v in the same way as above, we get

$$\left\|U_{a,b}\right\|_{\infty} \geq \left|2\left\|v\right\| - 1\right|.$$

Considering both estimates, we conclude the proof. □

Lemma 2. *Let $\mathcal{L}$ be a Lorentz algebra and let $a = \lambda + u$, $b = \mu + v \in \mathcal{L}$ be $\infty-$norm one elements. Then we have*

$$\left\|U_{a,b}\right\|_{\infty} \geq \left(1 - \left\|u\right\|\right)\left(1 - \left\|v\right\|\right) + \left\|u\right\|\left\|v\right\|.$$

Proof. Suppose first that λ has the same sign as μ. Choose $x = \rho + w \in \mathcal{L}$ such that $\rho = 0$ and $w = \frac{\left\|v\right\|u + \left\|u\right\|v}{\left\|\left\|v\right\|u + \left\|u\right\|v\right\|}$. Then $\left\|w\right\| = 1$ and so

$$\begin{aligned}\left\|U_{u,v}\right\|_{\infty} \geq \left\|U_{u,v}(x)\right\|_{\infty} &\geq \left|\lambda\left\langle v, \frac{\left\|v\right\|u + \left\|u\right\|v}{\left\|\left\|v\right\|u + \left\|u\right\|v\right\|}\right\rangle + \mu\left\langle u, \frac{\left\|v\right\|u + \left\|u\right\|v}{\left\|\left\|v\right\|u + \left\|u\right\|v\right\|}\right\rangle\right| + \\ &+ \left\|\lambda\mu\frac{\left\|v\right\|u + \left\|u\right\|v}{\left\|\left\|v\right\|u + \left\|u\right\|v\right\|} + \left\langle v, \frac{\left\|v\right\|u + \left\|u\right\|v}{\left\|\left\|v\right\|u + \left\|u\right\|v\right\|}\right\rangle u + \right. \\ &\left. + \left\langle u, \frac{\left\|v\right\|u + \left\|u\right\|v}{\left\|\left\|v\right\|u + \left\|u\right\|v\right\|}\right\rangle v - \left\langle u,v\right\rangle\frac{\left\|v\right\|u + \left\|u\right\|v}{\left\|\left\|v\right\|u + \left\|u\right\|v\right\|}\right\| \geq \\ &\geq \frac{1}{\left\|\left\|v\right\|u + \left\|u\right\|v\right\|}\left\|\lambda\mu\left(\left\|v\right\|u + \left\|u\right\|v\right) + \left\|v\right\|\left\langle v,u\right\rangle u + \right. \\ &\left. + \left\|u\right\|\left\|v\right\|^2 u + \left\|v\right\|\left\|u\right\|^2 v + \left\|u\right\|\left\langle v,u\right\rangle v - \left\|v\right\|\left\langle u,v\right\rangle u - \left\|u\right\|\left\langle u,v\right\rangle v\right\| = \\ &= \frac{1}{\left\|\left\|v\right\|u + \left\|u\right\|v\right\|}\left\|\lambda\mu\left(\left\|v\right\|u + \left\|u\right\|v\right) + \left\|u\right\|\left\|v\right\|^2 u + \left\|v\right\|\left\|u\right\|^2 v\right\| = \\ &= \frac{1}{\left\|\left\|v\right\|u + \left\|u\right\|v\right\|}\left\|\left(\lambda\mu + \left\|u\right\|\left\|v\right\|\right)\left(\left\|v\right\|u + \left\|u\right\|v\right)\right\| = \\ &= \left|\lambda\mu + \left\|u\right\|\left\|v\right\|\right| = \left(1 - \left\|u\right\|\right)\left(1 - \left\|v\right\|\right) + \left\|u\right\|\left\|v\right\|.\end{aligned}$$

If λ has the opposite sign as μ, then we choose $x = \rho + w \in \mathcal{L}$ such that $\rho = 0$ and $w = \frac{\left\|v\right\|u - \left\|u\right\|v}{\left\|\left\|v\right\|u + \left\|u\right\|v\right\|}$. In the same way as above, we get

$$\begin{aligned}\left\|U_{u,v}\right\|_{\infty} \geq \left\|U_{u,v}(x)\right\|_{\infty} &\geq \frac{1}{\left\|\left\|v\right\|u - \left\|u\right\|v\right\|}\left\|\left(\lambda\mu - \left\|u\right\|\left\|v\right\|\right)\left(\left\|v\right\|u - \left\|u\right\|v\right)\right\| = \\ &= \left|\lambda\mu - \left\|u\right\|\left\|v\right\|\right| = \left(1 - \left\|u\right\|\right)\left(1 - \left\|v\right\|\right) + \left\|u\right\|\left\|v\right\|.\end{aligned}$$

□

Proof of Theorem 1. Without loss of generality we may assume that $\|a\|_\infty = \|b\|_\infty = 1$. Let $a = \lambda + u$ and $b = \mu + v$. Denote by

$$\kappa = \max\{\|u\|, \|v\|\}.$$

If $\kappa \geq \frac{2}{3}$, by Lemma 1, we have

$$\|U_{a,b}\|_\infty \geq |2\kappa - 1| = \left|2\tfrac{2}{3} - 1\right| = \tfrac{1}{3}.$$

If $\kappa \leq \frac{2}{3}$, then the second lemma yields

$$\|U_{a,b}\|_\infty \geq (1 - \|u\|)(1 - \|v\|) + \|u\|\|v\| \geq \tfrac{1}{3}.$$

Considering both cases, we have

$$\|U_{a,b}\|_\infty \geq \tfrac{1}{3}\|a\|_\infty \|b\|_\infty.$$

□

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