

UNIVERSITY OF LJUBLJANA
INSTITUTE OF MATHEMATICS, PHYSICS AND MECHANICS
DEPARTMENT OF MATHEMATICS
JADRANSKA 19, 1000 LJUBLJANA, SLOVENIA

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EDGE-COLOURINGS OF CUBIC
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OF AUTOMORPHISMS

Primož Potočnik

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**Edge-colourings of cubic graphs admitting a solvable
vertex-transitive group of automorphisms**

PRIMOŽ POTOČNIK¹,
IMFM, Oddelek za matematiko,
Univerza v Ljubljani,
Jadranska 19, SI-1000 Ljubljana
Slovenija

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Abstract

It is proved that every connected cubic simple graph admitting a vertex-transitive action of a solvable group is either 3-edge-colourable, or isomorphic to the Petersen graph.

Key words: graph, solvable group, vertex-transitive graph, transitive action, edge-colouring, flow.

Ključne besede: graf, rešljiva grupa, točkovno tranzitiven graf, tranzitivno delovanje, barvanje povezav, pretok.

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1 Introduction

In this section all graphs are considered to be finite and simple. By the well-known theorem of Vizing the chromatic index of a graph with maximum degree Δ is either Δ or $\Delta + 1$. According to this result all graphs can be divided into two classes, *class 1* consisting of the graphs with the chromatic index being the same as the maximum degree, and *class 2* containing all graphs with the chromatic index being greater than the maximum degree. Though some sufficient conditions for the graph to belong to class 1 are known (for example, by an old result of König every bipartite graph belongs to class 1), an efficient characterization of class 1 graphs is a goal not very likely to be achieved in near future. The problem is unsolved even in the case of cubic graphs, where the characterization of class 1 graphs is closely related to two famous problems, namely to Tutte's conjecture on nowhere-zero 4-flows and to Lovász's question on Hamiltonian paths in vertex-transitive graphs.

Conjecture 1.1 (Tutte, [16]) *Every bridgeless cubic graph containing no subdivision of the Petersen graph admits a nowhere-zero 4-flow.*

Question 1.2 (Lovász, [8]) *Does every connected vertex-transitive graph contain a Hamiltonian path?*

The relation with Lovász's question is established through the fact that every Hamiltonian cubic graph is 3-edge-colourable, while the connection with Tutte's conjecture reflects in the fact that cubic graph belongs to class 1 if and only if it admits a nowhere-zero 4-flow. Tutte's conjecture motivated *the hunt of the snarks* (here *snark* stands for a bridgeless cubic class 2 graph) [2, 4, 5, 7, 11, 12, 13], as well as the search for sufficient conditions for cubic graphs to belong to class 1.

In view of Lovász's conjecture a natural direction of investigations leads to vertex-transitive graphs, that is graphs for which their automorphism group acts transitively on the set of vertices. There are only two known examples of connected cubic vertex-transitive graphs which are not 3-edge-colourable, namely the Petersen graph and its truncation (observe that the Coxeter graph is 3-edge-colourable even though it is not Hamiltonian). At this point the following question (asked by Riste Škrekovski – oral communication), arises naturally.

Question 1.3 *Is every connected cubic vertex-transitive graph which is not isomorphic neither to the Petersen graph nor to the truncation of the Petersen graph 3-edge-colourable?*

This question is a generalization of the conjecture made by Alspach and Zhang in 1991 saying that every connected Cayley graph admits a nowhere-zero 4-flow (or equivalently, is 3-edge-colourable). Alspach, Liu and Zhang [1] proved this conjecture in the case of cubic Cayley graphs of solvable groups (that is, cubic graphs with

the automorphism group containing a regular solvable group). Their result was improved by Nedela and Škoviera [12] who proved that every counter-example to the conjecture of Alspach and Zhang should be a regular cover over a Cayley graph of an almost simple group. Let us mention at this point that the conjecture of Alspach and Zhang (though closely related to the Tutte's 4-flow conjecture) remains open even in the case if the Tutte' conjecture holds for cubic graphs (see [14]), as many Cayley graphs do contain subdivisions of the Petersen graph.

In this article we give the following partial answer to Question 1.3 and thus generalize the result of Alspah, Liu and Zhang to a much wider class of graphs.

Theorem 1.4 *Let X be a connected cubic simple graph and suppose that its automorphism group contains a solvable subgroup acting transitively on the set of vertices of X . If X is not 3-edge-colourable then it is isomorphic to the Petersen graph.*

In Section 2 we define some basic notions needed later. We prove the above theorem in Section 3.

2 Preliminaries

Though our main interest lies in the graphs without loops and semiedges it will be convenient for us to use slightly more general definition of a graph, which arises naturally in the context of graph covers and quotients. We only give some very basic definition and results in this section and refer the reader to [9, 10] for all notions and concepts not defined here.

A *graph* is an ordered 4-tuple $(D, V; \text{beg}, \text{inv})$ where D and $V \neq \emptyset$ are disjoint finite sets of *darts* and *vertices*, respectively, $\text{beg} : D \rightarrow V$ is a mapping which assigns to each dart x its *initial vertex* $\text{beg } x$, and $\text{inv} : D \rightarrow D$ is an involution which interchanges every dart x and its *inverse dart* x^{-1} . The orbits of inv are called *edges*. The *initial vertices of an edge* are the initial vertices of the darts contained in the edge. An edge is called a *semiedge* if $\text{inv } x = x$, a *loop* if $\text{inv } x \neq x$ while $\text{beg } (x^{-1}) = \text{beg } x$, and is called a *link* otherwise. Two links are *parallel* if they have the same initial vertices. A graph with no semiedges, no loops and no parallel links is called a *simple graph*. *Degree* of a vertex v is the number of darts having v as their initial vertex. Graph is *cubic* if all its vertices have degree 3.

A mapping $c: D \rightarrow \{1, 2, \dots, n\}$ is a *proper n -edge colouring of the graph $X = (D, V, \text{beg}, \text{inv})$* if for every pair $x, y \in D$, such that $\text{beg } x = \text{beg } y$, we have $c(x) \neq c(y)$. If there exists a proper n -edge-colouring of the graph we say that the graph is *n -edge-colourable*. The minimal integer n for which a graph is n -edge-colourable is called the *chromatic index* of the graph X .

A *morphism of graphs* $f: X \rightarrow X'$ is a function $f: V_X \cup D_X \rightarrow V_{X'} \cup D_{X'}$ such that $f(V_X) \subseteq V_{X'}$, $f(D) \subseteq D_{X'}$, $f \text{ beg}_X = \text{beg}_{X'} f$ and $f \text{ inv}_X = \text{inv}_{X'} f$.

A graph morphism is *epimorphism (automorphism)* if it is a surjection (bijection, respectively). The group of automorphisms of a graph X is denoted by $\text{Aut } X$.

Let $N \leq \text{Aut } X$ be a subgroup of the automorphism group and let D_N and V_N denote the sets of orbits on darts and vertices of X , respectively. Further, let $\text{beg}_N[x] = [\text{beg } x]$ and $\text{inv}_N[x] = [\text{inv } x]$. This defines the *quotient graph* $X_N = (D_N, V_N; \text{beg}_N, \text{inv}_N)$ together with the natural epimorphism $p_N: X \rightarrow X_N$ called the *quotient projection relative to N* .

If the quotient projection $\wp_N: X \rightarrow X_N$ of connected graphs is also a local bijection on darts (that is, if for each vertex $v \in V_X$ the set of darts of X having v as the initial vertex is mapped bijectively onto the set of darts of X having $\wp(v)$ as the initial vertex) then the quotient projection \wp is a *regular covering projection* and the group N is the group of covering transformation. Observe that chromatic index of the covering graph is at most that of the base graph.

Regular covering projections can be combinatorialized as follows [6, 9, 10]. A *Cayley voltage space* on a connected graph $X = (D, V; \text{beg}, \text{inv})$ is an ordered pair $(N; \zeta)$, where N is a *voltage group* acting on itself by right multiplication, and $\zeta: D \rightarrow N$ is a surjective function such that $\zeta(x^{-1}) = (\zeta(x))^{-1}$. This function extends naturally to all walks in X . Note that homotopic walks have the same voltage. This allows us to view the function ζ as defined on the fundamental group $\pi(X, b)$ of X at base point b . Moreover, if the group N is abelian, the function ζ naturally extends to the function defined on the abelianization $H_1(X)$ of $\pi(X, b)$. Finally, if N is of prime exponent p then we can view ζ as defined on the first homology group $H_1(X, \mathbb{Z}_p)$ with coefficient in the prime field \mathbb{Z}_p . (We refer the reader to [10] for details).

Let $(N; \zeta)$ be a Cayley voltage space on a connected graph $X = (D, V; \text{beg}, \text{inv})$. With $(N; \zeta)$ we associate a *derived covering projection* $\wp_\zeta: \text{Cov}(N; \zeta) \rightarrow X$ as follows. The graph $\text{Cov}(N; \zeta)$ has $D \times N$ and $V \times N$ as the sets of darts and vertices, respectively, with $\text{beg}(x, \nu) = (\text{beg } x, \nu)$ and $\text{inv}(x, \nu) = (\text{inv } x, \nu\zeta(x))$. The corresponding projection \wp_ζ is defined as the projection onto the first component. This is indeed a regular covering projection with the group of covering transformation isomorphic to the group N . It is not difficult to see that every regular covering projection can be obtained in this way.

3 Proof of Theorem 1.4

Suppose that the statement of Theorem 1.4 is false and let \tilde{X} be the smallest graph (in terms of the number of its vertices) which satisfies the conditions of the theorem, but is neither isomorphic to the Petersen graph nor 3-edge-colourable. Let \tilde{G} denote the solvable vertex-transitive subgroup of $\text{Aut } \tilde{X}$ and let N denote the minimal (non-trivial) normal subgroup of \tilde{G} . By [15, Theorem 5.24] there exists a prime p and a positive integer k such that N is isomorphic to the elementary abelian group

\mathbb{Z}_p^k . Let X denote the quotient graph \tilde{X}_N , $\wp : \tilde{X} \rightarrow X$ the corresponding quotient projection, and G the quotient group \tilde{G}/N . Observe first that if the vertex set of X is a singleton then the graph \tilde{X} is a cubic Cayley graph of elementary abelian group and \tilde{X} is 3-edge-colourable by [1]. We can thus assume that X has at least two vertices. Clearly, the group G acts transitively (in the natural way) on the vertices of X , and is solvable. Furthermore, the connectedness of \tilde{X} implies the connectedness of X . Observe that X is cubic if and only if the quotient projection \wp is a covering projection. Since every cover of a 3-edge-colourable graph is also 3-edge-colourable, we can assume that if X is cubic it is not 3-edge-colourable. The minimality of \tilde{X} then implies that one of the following occurs:

- a) X is not cubic, or
- b) X is cubic, but neither simple nor 3-edge-colourable, or
- c) X is isomorphic to the Petersen graph.

Assume first that X is not cubic. Then its valence is either 1 or 2. If the valence of X is 1 then it is isomorphic to K_2 and \tilde{X} is bipartite and thus 3-edge-colourable, a contradiction. If the valence of X is 2 then X is isomorphic either to K_2 with the additional semiedge at every vertex, or to a cycle C_n . In the first case the \wp -preimage of the two semiedges of X is a 1-factor (or a disjoint union of two 1-factors) of \tilde{X} , and the \wp -preimage of the edge of X is a disjoint union of cycles of even length (or a 1-factor of \tilde{X} , respectively). In both cases \tilde{X} is 3-edge-colourable. If X is isomorphic to C_n then \wp -preimage of every second edge of C_n is a 1-factor and the \wp -preimage of other edges of C_n is disjoint union of two 1-factors. This shows that n is even and that \tilde{X} is bipartite, a contradiction.

We can thus assume that X is a cubic graph. Suppose first that X is not simple. There are only 2 infinite families and four sporadic cases of vertex-transitive cubic graphs (with more than one vertex) which are not simple. These are:

1. the dipole D_3 , having two vertices and three parallel edges between them;
2. the graph D_2' , having two vertices, two parallel edges between them, and a semiedge attached to every vertex;
3. the graph \bar{C}_{2n} obtained from the cycle C_{2n} by attaching an edge parallel to every second edge of the cycle;
4. the graph K_2° obtained from the graph K_2 by attaching a loop to each of the two vertices;
5. the graph K_2'' obtained from the graph K_2 by attaching a pair of semiedges to each of the two vertices;
6. the graph C_{2n}' obtained from the cycle C_{2n} by attaching a semiedge to every vertex of the cycle.

Observe that all of the above graphs, except for the graph K_2° , are 3-edge-colourable. We can thus assume that $X \cong K_2^\circ$ and that \tilde{X} is a connected regular cover over K_2° with elementary abelian group N of covering transformations. Since the Betti number of the graph K_2° is 2, the group N is isomorphic either to \mathbb{Z}_p or to \mathbb{Z}_p^2 . In the first case \tilde{X} is a generalized Petersen graph. It is known that all generalized Petersen graph, except for the Petersen graph itself, are 3-edge-colourable (see [3]). In the second case the graph \tilde{X} is isomorphic to the homological p -cover of the graph K_2° . It is easy to see that it is 3-edge-colourable as well, in fact, it is a Cayley graph of a semidirect product of the elementary abelian group \mathbb{Z}_p^2 and the group \mathbb{Z}_2 .

We are now left with the case where X is isomorphic to the Petersen graph. We begin by proving the following lemma on solvable vertex-transitive group of automorphisms of the Petersen graph.

Lemma 3.1 *Let Pet be the Petersen graph and G a solvable vertex-transitive subgroup of Aut Pet . Let $\mathbb{Z}_5^{(2)} = \{ij \mid i, j \in \mathbb{Z}_5\}$ denote the set of unordered pairs of elements of \mathbb{Z}_5 and let the symmetric group S_5 act on the set $\mathbb{Z}_5^{(2)}$ in the natural way. Denote by ρ and τ the elements $(0, 1, 2, 3, 4)$ and $(1, 2, 4, 3)$ of S_5 , respectively. Then the action of G on the vertex-set of Pet is isomorphic to the action of the group $\langle \rho, \tau \rangle$ on $\mathbb{Z}_5^{(2)}$.*

PROOF. Since G acts transitively on the set of 10 elements it contains an element r' of order 5 having exactly two orbits on the vertex-set of Pet . Since Pet is not bipartite there are at least two adjacent vertices, say u and v , contained in the same orbit of r' . This implies that some power of r' , call it r , maps the vertex u to the vertex v . Let w be the neighbour of u which is not contained in the same orbit of r as u . For every $i \in \mathbb{Z}_5$, let u_i and w_i denote the vertices $r^i(u)$ and $r^i(w)$, respectively. Define a bijection $f: V(\text{Pet}) \rightarrow \mathbb{Z}_5^{(2)}$ by the rules $f(u_i) = i(i+2)$ and $f(w_i) = (i+3)(i+4)$, for every $i \in \mathbb{Z}_5$. Note that any two vertices of Pet are adjacent if and only if their f -images are disjoint. The bijection f gives rise to a group-isomorphism $\phi: \text{Aut Pet} \rightarrow S_5$, such that $(f, \phi): (V(\text{Pet}), \text{Aut Pet}) \rightarrow (\mathbb{Z}_5^{(2)}, S_5)$ is an isomorphism of actions. Note that $\phi(G)$ contains the permutation ρ . Since $\phi(G)$ is a solvable subgroup of S_5 containing ρ and acting on the set $\mathbb{Z}_5^{(2)}$ transitively it must be the group of order 20 generated by ρ and τ . The pair $(f, \phi|_G)$ is then the isomorphism of the actions $(V(\text{Pet}), G)$ and $(\mathbb{Z}_5^{(2)}, \langle \rho, \tau \rangle)$. \blacksquare

In view of lemma 3.1 we can assume that the vertices of X are labelled by the elements of the set $\mathbb{Z}_5^{(2)}$ and that the group G is generated by the elements

$$\rho = (0, 1, 2, 3, 4) \quad \text{and} \quad \tau = (1, 2, 4, 3)$$

of the symmetric group S_5 , acting in the natural way on the set $\mathbb{Z}_5^{(2)}$. Since the graph \tilde{X} is a regular N -cover over the graph $X \cong \text{Pet}$, we can identify \tilde{X} with the

derived covering graph $\text{Cov}(N, \zeta)$, where ζ is a voltage assignment, defined on the homology group $H_1(\text{Pet}, \mathbb{Z}_p)$ of the Petresen graph Pet (with coefficients in \mathbb{Z}_p). We shall denote by T the spanning tree of Pet containing all the edges of Pet , except the underlying edge of the dart $\vec{e} = 12 \rightarrow 34$ and the underlying edges of the darts $\vec{x}_i = i(i+2) \rightarrow (i+1)(i+3)$, $i \in \mathbb{Z}_5$. By identifying each dart \vec{x} (of the above six) with the oriented cycle of Pet containing no other darts outside T but \vec{x} , we can consider them as the elements of the homology group $H_1(\text{Pet}, \mathbb{Z}_p)$. Clearly, the set $\{\vec{e}, \vec{x}_0, \dots, \vec{x}_4\}$ is then a basis of $H_1(\text{Pet}, \mathbb{Z}_p)$ (viewed as the vector space over \mathbb{Z}_p), and the voltage assignment ζ is uniquely determined by the following images (see Figure 1):

$$e = \zeta(\vec{e}) \quad \text{and} \quad x_i = \zeta(\vec{x}_i), \quad i \in \mathbb{Z}_5.$$

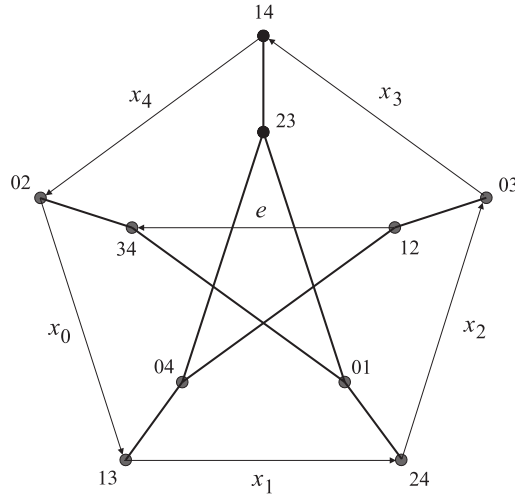


Figure 1

Since \tilde{X} is a connected graph, the voltages e, x_0, \dots, x_4 generate the elementary abelian group \mathbb{Z}_p^k . This implies immediately that $k \leq 6$. Furthermore, the fact that the group $G = \langle \rho, \tau \rangle$ lifts along the covering projection $\wp: \tilde{X} \rightarrow \text{Pet}$ implies by [10, Theorem 5.2] the existence of the group homomorphism $\#: G \rightarrow \text{Aut } N$, which satisfies the rule

$$\alpha^\# \circ \zeta = \zeta \circ \alpha,$$

for every $\alpha \in G$ (where α is considered as acting on $H_1(\text{Pet}, \mathbb{Z}_p)$). Using this formula we can deduce that the automorphisms $\rho^\#$ and $\tau^\#$ map the elements e, x_0, \dots, x_4 as it is shown in Table 1.

Table 1

	e	x_0	x_1	x_2	x_3	x_4
$\rho^\#$	e	$x_1 - e$	$x_2 + e$	$x_3 - e$	x_4	$x_0 + e$
$\tau^\#$	$-\sum_{i \in \mathbb{Z}_5} x_i$	$x_0 + x_3 + x_4$	$-x_3 - x_4 + e$	$x_2 + x_3 + x_4$	$-x_2 - x_3$	$-x_0 - x_4$

If we consider the elementary abelian group N as the vector space over the field \mathbb{Z}_p the group homomorphism $\#: G \rightarrow \text{Aut } N$ defines a linear representation of the group G . The minimality of the normal subgroup N in the group \tilde{G} implies that this representation is irreducible. In other words, linear transformations $\rho^\#$ and $\tau^\#$ have no common nontrivial invariant subspaces. It follows immediately from Table 1 that the subspace of N spanned by e and $\sum_{i \in \mathbb{Z}_5} x_i$ is invariant for both $\rho^\#$ and $\tau^\#$. This implies that either $N \cong \mathbb{Z}_p$, or $N \cong \mathbb{Z}_p^2$, or $N \cong \mathbb{Z}_p^k$, $k \geq 3$, and $e = \sum_{i \in \mathbb{Z}_5} x_i = 0$.

Case 1 $N \cong \mathbb{Z}_p$. Since the automorphism group $\text{Aut } N$ is isomorphic to the multiplicative group \mathbb{Z}_p^* (acting on N by multiplication), there exist two elements $r, t \in \mathbb{Z}_p$, such that $\rho^\#(a) = ra$ and $\tau^\#(a) = ta$, for each $a \in N$. Moreover, since $\tau\rho\tau^{-1} = \rho^2$ and since $\#: G \rightarrow \text{Aut } N$ is a group homomorphism, it follows that $t\tau t^{-1} = r^2$ and thus $r = 1$. It follows from Table 1 that $x_2 = x_0$ and $x_1 = x_3 = x_4 = x_0 + e$, in this case. Furthermore, $tx_0 = 3x_0 + 2e$ and $t(x_0 + e) = -2x_0 - e$, which gives $5x_0 = -(3+t)e$. We are going to distinguish two subcases: $e \neq 0$ and $e = 0$.

If $e \neq 0$ then the two disjoint 5-cycles of Pet induced by the orbits of ρ lift into two disjoint $5p$ cycles of the graph \tilde{X} . The lift $\tilde{\rho} \in \text{Aut } \tilde{X}$ of the automorphism $\rho \in \text{Aut } X$ generates together with the group N a subgroup $\langle \tilde{\rho}, N \rangle$ of $\text{Aut } \tilde{X}$, which acts transitively on each of the two disjoint cycles of length $5p$. This implies that the group $\langle \tilde{\rho}, N \rangle$ is cyclic of order $5p$. It is easy to see that the graph \tilde{X} is then isomorphic to a generalized Petersen graph on $10p$ vertices, and is therefore 3-edge-colourable by [3].

Assume now that $e = 0$. In this case the equality $5x_0 = -(3+t)e$ implies $p = 5$. The graph \tilde{X} is then isomorphic to the \mathbb{Z}_5 -cover of Pet defined by voltages $e = 0$ and $x_0 = x_1 = x_2 = x_3 = x_4 = 1$. It is an easy exercise to show that this vertex-transitive graph on 50 vertices is 3-edge-colourable.

Case 2 $N \cong \mathbb{Z}_p^2$. In order to deal with this case we shall first prove the following general lemma on irreducible representations of the group G .

Lemma 3.2 *Let $G = \langle \rho, \tau \mid \rho^5 = \tau^4 = \rho^{-2}\tau\rho\tau^{-1} = 1 \rangle$ and let $\Phi: G \rightarrow \text{Aut } \mathbb{Z}_p^2$ be an irreducible representation. Then $\Phi(\rho) = \text{id}$.*

PROOF. Let $\bar{g} = \Phi(g)$ for all $g \in G$. Suppose first that $\bar{\rho}$ has no eigenvectors. Since there are $p+1$ 1-dimensional subspaces of the vector space \mathbb{Z}_p^2 , and since the order of $\bar{\rho}$ is 5, this implies that $p \equiv -1 \pmod{5}$. Let v be an arbitrary non-zero element of \mathbb{Z}_p^2 and let $v_1 = \bar{\rho}(v)$. Then $\{v, v_1\}$ is a basis of \mathbb{Z}_p^2 relative to which the automorphism $\bar{\rho}$ is represented by the matrix

$$\begin{pmatrix} 0 & \alpha \\ 1 & \beta \end{pmatrix}.$$

Since $p \equiv -1 \pmod{5}$ and since $\det(\bar{\rho})^5 = 1$, we have that $-\alpha = \det(\bar{\rho}) = 1$. Furthermore, it follows from $\bar{\rho}^5 = \text{id}$ that $\beta^2 + \beta - 1 = 0$. Then $\bar{\rho}^2$ is represented by

the matrix

$$\begin{pmatrix} -1 & -\beta \\ \beta & -\beta \end{pmatrix}.$$

Since $\bar{\rho}$ and $\bar{\rho}^2$ are conjugate (by $\bar{\tau}$), they must have the same characteristic polynomial. Therefore, $x^2 - \beta x + 1 = x^2 + (\beta + 1)x + (\beta^2 + \beta)$, and thus $\beta + 1 = -\beta$. If we use this equality together with $\beta^2 + \beta - 1 = 0$ we get $5 = 0$ and thus $p = 5$, contradicting the congruence $p \equiv -1 \pmod{5}$.

We can therefore assume that $\bar{\rho}$ has an eigenvector. Let $\bar{\rho}v = \lambda v$ for some non-zero element $v \in \mathbb{Z}_p^2$ and some $\lambda \in \mathbb{Z}_p^*$. Since $\bar{\rho}^5 = \text{id}$ we have $\lambda^5 = 1$. It follows from the equality $\bar{\tau}\bar{\rho}\bar{\tau}^{-1} = \bar{\rho}^2$ that the vector $v_1 = \bar{\tau}(v)$ is an eigenvector of $\bar{\rho}$, with the corresponding eigenvalue λ^3 . The spectrum of $\bar{\rho}$ is thus the set $\{\lambda, \lambda^3\}$. Since, $\bar{\rho}$ and $\bar{\rho}^2$ are conjugate, they have the same spectrum, implying the equality $\{\lambda, \lambda^3\} = \{\lambda^2, \lambda\}$, and so $\lambda = 1$. Since $\{v, v_1\}$ is a basis for \mathbb{Z}_p^2 (for otherwise v would be a common eigenvector for $\bar{\rho}$ and $\bar{\tau}$), this shows that $\bar{\rho} = \text{id}$. \blacksquare

By the above lemma we can assume that $\rho^\# = \text{id}$, which implies $x_2 = x_0$ and $x_1 = x_3 = x_4 = x_0 + e$. This shows that the set $\{x_0, e\}$ is a basis of the vector space N . Let $K \leq N$ be the subgroup generated by the element $x_0 + e$, and let $q: N \rightarrow N/K$ be the corresponding quotient projection. Observe that K is invariant under the action of $\rho^\#$. In view of [10, Corollary 5.3 and Proposition 4.1] the graph \tilde{X} is a regular cover over the covering graph $Y = \text{Cov}(\text{Pet}, q\zeta)$ and the automorphism $\rho \in \text{Aut Pet}$ lifts into an automorphism of the graph Y . If $p > 2$ the graph Y consists of two disjoint $5p$ cycles and a matching between them. Clearly, the lift of the automorphism ρ acts transitively on the two disjoint $5p$ -cycles. This implies that Y is a generalized Petersen graph on $10p$ vertices. This is true even in the case $p = 2$, where Y is isomorphic to the generalized Petersen graph $GP(10, 2)$. The graph Y is then 3-edge-colourable by [3]. Being a cover over Y the graph \tilde{X} is also 3-edge-colourable, a contradiction.

Case 3 $N \cong \mathbb{Z}_p^k$, $k \geq 3$, $e = \sum_{i \in \mathbb{Z}_5} x_i = 0$. Since the six voltages e, x_0, \dots, x_4 generate the group N we have that $k \leq 4$. The following lemma deals with the case $k = 3$.

Lemma 3.3 *Let $G = \langle \rho, \tau \mid \rho^5 = \tau^4 = \rho^{-2}\tau\rho\tau^{-1} = 1 \rangle$ and let $\Phi: G \rightarrow \text{Aut}\mathbb{Z}_p^3$ be a linear representation of the group G . Then Φ is reducible.*

PROOF. Suppose that Φ is irreducible. Let \bar{g} denote the linear transformation $\Phi(g)$, for every $g \in G$. Since there are $p^2 + p + 1$ 1-dimensional subspaces in \mathbb{Z}_p^3 and since, for every prime p , the number $p^2 + p + 1$ is not divisible by 5, there exists an eigenvector v of the linear transformation $\bar{\rho}$. Let λ be the corresponding eigenvalue. Since $\bar{\tau}\bar{\rho}\bar{\tau}^{-1} = \bar{\rho}^2$, the vectors $\bar{\tau}(v)$ and $\bar{\tau}^2(v)$ are eigenvectors of $\bar{\rho}$ with the corresponding eigenvalues λ^3 and λ^4 . Since $\bar{\rho}$ and $\bar{\tau}$ have no common invariant subspaces, the vectors $v, \bar{\tau}(v)$ and $\bar{\tau}^2(v)$ form a basis of \mathbb{Z}_p^3 . This implies that the

spectrum of $\bar{\rho}$ equals $\{\lambda, \lambda^3, \lambda^4\}$. Furthermore, since $\bar{\rho}$ and $\bar{\rho}^2$ are conjugate, they have the same spectrum, showing that $\{\lambda, \lambda^3, \lambda^4\} = \{\lambda^2, \lambda, \lambda^3\}$, and so $\lambda = 1$ and $\bar{\rho} = \text{id}$. Similarly, since the number $p^2 + p + 1$ is odd, the transformation $\bar{\tau}$ has a fixed point in its action on 1-dimensional subspaces of \mathbb{Z}_p^3 . The eigenvector of $\bar{\tau}$ is thus a common eigenvector of $\bar{\rho}$ and $\bar{\tau}$, a contradiction. ■

We are now left with the case $k = 4$. We can assume without loss of generality that x_0, x_1, x_2 and x_3 are linearly independent and that $x_4 = -x_0 - x_1 - x_2 - x_3$. Let $K \leq N$ be a subgroup generated by x_1, x_3 and $x_0 + x_2$, and let $q: N \rightarrow N/K$ be the corresponding quotient projection. In view of [10, Proposition 4.1] the graph \tilde{X} is a regular cover over the covering graph $Y' = \text{Cov}(\text{Pet}, q\zeta)$. By [10, Theorem 6.2] the graph Y' is isomorphic to the covering graph $Y = \text{Cov}(\text{Pet}, \zeta_1)$ associated with the Cayley voltage space (\mathbb{Z}_p, ζ_1) , where ζ_1 is defined by $\zeta_1(\vec{e}) = \zeta_1(\vec{x}_1) = \zeta_1(\vec{x}_3) = \zeta_1(\vec{x}_4) = 0$, $\zeta_1(\vec{x}_0) = 1$ and $\zeta_1(\vec{x}_2) = -1$. Recall that the vertex set of the graph Y is the set $\{(u, i) \mid u \in V(\text{Pet}), i \in \mathbb{Z}_p\}$. The s^{th} layer, $s \in \mathbb{Z}_p$, of the graph Y is the set of darts $x \in D(Y)$ having their initial vertex $\text{beg}_Y(x)$ contained in the set $\{(u, s) \mid u \in V(\text{Pet})\}$. Let $\mathcal{C} = \{1, 2, 3\}$ denote the set of three colours, and let π be a permutation of the set \mathcal{C} . The *colouring of type π* of a layer of Y is the mapping from this layer to the set \mathcal{C} , which is schematically shown on Figure 2.

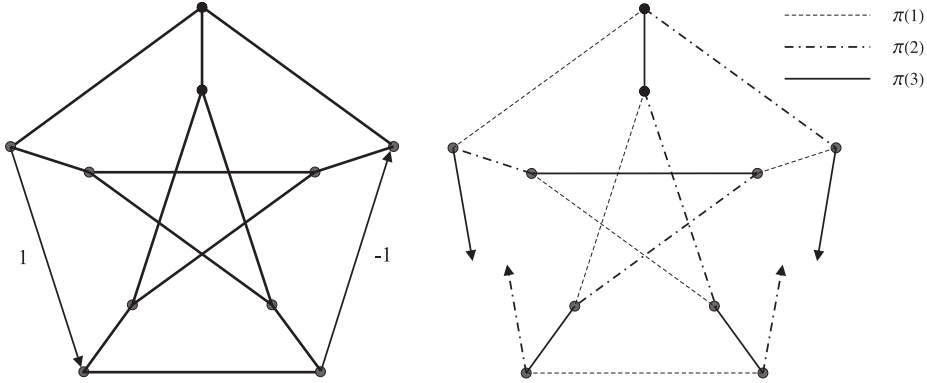


Figure 2: The graph Y and the colouring of a layer of the graph Y of type π .

If $p = 2$ then define the colouring of the graph Y in such a way that the 0^{th} layer receives the colouring of type id and the 1^{st} layer the colouring of type $(2, 3)$. This is clearly a proper 3-edge-colouring of the graph Y .

If p is odd then let the 0^{th} layer receive the colouring of type $(1, 2, 3)$, the 1^{st} layer the colouring of type $(1, 3, 2)$, the s^{th} layer, for $s \in \{2, 4, \dots, p-1\}$, the colouring of type id , and the s^{th} layer, for $s \in \{3, 5, \dots, p-2\}$, the colouring of type $(2, 3)$. Such a colouring is a proper 3-edge-colouring of the graph Y . Since \tilde{X} is a regular cover over Y , it is 3-edge-colourable as well. This completes the proof of Theorem 1.4.

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