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# WEAKLY REGULAR EMBEDDINGS OF STEIN SPACES WITH ISOLATED SINGULARITIES

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## 1. Introduction

The motivation for this paper was the following question: Let $M$ be a smooth, compact, strongly pseudoconvex, integrable $CR$-manifold of dimension $2n-1 \geq 5$ (and of $CR$-dimension $n-1 \geq 2$). Find the smallest integer $N = N(n)$ such that $M$ admits a $CR$ embedding into $\mathbf{C}^N$. By the results of Ohsawa and Rossi ([Oh1],[Oh2],[Ros]) there exist a pure $n$-dimensional Stein space $X$ with isolated singular points and a relatively compact domain $D \subset X$ such that $\partial D = M$. Therefore our problem reduces to the following problem: Let $X$ be an $n$-dimensional Stein space with isolated singular points. Find the smallest integer $N$ such that there exists a proper injective map $f: X \to \mathbf{C}^N$, which is regular on $\mathrm{Reg}(X)$. It turns out that the dimension $N$ can be expressed in terms of Whitney cones $C_5$ (for the definition see [Chi] or next chapter):

**Theorem 1.1.** *Let $X$ be an $n$-dimensional Stein space with isolated singular points. Let $N(X) = \max\{[n/2]+n+1, 3, \max\{\dim C_5(x,X),\ x \in X\}\}$ . Then there exists a proper injective map $f: X \to \mathbf{C}^{N(X)}$, which is regular on* $\mathrm{Reg}(X)$.

**Remark 1.** Since we are not interested in regularity at singular points we may (and will) with no loss of generality assume that the space is reduced. By [ABT] there exists a proper holomorphic injective map $f: X \to \mathbf{C}^N, N \geq 2n+1$, which is regular on $\mathrm{Reg}(X)$; the dimension $N(X)$ from the theorem 1.1., however, is at most $2n$, because $\dim C_5(x,X) \leq 2n$ for all $x \in X$ and all $n$-dimensional Stein spaces $X$.

**Remark 2.** In the case of normal Stein spaces any two weaky holomorphic embeddings are biholomorphically equivalent.

The paper is organized as follows: the second section contains the definition and some properties of Whitney cones and the last section consists of the proof of the main theorem.

**Definitions and notation.**
For $y \in \mathbf{C}^n$ let $|y| := \sup\{|y_i|, 1 \leq i \leq n\}$ denote the sup norm and $\|y\|$ the euclidean norm. By $B_n(r)$ we denote the ball in $\mathbf{C}^n$ with radius $r$ and center 0.

Let $X$ be a complex space, $K \subset X$ a compact subset and $f : X \to \mathbf{C}^n$ a continuous map. We will use the notations $|f|_K := \max\{|f(x)|,\ x \in K\}$ and $\|f\|_K := \max\{\|f(x)\|,\ x \in K\}$. By $\mathcal{O}(X)$ we denote the space of all holomorphic functions on a complex space $X$ equipped with the standard topology of uniform convergence on compact sets. For an analytic set $Y \subset X$ let $\Gamma(X, \mathcal{J}(Y))$ denote the space of holomorphic functions on $X$ which vanish on $Y$. By $TX$ we denote the complex tangent space of $X$ and with $T_x X$ the complex tangent space of $X$ at the point $x$.

A holomorphic map $f : X \to Y$ is *almost proper* if for each compact set $K \subset Y$ the connected components of $f^{-1}(K)$ are compact. A *stratification of a complex manifold* $X$ is a finite descending chain of analytic sets $A_m := X \supset A_{m-1} \ldots \supset A_0$ such that $A_i \setminus A_{i-1}$ is a complex manifold ($i = 1, \ldots, m$).

## 2. Some properties of tangent cones

The Whitney tangent cones that will play an important role in our work are the cones $C_3, C_4$ and $C_5$.

**Definition 2.1.** *Let $X \subset \mathbf{C}^m$ be an analytic set, $x \in X$. Then*

(*a*) $C_3(x, X) = \{v \in \mathbf{C}^m,$ *there exist a sequence* $x_j \in X,\ x_j \to x$ *and a sequence* $\lambda_j \in \mathbf{C}$ *such that* $\lambda_j(x_j - x) \to v\}$,

(*b*) $C_4(x, X) = \{v \in \mathbf{C}^m,$ *there exist a sequence* $z_j \in \mathrm{Reg}(X),\ z_j \to x$ *and a sequence* $v_j \in T_{z_j} X$ *such that* $v_j \to v\}$,

(*c*) $C_5(x, X) = \{v \in \mathbf{C}^m,$ *there exist sequences* $z_j, w_j \in X,\ z_j, w_j \to x$ *and a sequence* $\lambda_j \in \mathbf{C}$ *such that* $\lambda_j(z_j - w_j) \to v\}$,

(*d*) $C_i(X) = \{(x, v),\ x \in X, v \in C_i(x, X)\},\ i = 3, 4, 5.$

It is easy to see that $C_3(X), C_4(X), C_5(X) \subset TX$. Using the fact that every analytic set is locally biholomorphic to an analytic set in some $\mathbf{C}^m$, we can extend the above definition of cones to an arbitrary complex space X. A more detailed discussion on this subject can be found in [Chi] and [Stu]. Let us state some simple properties of Whitney cones ([Chi]):

(*i*) the cones $C_3(x, X), C_4(x, X)$ and $C_5(x, X)$ are biholomorphically invariant, projective algebraic sets with dim $C_3(x, X) = n$ and $n \leq$ dim $C_i(x, X) \leq 2n$, $i = 4, 5$,

(*ii*) $C_3(x, X) \subset C_4(x, X) \subset C_5(x, X)$,

(*iii*) $C_4(X)$ is the closure of $TX|_{\mathrm{Reg}(X)}$,

$(iv)$ if $x \in \operatorname{Reg}(X)$ then $\dim C_4(x, X) = \dim C_5(x, X) = n$,

$(v)$ if $\dim C_5(x, X) = n$ then $x \in \operatorname{Reg}(X)$.

**Example.** Let $X = (C^n \times 0) \cup (0 \times \mathbf{C}^n) \subset \mathbf{C}^{2n}$. Then $C_3(0, X) = C_4(0, X) = X$ and $C_5(0, X) = \mathbf{C}^{2n}$.

**Proposition 2.1.** [Chi] *Let $X \subset \mathbf{C}^m$ be an analytic set, $L = \mathbf{C}^{m-k} \times 0 \subset \mathbf{C}^m$, $0 \in X$ and suppose that $C_3(0, X) \cap L = \{0\}$. Then there exists an open set $U \subset \mathbf{C}^m$ such that the orthogonal projection $\pi_L : U \cap X \to \mathbf{C}^k$ is proper.*

**Remark.** The condition $C_3(0, X) \cap L = \{0\}$ implies that the neighbourhood of 0 lies in some cone. The condition is fulfilled for almost every $(m - k)$-dimensional linear subspace $L \subset \mathbf{C}^m$. Clearly, the projection along any $L$ with $\dim L \le m - n$ satisfying $C_3(0, X) \cap L = \{0\}$ is proper also.

**Proposition 2.2.** [Chi] *Let $X \subset \mathbf{C}^m$ be a pure $n$-dimensional analytic set, $L = \mathbf{C}^{m-k} \times 0 \subset \mathbf{C}^m$ and $0 \in X$. If $C_4(0, X) \cap L = \{0\}$ then there exists an open set $U \subset \mathbf{C}^m$ such that the orthogonal projection $\pi_L : U \cap X \to \mathbf{C}^k$ is a $p$-sheeted cover with $br(\pi_L, X \cap U) = (X \setminus \operatorname{Reg}(X)) \cap U$.*

**Remark.** In the case of a general $n$-dimensional analytic set such projection is of course not a cover; it is, however, proper (because $C_3(0, X) \subset C_4(0, X)$) and regular on $\operatorname{Reg}(X) \cap U$.

**Corollary 2.1.** *Let $X$ be a complex space, $x \in X$ and $\dim C_4(x, X) = k$. Then there exists an open set $U$ and a proper holomorphic map $f : U \to \mathbf{C}^k$, which is regular on $\operatorname{Reg}(X) \cap U$. Every holomorphic map $f : X \to \mathbf{C}^k$ with $\operatorname{Ker} Df(x) \cap C_4(x, X) = \{0\}$ is regular on $\operatorname{Reg}(X) \cap U$ for a suitable open neighbourhood $U$ of $x$.*

**Proof.** For the first part of the corollary note that the condition $\dim C_4(x, X) = k$ implies the existence of a $(m - k)$-dimensional linear subspace $L$ such that $L \cap C_4(x, X) = \{0\}$. The rest follows from the proposition 2.2. and the remark below. For the second one we may assume that $X \subset \mathbf{C}^m$ since the statement is local. If the statement were false, then there would be sequences $x_j \in \operatorname{Reg}(X)$, $x_j \to x$ and $v_j \in T_{x_j} X$, $\|v_j\| = 1$, such that

$$Df(x_j)(v_j) = 0.$$

By passing to a subsequence we may assume that $v_j \to v$, but $v$ is in $C_4(x, X)$ by definition. Therefore $Df(x)(v) = 0$, which is a contradiction. ♠

**Proposition 2.3.** [Chi] *Let $X \subset \mathbf{C}^m$ be a pure $n$-dimensional analytic set, $L = \mathbf{C}^{m-k} \times 0 \subset \mathbf{C}^m$ and $0 \in X$. If $C_5(0,X) \cap L = \{0\}$ then there exists open set $U \subset \mathbf{C}^m$ such that after a unitary change of coordinates the orthogonal projection $\pi_L : U \to \mathbf{C}^k$ is an almost one-sheeted cover over some analytic subset of $\mathbf{C}^k$, i.e. it is a homeomorphism with $br(\pi_L, X \cap U) = (X \setminus \mathrm{Reg}(X)) \cap U$.*

**Remark.** As before in the case of a general $n$-dimensional analytic set such projection is not a cover; it is, however, proper (because $C_3(0,X) \subset C_5(0,X)$), regular on $\mathrm{Reg}(X) \cap U$ and injective.

**Corollary 2.2.** *Let $X$ be a complex space, $x \in X$ and $\dim C_5(x,X) = k$. Then there exists an open neighbourhoood $U$ of $x$ and a proper, injective holomorphic map $f : U \to \mathbf{C}^k$, which is regular on $\mathrm{Reg}(X) \cap U$. Every holomorphic map $f : X \to \mathbf{C}^k$ with $\mathrm{Ker}\, Df(x) \cap C_5(x,X) = \{0\}$ is injective, proper and regular on $\mathrm{Reg}(X) \cap U$ for some neighbourhood $U$ of $x$.*

**Proof.** Similarly as in the previous corollary the first statement of this one is obvious. We may assume that $X \subset \mathbf{C}^m$. Because $C_4(x,X) \subset C_5(x,X)$, the regularity of the map on $\mathrm{Reg}(X) \cap U$ for some small neighbourhood $U$ of $x$ follows from 2.1. If the map were not injective in any neighbourhood of $x$ then there would be sequences $x_j, y_j \in X$, $x_j, y_j \to x$ and $x_j \neq y_j$ such that $f(x_j) - f(y_j) = 0$. The Taylor series expansion gives us

$$f(x_j) - f(y_j) = Df(x_j)(x_j - y_j) + o(|x_j - y_j|) = 0,$$

which means that

$$Df(x_j)(\frac{x_j - y_j}{|x_j - y_j|}) \to 0.$$

By passing to a subsequence we may assume that $\frac{x_j - y_j}{|x_j - y_j|} \to v$ which belongs to $C_5(x,X)$. But then $Df(x)(v) = 0$ which contradicts the assumption. ♠

**Definition 2.2.** *Let $X$ be a complex space, $x \in X$ and $f : X \to \mathbf{C}^m$ a holomorphic map. The map $f$ is* **weakly regular at** *$x$ if $C_5(x,X) \cap \mathrm{Ker}\, Df(x) = \{0\}$ and* **weakly regular** *if $C_5(X) \cap \mathrm{Ker}\, Df = 0$, where $0$ is the zero section in $TX$.*

On a complex manifold the notions weakly regular and regular coincide. One of the key features of weakly regular maps that will be used in the sequel is local injectivity. Let us state two more lemmas describing properties of injective weakly regular maps.

**Lemma 2.1.** *Let $X$ be a complex space, $K \subset X$ a compact set and $f: X \to \mathbf{C}^m$ a holomorphic map, which is weakly regular and injective on $K$. Then there exists an open neighbourhood $U \subset X$ of $K$ such that $f$ is injective and weakly regular on $K$.*

**Proof.** Weak regularity obviously is an open condition. Assume that the map is not injective. Then there are sequences $x_j, y_j \in X$, $x_j \neq y_j$ such that $x_j \to x \in K$, $y_j \to y \in K$ and $f(x_j) = f(y_j)$. Injectivity of $f$ on $K$ implies $x = y$ and since $f$ is weakly regular on $K$ it is injective in a neighbourhood of $x$, which contradicts the existence of the above sequences.

♠

**Lemma 2.2.** *Let $X$ be a complex space, $K \subset X$ a compact set and $f: X \to \mathbf{C}^m$ a holomorphic map, which is weakly regular and injective on $K$. Then there exists $\varepsilon > 0$ such that any holomorphic map $g: X \to \mathbf{C}^m$ satisfying $|g - f|_K < \varepsilon$, is injective and weakly regular on $K$.*

**Proof.** Every map $g$ close enough to $f$ on $K$ is weakly regular on $K$ and therefore locally injective. The next step is to prove a local result:

> **Claim.** Take $x \in X$ and assume that the map $f$ is weakly regular (and therefore injective) in a small compact neighbourhood $U$ of $x$. Then there exists $\varepsilon > 0$ such that if $|g - f|_U < \varepsilon$ then $g$ is injective and weakly regular on $U$.
>
> **Proof.** Note that any map close to $f$ is weakly regular at $x$ and therefore injective in some neighbourhood of $x$. We need to prove that the map is injective on $U$. Assume the converse. Then there are sequences $\varepsilon_j \to 0$, $g_j: X \to \mathbf{C}^m$ and $x_j, y_j \in U$ such that $|g_j - f| < \varepsilon_j$ and $g_j(x_j) - g_j(y_j) = 0$. We may assume that $x_j \to x$, $y_j \to y$ and $(x_j - y_j)/|x_j - y_j| \to v$. Injectivity of $f$ implies $x = y$. The Taylor series expansion gives
>
> $$Dg_j(x_j)(x_j - y_j) = o_j(|x_j - y_j|).$$
>
> Because of the Cauchy estimates there is $o(|x_j - y_j|)$ such that
>
> $$|o_j(|x_j - y_j|)| < |o(|x_j - y_j|)| \forall j.$$
>
> Dividing the above equation by $|x_j - y_j|$ and passing to the limit we get $Df(x)(v) = 0$ which contradicts the fact that $f$ is weakly regular.

We have proved that there exists an open neighbourhood $V$ of the diagonal $\Delta \subset K\times K$ such that if $g$ is close enough to $f$, the map $g(x) - g(y) : X \times X \to \mathbf{C}^m$ will have no zeroes in $V$ except the diagonal $\Delta$. Injectivity of $f$ implies that $\min\{|f(x)-f(y)|, (x,y) \in K \times K \setminus V\} > 0$. The same holds for each map $g$ close enough to $f$ on $K$ which means that any such $g$ is injective on $K$. ♠

## 3. Proof of the main theorem

Let $X$ be a $n$-dimensional Stein space with $S = \{s_j\} = X \setminus \mathrm{Reg}(X)$ discrete and let $N = N(X) = \max\{[n/2] + n + 1, 3, \max\{\dim C_5(x,X),\ x \in X\}\}$. We are looking for a holomorphic, weakly regular injective map $F = (H,G)\colon X \to \mathbf{C}^n \times \mathbf{C}^{N-n}$. We first construct an almost proper holomorphic map $H\colon X \to \mathbf{C}^n$, having certain additional properties (a generic almost proper map) and then the map $G\colon X \to \mathbf{C}^{N-n}$ such that $F = (H,G)$ has the desired properties.

By definition of $N$ there exist injective weakly regular holomorphic maps $\Phi_j\colon U_j \to \mathbf{C}^N$, defined on small neighbourhoods $U_j$ of $s_j$. For simplicity let's assume that $\Phi_{j,N}(s_j) = j$. Since $X$ is Stein there exists a holomorphic map $\Phi\colon X \to \mathbf{C}^N$ which coincides with $\Phi_j$-s to the second order on $S$. Define $\varphi = (\Phi_1, \ldots, \Phi_n)$. The following theorem gives a generic almost proper map $H$ which coincides with $\varphi$ on $S$ to the second order.

**Proposition 3.1.** [Pre1] (Generic almost proper maps). *Let $X$ be a Stein $n$-manifold, $Y \subset X$ a discrete set, $\varphi : X \to \mathbf{C}^n$ a holomorphic map and $q' = [\frac{n+1}{2}]$. For each $y \in Y$ let a number $m_y \in \mathbf{N}$ be given. The set of all almost proper holomorphic maps $H : X \to \mathbf{C}^n$ satisfying*

(1) $(H - \varphi)_y \in \mathcal{J}(Y)^{m_j}$ *for each* $y \in Y$ *and*

(2) $\dim\{x \in X \setminus Y,\ rank_x H \le n - i\} < 2(q' - i + 1),\ i = 1, \ldots, n$

*is residual in the set $\mathcal{G}$ of all holomorphic maps $G$ satisfying $(G - \varphi)_y \in \mathcal{J}(Y)^{m_j}$ for each $y \in Y$.*

**Remark.** This theorem is stated for Stein manifolds but in fact it also holds for Stein spaces $X$ with $Y = X\setminus\mathrm{Reg}(X)$ discrete. The first statement holds for general Stein spaces and the second is proved by using Thom's transversality theorems on the set $X \setminus Y$. In our case $m_j$ is 2 for each $j$. The maps $H$ and $\varphi$ will be fixed through the rest of the section.

The construction of the map $G$ requires more work. We follow the proof of embedding theorem in [Sch1]. Since the full proof is quite long and complicated we will only explain how to modify the theorems in such a way that they hold for weakly regular maps. The main tool in the proofs of embedding theorem is h-principle:

**Definition 3.1.** [Gr] *Let $Z$ and $X$ be complex spaces, $h: Z \to X$ a surjective submersion and let $U \subset X$ be an open set. The submersion $h$ admits a spray over $U$ if for some $m \in \mathbf{N}$ there exists a holomorphic map $s: h^{-1}(U) \times \mathbf{C}^m \to h^{-1}(U)$ such that*

*$s(z,0) = z$ for each $z \in h^{-1}(U)$,*

*$s(z, \mathbf{C}^m) \subset h^{-1}(h(z))$ for each $z \in h^{-1}(U)$, and*

*$\frac{\partial}{\partial t} s(z,t)|_{t=0}: \mathbf{C}^m \to \mathrm{Ker} D_z h$ is surjective.*

**Theorem 3.1.** (The h-principle for Stein spaces) ([Gr], [FP1], [FP2], [Pre2]).
*Let $X$ be a Stein space, $Z$ a complex space and $h: Z \to X$ a holomorphic submersion (with constant corank) onto $X$. Assume that each $x \in X$ has a neighbourhood $U \subset X$ such that $h$ admits a spray over $U$. Let $d$ be a metric on $Z$ compatible with the complex space topology. Then the following hold:*

*(a) Each continuous section $f_0: X \to Z$ can be deformed to a holomorphic section $f_1: X \to Z$ through a continuous one-parameter family of continuous sections (a homotopy) $f_t: X \to Z$, $t \in [0,1]$.*

*(b) If $K \subset X$ is a compact holomorphically convex set and the initial section $f_0$ is holomorphic in a neighbourhood of $K$, then for each $\varepsilon > 0$ there exists a homotopy $f_t: X \to Z$, $t \in [0,1]$, such that $d(f_t(x), f_0(x)) < \varepsilon$ for each $x \in K$ and $t \in [0,1]$, each $f_t$ is holomorphic in a neighbourhood of $K$ and $f_1$ is holomorphic on $X$. In this case it suffices to assume that the submersion $h: Z \to X$ has a spray over small open subsets of $X \setminus K$.*

For $R > 0$ let $X^R$ be an arbitrary union of finitely many connected components of the set $H^{-1}(B_n(R)) \subset X$ and let $Z^R = H(X^R) = B_n(R)$. Note that the map $H : X^R \to B(R)$ is proper. Let $\{X^k\}$ be a normal exhaustion of $X$ and let for each $k$ the set $U_k$ be an open Stein neighbourhood of $X^k$ contained in $X^{k+1}$. By the above definition the set $X^k$ is Runge in $X^{k+1}$. We may assume that $S \cap (\partial X^k \cup \partial U_k) = \emptyset$. By $\psi(z) = \|z\|^2$ we denote the euclidean norm on $\mathbf{C}^n$.

We will construct a sequence of maps $G_k: U_k \to \mathbf{C}^{N-n}$ and a decreasing sequence $\varepsilon_j \to 0$ such that

(1) $(H, G_k): U_k \to \mathbf{C}^N$ is weakly regular and injective,

(2) $\|G_k - G_{k-1}\|_{X^{k-1}} < 2^{-k}\varepsilon_{k-1}$,

(3) if $G'\colon U_k \to \mathbf{C}^{N-n}$ satisfies $\|G' - G_k\|_{X^k} < \varepsilon_k$ then $(H, G')\colon X \to \mathbf{C}^N$ is weakly regular and injective,

(4) $\inf\{\|G_k(x)\|, x \in (H^{-1}(B_n(k-1)) \setminus X^{k-1}) \cap X^k\} > k - 1$,

(5) $DG_k(s_j) = D(\Phi_{n+1}, \ldots, \Phi_N)(s_j)$ for each $j, k$ whenever the expression makes sense.

Note that the sequence $G_k$ converges uniformly on compact sets to a map $G$ such that the map $(H, G) : X \to \mathbf{C}^N$ is weakly regular and injective by (1), (2), (3) and proper by (4). The condition (5) could be omitted since weak regularity is stable under small perturbations; in fact the construction is such that we get (5) for free.

Before proceeding to the construction of maps $G_k$ we define stratifications of $X$ and $\mathbf{C}^n$ which we will need in the sequel.

**Lemma 3.1.** ([Sch1],[Pre1]) *There are stratifications $X_n := X^R \supset X_{n-1} \ldots \supset X_0 \supset X_{-1} = \emptyset$ and $Z_n := Z^R \supset Z_{n-1} \ldots \supset Z_0 \supset Z_{-1} = \emptyset$, with $X_0, Z_0 \neq \emptyset$, satisfying*

(1) *$X_0 \supset X^R \cap S$ and $Z_0 \supset H(S \cap X^R)$,*

(2) *$X_j = H^{-1}(Z_j) \cap X^R$,*

(3) *the sets $X_j$ and $Z_j$ have dimension at most $j$ and the sets $X_j^* := X_j \setminus X_{j-1}$, $Z_j^* = Z_j \setminus Z_{j-1}$ are complex $j$-dimensional manifolds (or empty),*

(4) *if $X_j^*$ is not empty, the map $H : X_j^* \to Z_j^*$ is an immersion for $j \in \{0, \ldots, n\}$,*

(5) *the rank of $H$ is constant on each connected component of the set $X_j^*$ for each $j \in \{0, \ldots, n\}$.*

We quote some more results from [Sch1] (the almost proper map $H$ is fixed). The original theorems are dealing with immersions and injective immersions; in our case the term immersion will be replaced with the term weakly regular. Let $q = N - n$, fix some $R > 0$ and let $\{X_j\}$ and $\{Z_j\}$ be stratifications from lemma 3.1.

**Theorem 3.2.** *Choose $j \in \{1, \ldots, n\}$ and $r, r', r'' \in \mathbf{R} \setminus \{0\}$ such that $r'' > 0$, $r' < r'' < r < R$. Let $r'^2$ and $r''^2$ be regular values for $\psi|_{Z_j^*}$ and $(X_{j-1} \cap \overline{X^{r''}}) \cup (X_j \cap \overline{X^r}) \neq \emptyset$. Let $f\colon X^r \to \mathbf{C}^q$ be a holomorphic map such that the map $(H, f)\colon X^r \to \mathbf{C}^N$ is weakly regular on $(X_{j-1} \cap \overline{X^r}) \cup (X_j \cap \overline{X^{r'}})$. Then:*

*(a) If $r' < 0$ then there exists a holomorphic map $f'\colon X^{r''} \to \mathbf{C}^q$ such that $f' - f|_{X^{r''}} \in \Gamma(X^{r''}, \mathcal{J}(X_{j-1})^2 \cap \mathcal{J}(X_j))^q$ and $(H, f')$ is weakly regular on $X_j \cap X^{r''}$.*

*(b) If $r' > 0$ then the map $f$ can be approximated arbitrarily well on the set $X^{r'}$ by a holomorphic map $f'\colon X^{r''} \to \mathbf{C}^q$ such that $f' - f|_{X^{r''}} \in \Gamma(X^{r''}, \mathcal{J}(X_{j-1})^2 \cap \mathcal{J}(X_j))^q$ and $(H, f')$ is weakly regular on $X_j \cap X^{r''}$.*

**Proof.** Since the sheaf $\mathcal{J}(X_{j-1})^2 \cap \mathcal{J}(X_j)$ is coherent, there exist finitely many holomorphic sections $f_1, \dots f_M \in \Gamma(X, \mathcal{J}(X_{j-1})^2 \cap \mathcal{J}(X_j))^q$ generating $\Gamma(X^{r''}, \mathcal{J}(X_{j-1})^2 \cap \mathcal{J}(X_j))^q$. We are looking for the map $f'$ of the form

$$f' = f + \sum \alpha_j f_j,$$

where $\alpha = (\alpha_1, \dots, \alpha_m)$ is considered to be a section of the trivial bundle $X^r \times \mathbf{C}^M$. Denote by $K = \mathrm{Ker} DH \subset TX$ the kernel of $DH$; note that the definition of $X_j^*$ implies that $K|_{X_j^*}$ is a vector bundle. Let $\Sigma \subset (X_j \cap X^r) \times \mathbf{C}^M =: V$ be the set of all $(x, a_1, \dots, a_M)$, such that the map $(H, f + \sum a_j f_j)$ is not weakly regular in $x$. Let $p : V \to (X_j \cap X^r)$ be the trivial projection. Then because $(H, f)$ is weakly regular on $X_{j-1} \cap X^r$ and the maps $f_j$ vanish to the second order on $X_{j-1}$, the set $\Sigma$ is a subset of $X_j^* \times \mathbf{C}^M$. Since $K|_{X_j^*}$ is a vector bundle, $\Sigma$ is the set of all $(x, a_1, \dots, a_M) \in X_j^* \times \mathbf{C}^M$, such that the map $K_x \to \mathbf{C}^{q'}$, $v \to Df(x)v + \sum a_i Df_j(x)v$ is not injective; therefore $\Sigma$ is analytic in $X_j^* \times \mathbf{C}^M$. But we want $\Sigma$ to be analytic in $V$ which means we have to prove that $\Sigma$ is closed in $V$. Now, since $(H, f)$ is weakly regular on $X_{j-1} \cap X^r$ any map of the form $(H, f + \sum \alpha_j f_j)$ is weakly regular on the same set and because weak regularity is an open condition, the map $(H, f + \sum \alpha_j f_j)$ is also weakly regular in some neighbourhood of $X_{j-1} \cap X^r$ which means that $\Sigma$ is a closed in $V$. As in [Sch1] we prove that the projection $p : V \setminus \Sigma \to (X_j^* \cap X^r)$ is a locally trivial fibration which admits a spray.

Our goal is to find a holomorphic section $\alpha$ of $(X^r \times \mathbf{C}^M) \setminus \Sigma$. The zero section defined on $(X_{j-1} \cap \overline{X^r}) \cup (X_j \cap \overline{X^{r'}})$ is a section of $((X^r \cap X_j) \times \mathbf{C}^M) \setminus \Sigma$ because the map $(H, f)$ is weakly regular on $(X_{j-1} \cap \overline{X^r}) \cup (X_j \cap \overline{X^{r'}})$. And since weak regularity is an open condition the zero section defined in a neighbourhood of the set $(X_{j-1} \cap \overline{X^r}) \cup (X_j \cap \overline{X^{r'}})$ is a section of $((X^r \cap X_j) \times \mathbf{C}^M) \setminus \Sigma$ as well. As in [Sch1] this section can be extended to a continuous section of $((X^r \cap X_j) \times \mathbf{C}^M) \setminus \Sigma$. Then the h-principle applies (if $r' < 0$ we use the existence version and if $r' > 0$ the approximation version) which yields a holomorphic section $\alpha'$ of $((X^r \cap X_j) \times \mathbf{C}^M) \setminus \Sigma$. This section can be trivially extended to a holomorphic section $\alpha$ of $(X^r \times \mathbf{C}^M) \setminus \Sigma$. ♠

**Remark.** The maps $f_j$ used in the theorem vanished to the first order on $X_j$ which means that if the initial map $f$ is such that $(H, f)$ is injective on $X_j \cap X^r$, the map $(H, f')$ is also injective on $X_j \cap X^r$.

**Theorem 3.3.** *Choose $j \in \{1, \ldots, n\}$ and $r, r', r'' \in \mathbf{R} \setminus \{0\}$ such that $r'' > 0$, $r' < r'' < r < R$. Let $r'^2$ and $r''^2$ be regular values for $\psi|_{Z_j^*}$ and $(X_{j-1} \cap \overline{X^{r''}}) \cup (X_j \cap \overline{X^r}) \neq \emptyset$. Let $f: X^r \to \mathbf{C}^q$ be a holomorphic map such that the map $(H, f): X^r \to \mathbf{C}^N$ is injective and weakly regular on $(X_{j-1} \cap \overline{X}_r) \cup (X_j \cap \overline{X}_{r'})$. Then:*

*(a) If $r' < 0$ then there exists a holomorphic map $f': X^{r''} \to \mathbf{C}^q$ such that $f' - f|_{X^{r''}} \in \Gamma(X^{r''}, \mathcal{J}(X_{j-1})^2)^q$ and such that $(H, f')$ is injective and weakly regular on $X_j \cap X^{r''}$.*

*(b) If $r' > 0$ then the map $f$ can be approximated arbitrarily well on the set $X^{r'}$ by a holomorphic map $f': X^{r''} \to \mathbf{C}^q$ such that $f' - f|_{X^{r''}} \in \Gamma(X^{r''}, \mathcal{J}(X_{j-1})^2)^q$ and $(H, f')$ is injective and weakly regular on $X_j \cap X^{r''}$.*

**Proof.** Since the sheafs $\mathcal{F} = H_* \mathcal{O}(X^R)$ and $\mathcal{J}(Z_{j-1})^2 \mathcal{F}$ are coherent, there exist finitely many holomorphic sections $\psi_1, \ldots \psi_M \in \Gamma(Z, \mathcal{J}(Z_{j-1})^2 \mathcal{F})^q$ generating $\Gamma(Z^{r''}, \mathcal{J}(Z_{j-1})^2 \mathcal{F})^q$. Let $f_j \in \Gamma(X^r, \mathcal{O}(X))^q$ be liftings of the sections $\psi_j$. We are looking for the map $f'$ of the form

$$f' = f + \sum (\alpha_j \circ H) \cdot f_j,$$

where $\alpha = (\alpha_1, \ldots, \alpha_m)$ is considered to be a section of the trivial bundle $B(r) \times \mathbf{C}^M$. Let $\Sigma \subset (Z_j \cap B(r)) \times \mathbf{C}^M =: V$ be the set of all $(z, a_1, \ldots, a_M)$, such that the map $H^{-1}(z) \to \mathbf{C}^n$, $x \to f(x) + \sum a_j f_j(x)$ is not injective. Let $p : V \to (Z_j \cap B(r))$ be the trivial projection. Then because $(H, f)$ is injective on $(X_{j-1} \cap \overline{X}_r) \cup (X_j \cap \overline{X}_{r'})$ and the maps $f_j$ vanish to the second order on $X_{j-1}$, the set $\Sigma$ is an analytic subset of $Z_j^* \times \mathbf{C}^M$. As above we want $\Sigma$ to be analytic subsev of $V$, i.e. closed in $V$. Since $(H, f)$ is injective and weakly regular on $X_{j-1} \cap X^r$ and the maps $f_j$ vanish to the second order on $X_{j-1}$, any map of the form $(H, f + \sum(\alpha_j \circ H) \cdot f_j)$ is injective and weakly regular on the same set and because being injective and weakly regular is an open condition, such map is also injective and weakly regular in some neighbourhood of $X_{j-1} \cap X^r$. This means that the set $\Sigma$ is a closed in $V$. As in [Sch1] we prove that $p : V \setminus \Sigma \to (Z_j^* \cap B(r))$ is a locally trivial fibration which admits a spray.

Our goal is to find a holomorphic section $\alpha$ of the submersion $(B(r) \times \mathbf{C}^M) \setminus \Sigma$. The zero section defnined on $(Z_{j-1} \cap \overline{B}(r)) \cup (Z_j \cap \overline{B}(r'))$ is a section of $((B(r) \cap Z_j) \times \mathbf{C}^M) \setminus \Sigma$ because the map $(H, f)$ is weakly regular and injective on $(X_{j-1} \cap \overline{X^r}) \cup (X_j \cap \overline{X^{r'}})$. And since being injective and weakly regular is an open condition, the zero section defined in a neighbourhood of the set $(Z_{j-1} \cap \overline{B}(r)) \cup (Z_j \cap \overline{B}(r'))$ is a section of $((B(r) \cap Z_j) \times \mathbf{C}^M) \setminus \Sigma$ as well. As in [Sch1] this section can be extended to a continuous section of $((B(r) \cap Z_j) \times \mathbf{C}^M) \setminus \Sigma$. Then the h-principle applies (if $r' < 0$ we use the existence version and if $r' > 0$ the approximation version) which yields a holomorphic section $\alpha'$ of $((B(r) \cap Z_j) \times \mathbf{C}^M) \setminus \Sigma$. This section can be trivially extended to a holomorphic section $\alpha$ of $(B(r) \times \mathbf{C}^M) \setminus \Sigma$ .

♠

At the beginning of this section we defined the map $\Phi$. Now let's define

$$f = (\Phi_{n+1}, \dots, \Phi_N).$$

The map $(H, f)$ clearly is weakly regular on $S$ and injective in a small neighbourhood of $S$, since $\Phi_N(s_j) = j$. Using in turn theorems 3.3. and 3.2. we can proceed by induction over the strata starting with $X_0$ (as in [Sch1]) and using the fact that being weakly regular or injective and weakly regular is an open condition) to obtain the following results:

**Theorem 3.4.** [Sch1] (Existence). *Let $R > 0$ and let $X^R$ be the union of a finite number of connected components of the set $H^{-1}(B_n(R))$. For $r \in (0, R)$ let $X^r := X^R \cap H^{-1}(B_n(r))$. There exists a holomorphic map $G\colon X^r \to \mathbf{C}^q$ satisfying the conditions*

$\alpha(r)$ : *the map $(H, G)\colon X^r \to \mathbf{C}^N$ is injective and weakly regular and*

$\beta(r)$ : *$(H, G)$ coincides with $\Phi$ to the second order $S \cap X^r$.*

**Theorem 3.5.** [Sch1] (Approximation). *Let $R, r > 0$, $X^R$ and $X^r$ be as in theorem 3.4. Choose $r' \in (r, R)$ and set $X^{r'} := X^R \cap H^{-1}(B_n(r'))$.*

*If a holomorphic map $G\colon X^r \to \mathbf{C}^q$ satisfies $\alpha(r)$ and $\beta(r)$ from theorem 3.4., it can be approximated arbitrarily well on the set $X^r$ by a map $G'\colon X^{r'} \to \mathbf{C}^q$ satisfying $\alpha(r')$ and $\beta(r')$ from theorem* 3.4.

**Remark.** Note that the induction preserves the derivatives of $\Phi$ at the points in $S$ since they are contained in $X_0$ and and the maps $f_j$ in the theorems 3.2. and 3.3. vanish to the second order on $X_0$.

**Proof of the main theorem.** Now we can construct the maps $G_j$ and the sequence $\varepsilon_j \to 0$ with the properties (1) – (5) listed above.

<u>$k = 1$.</u> By the existence theorem 3.4. there exists a map $G_1\colon X^1 \to \mathbf{C}^q$ with properties (1) and (5). By 2.2. there is an $\varepsilon_1 > 0$ such that (3) holds.

<u>$k \to k+1$.</u> Assume that $G1, \dots, G_k$ and $\varepsilon_1, \dots, \varepsilon_k$ have already been constructed. Let $X^{k'} := X^{k+1} \cap (H^{-1}(B_k) \setminus X^k)$, i.e. $X^{k'}$ is the union of those connected components of $H^{-1}(B(k))$ which lie in $X^{k+1}$ but not in $X^k$. By existence theorem 3.4. there is a map

$G'_k$ satisfying (1). By adding a sufficiently large positive constant we may assume that $\|G'\|_{X^{k'}} > 2k$ and that the map $G': X^k \cup X^{k'} \to \mathbf{C}^q$, defined by $G'|_{X^k} = G_k$, $G'_{X^{k'}} = G'_k$ is such that $(H, G'): X^k \cup X^{k'} \to \mathbf{C}^N$ is injective. Now the assumptions of the theorem 3.4. are fulfilled so there exists a map $G_{k+1}: X^{k+1} \to \mathbf{C}^N$ satisfying $(1), (2), (4)$ and $(5)$. As above there exists $\varepsilon_{k+1} \in (0, \varepsilon_k)$ such that (2) holds as well. ♠

## 4. * References


[ABT] Aquistapace, F., Broglia F., Tognolli, A.: A Relative Embedding Theorem for Stein Spaces. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 2 507-522 (1975)

[Bi] Bishop, E.: Mappings of partially analytic spaces. Amer. J. Math. 83, 209-242 (1961)

[Chi] Chirka, E.M.: Complex analytic sets. Kluwer academic publishers (1989)

[EG1] Eliashberg, Y. in Gromov, M.: Nonsingular mappings of Stein manifolds. Funct. Anal. and Appl. 5, 156-157 (1971)

[EG2] Eliashberg, Y. in Gromov, M.: Embeddings of Stein manifolds. Annals of Math. 136, 123-135 (1992)

[GR] Gunning, C.R., Rossi, H.: Analytic functions of several complex variables. Prentice–Hall (1965)

[FP1] Forstnerič, F., Prezelj, J.: The Oka's principle for sections of holomorphic fiber bundles with sprays. Math. Ann., to appear

[FP2] Forstnerič. F., Prezelj, J.: The Oka's principle for holomorphic submersions with sprays. Preprint, 1999

[Gr] Gromov, M.: Oka's principle for holomorphic sections of elliptic bundles. J. of the Amer. Math. Soc. 2, 851–897 (1989)

[Oh1] Ohsawa, T.: Holomorphic embedding of compact s.p.c. manifolds into complex manifolds as real hypersurfaces. Differential geometry of submanifolds (Kyoto, 1984), 64–76, Lecture Notes in Math., 1090 Springer (1984)

[Oh2] Ohsawa, T.: Global realization of strongly pseudoconvex $CR$-manifolds. Publ. Res. Inst. Math. Sci. 20 (1984)

[Pre1] Prezelj, J.: Interpolation of embeddings of Stein manifolds on discrete sets. IMFM Preprint series (1999)

[Pre2] Prezelj, J.: Homotopski princip za submerzije s sprayem nad Steinovimi prostori. PhD thesis, University of Ljubljana (2000)

[Ros] Rossi, H.: Attaching analytic spaces to an analytic space along a pseudoconcave boundary. Proc. Conf. Complex Analysis (Minneapolis, 1964), 242–256, Springer (1965)

[Sch1] Schürmann, J.: Einbettungen Steinscher Räume in affine Räume minimaler Dimension. Schriftenreihe des Math. Inst. Univ. Münster, 3. Serie, Heft 7 (1992)

[Sch2] J. Schürmann: Embeddings of Stein spaces into affine spaces of minimal dimension. Math. Ann. 307, 381–399 (1997)

[Stu] Stutz, J.: Analytic sets as branched coverings. Trans. of the AMS, 166, 241–259 (1972)