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CHROMATIC NUMBER OF A NONNEGATIVE MATRIX

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Abstract

A new set of combinatorial invariants of nonnegative matrices with zero diagonal is introduced. They generalize the notion of the chromatic and the circular chromatic number of a graph and the notion of an optimal traveling salesman tour (metric case). An extension of Wilf's eigenvalue upper bound on the chromatic number is derived, and some directions for further research are suggested.

1 Introduction

A set of new combinatorial invariants, called the *chromatic number*, is introduced for nonnegative square real matrices with zero diagonal. The chromatic number is invariant under permutation similarity and monotone with respect to matrix entries. The main motivation is the study of two special cases, called the *span* and the *circular span*. They generalize the notion of the chromatic number of a graph in the sense that the span of a symmetric 01-matrix A is equal to the chromatic number of the graph with adjacency matrix A. The span is related to the channel assignment problem [5], and the circular span corresponds to the notion of the circular chromatic number [6, 9]. In the case, when the triangular inequality holds for the entries of A, the circular span corresponds to the optimum length of the traveling salesman tour which has A as the cost matrix.

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Wilf [7] proved that the chromatic number of a graph cannot be larger than the maximum eigenvalue of its adjacency matrix increased by 1. Wilf's bound is extended to the generalized chromatic number of a matrix and several possibilities for further research are outlined.

2 Colorings in distance spaces

A *distance space* is a pair $\mathcal{D} = (D, d)$ where D is a set and $d : D \times D \to \mathbb{R}$ is a function such that $d(x, y) \geq 0$ for every $x, y \in D$.

Let $\mathcal{D} = (D, d)$ be a distance space, let V be a set and $A : V \times V \to \mathbb{R}$. A mapping $c : V \to D$ is called a $\mathcal{D}$*-coloring* of A if for any distinct $x, y \in V$,

$$d(c(x), c(y)) \geq A(x, y). \tag{1}$$

For a real number $p > 0$, let $p\mathcal{D}$ denote the distance space (D, pd). The $\mathcal{D}$*-chromatic number* of A, denoted by $\chi_{\mathcal{D}}(A)$, is the infimum of the set of all real numbers p for which there exists a $p\mathcal{D}$-coloring of A. In all interesting cases, the distance space will contain a subset D' such that $|D'| \geq |V|$ and for any distinct elements x, y of D', $d(x, y) > 0$. Clearly, in such a case, $\chi_{\mathcal{D}}(A) < \infty$.

It will be shown in the next section that the infimum in the definition of $\chi_{\mathcal{D}}(A)$ is attained, i.e., there exists a $p\mathcal{D}$-coloring of A for $p = \chi_{\mathcal{D}}(A)$, if some rather natural conditions on $\mathcal{D}$ hold.

In this paper we shall restrict our attention to the case when V is finite. Then we may assume that $V = \{1, 2, \ldots, n\}$ and that A is a nonnegative matrix. We may also assume that A has zero diagonal since the diagonal entries have no effect on coloring properties. We shall write $V = V(A)$ for the set of vertex-column indices of A.

The $\mathcal{D}$-chromatic number is a *combinatorial matrix invariant* since it is preserved by the permutation similarity of matrices, i.e., if P is a permutation matrix, then $\chi_{\mathcal{D}}(P^T AP) = \chi_{\mathcal{D}}(A)$.

Our main interest will be focused on two special cases of distance spaces. Let $\mathcal{I}$ denote the distance space whose underlying set is the unit interval $I = [0, 1] \subset \mathbb{R}$ and the corresponding mapping is the distance in I. The $\mathcal{I}$-chromatic number is also called the *span* and corresponds to the span studied in the channel assignment problem (cf. [5]). Since the distance in I is symmetric, the span really makes sense for symmetric matrices. More precisely, if A is a nonnegative matrix and $\bar{A}$ has entries $\bar{a}_{ij} = \max\{a_{ij}, a_{ji}\}$, then $\bar{A}$ is symmetric and $\chi_{\mathcal{I}}(A) = \chi_{\mathcal{I}}(\bar{A})$.

The second distance space $\mathcal{S} = (S^1, d^+)$ of particular importance has the underlying set $S^1 = \mathbb{R}/\mathbb{Z}$ which may be identified either with the interval $[0,1)$ or with a circle of perimeter 1 in $\mathbb{R}^2$. In the latter case, the distance function d^+ is defined as the distance on the circle in the clockwise direction, while for $x, y \in [0,1)$ we define

$$d^+(x,y) = \begin{cases} y - x & \text{if } y \geq x \\ 1 + y - x & \text{if } x > y. \end{cases}$$

The $\mathcal{S}$-chromatic number $\chi_{\mathcal{S}}(A)$ is related to circular colorings of graphs and is therefore also called the *circular chromatic number* or the *circular span* of A. The theory of circular colorings has become an important branch of chromatic graph theory with many interesting results, leading to new methods and exciting new results. We refer to the survey article by Zhu [9]. An extension of circular colorings to edge-weighted graphs was recently introduced by the author [6].

The *travelling salesman problem* (TSP) is one of the best-known NP-hard combinatorial optimization problems, and there is an extensive literature on both its theoretical and practical aspects [4, 3]. In the *metric travelling salesman problem* (MTSP), it is assumed that the cost matrix $A = [a_{uv}]$ satisfies the triangular inequality:

$$a_{uv} \leq a_{uw} + a_{wv}, \qquad u, w, v \in V. \tag{2}$$

The circular span generalizes the metric traveling salesman problem in the following sense. Let $A = [a_{uv}]_{u,v \in V}$ be the cost matrix for a MTSP, where none of the costs a_{uv} $(u \neq v)$ is zero. Then every $p\mathcal{S}$-coloring c of A determines a tour of the traveling salesman of cost $\leq p$ which is obtained by taking the tour through V as determined by the cyclic clockwise order of $c(V)$ on S^1. Conversely, (2) implies that every travelling salesman tour of length p yields a $p\mathcal{S}$-coloring of A. Therefore, $\chi_{\mathcal{S}}(A)$ is the optimum for the considered MTSP. (The same conclusion also holds if some of the costs are zero but that case needs a slightly different argument.) This example shows that computation of the circular span is NP-hard.

The span and the circular span are closely related. It is a simple exercise to show that the following inequalities hold:

Proposition 2.1 *Let A be a symmetric nonnegative matrix with zero diagonal. Then*

$$\chi_{\mathcal{I}}(A) < \chi_{\mathcal{S}}(A) \leq \chi_{\mathcal{I}}(A) + \max\{a_{ij} \mid i, j \in V(A), i \neq j\}.$$

3 Basic results

There is a natural partial order $\preceq$ on the set $\mathbb{M}_0$ of all nonnegative square real matrices with zero diagonal. For $A, B \in \mathbb{M}_0$, we write $A \preceq B$ if there is a 1-1 mapping $\pi : V(A) \to V(B)$ such that for every distinct indices $i, j \in V(A)$, $A_{ij} \leq B_{\pi(i)\pi(j)}$.

Proposition 3.1 *For any distance space $\mathcal{D}$, the function $\chi_{\mathcal{D}}$ is $\preceq$-monotone.*

The distance function d is *weakly continuous* if for every sequence $x_1, x_2, x_3, \ldots$ of elements of D there exists an $x_0 \in D$ such that for every sequence $y_1, y_2, y_3, \ldots$ of elements of D, there are indices $i_1 < i_2 < i_3 < \cdots$ such that

$$\liminf_{n\to\infty} d(x_0, y_{i_n}) \geq \liminf_{n\to\infty} d(x_{i_n}, y_{i_n}) \tag{3}$$

and

$$\liminf_{n\to\infty} d(y_{i_n}, x_0) \geq \liminf_{n\to\infty} d(y_{i_n}, x_{i_n}). \tag{4}$$

Suppose that d is weakly continuous. Let us assume the above notation. By taking the sequence $y_{i_1}, y_{i_2}, y_{i_3}, \ldots$ playing the role of $x_1, x_2, x_3, \ldots$, and the constant sequence $x_0, x_0, x_0, \ldots$ playing the role of the other sequence in the definition of weak continuity, we see that there is also a point $y_0 \in D$ and a subsequence $j_1 < j_2 < j_3 < \cdots$ of $i_1 < i_2 < i_3 < \cdots$ such that

$$d(x_0, y_0) \geq \liminf_{n\to\infty} d(x_0, y_{j_n}) \geq \liminf_{n\to\infty} d(x_{j_n}, y_{j_n}) \tag{5}$$

and

$$d(y_0, x_0) \geq \liminf_{n\to\infty} d(y_{j_n}, x_0) \geq \liminf_{n\to\infty} d(y_{j_n}, x_{j_n}). \tag{6}$$

A metric space is *sequentially compact* if every sequence has a convergent subsequence. Clearly, every compact metric space is sequentially compact.

Lemma 3.2 *If $\mathcal{D} = (D, d)$ is a sequentially compact metric space, then d is weakly continuous.*

Proof. Let $x_1, x_2, x_3, \ldots$ be a sequence of elements of D. Since D is sequentially compact, there exists a convergent subsequence $x_{j_1}, x_{j_2}, x_{j_3}, \ldots$ ($j_1 < j_2 < j_3 < \cdots$) with limit x_0, say. Given a sequence $y_1, y_2, y_3, \ldots$ in D, there is a convergent subsequence of $y_{j_1}, y_{j_2}, y_{j_3}, \ldots$. Let $i_1 < i_2 < i_3 < \cdots$ be the indices corresponding to that subsequence, and let y_0 be the limit. Since the distance is continuous function in its metric topology, $\lim_{n\to\infty} d(x_{i_n}, y_{i_n}) = \lim_{n\to\infty} d(x_0, y_{i_n}) = d(x_0, y_0)$. This proves (3), and (4) follows by symmetry. □

Our next goal is to prove that the infimum in the definition of $\chi_{\mathcal{D}}(A)$ is attained if d is weakly continuous.

Proposition 3.3 *Let $\mathcal{D} = (D, d)$ be a distance space. If d is weakly continuous, then for every $A \in \mathbb{M}_0$, there is a $\chi_{\mathcal{D}}(A)\mathcal{D}$-coloring of A.*

Proof. Let $p_1 > p_2 > p_3 > \cdots$ be a sequence of real numbers with $\lim_{i\to\infty} p_i = p := \chi_{\mathcal{D}}(A)$, and let c_i be a $p_i\mathcal{D}$-coloring of A, $i = 1, 2, \ldots$. Let $V(A) = \{v_1, \ldots, v_n\}$, and consider n sequences $c_1(v_k), c_2(v_k), c_3(v_k), \ldots$, $k = 1, \ldots, n$. Since d is weakly continuous, there is a point $r_1 \in D$, corresponding to the sequence of v_1, such that for each of the remaining $n-1$ sequences, there is a subsequence satisfying (3)–(4). By taking first the subsequence of $c_1(v_2), c_2(v_2), c_3(v_2), \ldots$ and later considering only subsequences of previously obtained subsequence, we achieve properties (3)–(4) for subsequences with the same indices for all initial sequences $c_1(v_k), c_2(v_k), c_3(v_k), \ldots$, $k = 2, \ldots, n$. By repeating the same argument for $k = 2, \ldots, n$, we get points $r_1, \ldots, r_n$ in D and indices $i_1 < i_2 < i_3 < \cdots$ such that

$$d(r_k, r_l) \geq \liminf_{m\to\infty} d(c_{i_m}(v_k), c_{i_m}(v_l)) \tag{7}$$

for any distinct $k, l \in \{1, \ldots, n\}$.

For $v_k \in V(A)$, let $c(v_k) = r_k$. We claim that c is a $p\mathcal{D}$-coloring of A. Take any distinct vertices $v_k, v_l \in V(A)$. Since c_{i_m} is a p_{i_m}-coloring of A, (7) implies

$$d(c(v_k), c(v_l)) = d(r_k, r_l) \geq \liminf_{m\to\infty} \frac{1}{p_{i_m}} a_{uv} = \frac{1}{p} a_{uv}.$$

This proves that c is a $p\mathcal{D}$-coloring of A. □

The condition on weak continuity is needed to get the conclusion of Proposition 3.3. For example, if D is the half-open interval $[0, 1)$ and d is the usual distance on D, then $\chi_{\mathcal{D}}(A) = 1$ for $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ but there is no $\mathcal{D}$-coloring of A.

Proposition 3.3 can be applied for the span but not for the circular span since the distance space $\mathcal{S}$ is not weakly continuous. Indeed, if $A =$

$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, then $\chi_{\mathcal{S}}(A) = 1$ but there is no $\mathcal{S}$-coloring of A. This example can be generalized as follows. Let A be an $n \times n$ matrix corresponding to a metric TSP with a single off-diagonal element (say a_{12}) being 0; so, $a_{21} \neq 0$. Assume, moreover, that every minimum TST tour contains the transition from 1 to 2. In that case, $\chi_{\mathcal{S}}(A)$ is equal to the optimum tour length, but there is no $\chi_{\mathcal{S}}(A)\mathcal{S}$-coloring of A.

On the other hand, the proof of Proposition 3.3 can be repeated to show the following:

Proposition 3.4 *If $A \in \mathbb{M}_0$ is a matrix with symmetric support (i.e., $a_{uv} = 0$ if and only if $a_{vu} = 0$), then there is a $\chi_{\mathcal{S}}(A)\mathcal{S}$-coloring of A.*

A more complicated proof of Proposition 3.4 is given in [6]; however, that proof provides additional information on $\chi_{\mathcal{S}}(A)$.

4 Critical elements and eigenvalue upper bound

Let $\mathcal{D} = (D, d)$ be a distance space. A finite set $C = \{(x_i, q_i, r_i) \in D \times \mathbb{R}^2 \mid i = 1, \ldots, k\}$ is a *cover* of $\mathcal{D}$ if for every $x \in D$ there exists $(x_i, q_i, r_i) \in C$ such that either $d(x, x_i) < q_i$ or $d(x_i, x) < r_i$. The sum $\tau(C) = \sum_{i=1}^{k}(q_i + r_i)$ is called the *diameter* of C. The infimum $\tau(\mathcal{D}) = \inf \tau(C)$ taken over all covers C of $\mathcal{D}$ is called the *covering diameter* of $\mathcal{D}$. The two special cases $\mathcal{I}$ and $\mathcal{S}$ have $\tau(\mathcal{I}) = 1$ and $\tau(\mathcal{S}) = 1$.

Let F be a combinatorial matrix invariant, and let $A \in \mathbb{M}_0$. We say that $i \in V(A)$ is *(F, p)-critical* if $F(A) = p$ but $F(A^{(i)}) < p$, where $A^{(i)}$ denotes the matrix obtained from A by deleting the ith row and the ith column. If p is not important, or if F is clear from the context, we simply say that A is *F-critical*, or *p-critical*, respectively.

Lemma 4.1 *Let $\mathcal{D}$ be a distance space and $A \in \mathbb{M}_0$. If $i \in V(A)$ is $(\chi_{\mathcal{D}}, p)$-critical, then*

$$d_i = \sum_{j \in V(A) \setminus \{i\}} (a_{ij} + a_{ji}) \geq p\, \tau(\mathcal{D}).$$

Proof. Let $V^{(i)} = V(A) \setminus \{i\}$, and let $c : V^{(i)} \to D$ be a $p_1\mathcal{D}$-coloring of $A^{(i)}$, where $p_1 < p$. Since c cannot be extended to a $p_1\mathcal{D}$-coloring of A, for every point $x \in D$, which could serve as the color of i, there exists $j \in V^{(i)}$ such that either $p_1 d(x, c(j)) < a_{ij}$ or $p_1 d(c(j), x) < a_{ji}$. This implies that the set $C = \{(c(j), a_{ij}/p_1, a_{ji}/p_1) \mid j \in V^{(i)}\}$ is a cover of $\mathcal{D}$. In particular,

$\tau(\mathcal{D}) \leq \tau(C) = d_i/p_1$. Hence, $d_i \geq p_1\tau(\mathcal{D})$. This implies the lemma since p_1 is arbitrarily close to p. $\square$

Matrices in $\mathbb{M}_0$ are nonnegative. By the Perron-Frobenius Theorem, every $A \in \mathbb{M}_0$ has a real nonnegative eigenvalue $\lambda_{\max}(A)$ such that every eigenvalue λ of A satisfies $|\lambda| \leq \lambda_{\max}(A)$. Moreover, $\lambda_{\max}(A)$ has an eigenvector with nonnegative coordinates. This implies, in particular, that $\lambda_{\max}(A)$ is $\preceq$-monotone combinatorial matrix invariant.

The following corollary of Lemma 4.1 is an analogue of the Wilf bound [7] for the usual chromatic number of the graph.

Theorem 4.2 *Let $\mathcal{D}$ be a distance space and $A \in \mathbb{M}_0$. Then*

$$\chi_{\mathcal{D}}(A) \leq \frac{2}{\tau(\mathcal{D})}\lambda_{\max}(A).$$

Proof. It is easy to see that $V(A)$ contains a subset U such that in the principal submatrix B of A corresponding to U, every $i \in U$ is $\chi_{\mathcal{D}}(B)$-critical and $\chi_{\mathcal{D}}(B) = \chi_{\mathcal{D}}(A)$. By Lemma 4.1,

$$d'_i = \sum_{j\in U\setminus\{i\}} (a_{ij} + a_{ji}) \geq \chi_{\mathcal{D}}(B)\cdot\tau(\mathcal{D}) = \chi_{\mathcal{D}}(A)\cdot\tau(\mathcal{D}) \tag{8}$$

for every $i \in U$. Now, let $x \in \mathbb{R}^{V(A)}$ be the vector whose entry x_i is 1 if $i \in U$ and 0 otherwise. Then $\|x\|^2 = |U|$ and $(Ax, x) = \sum_{i\in U}\sum_{j\in U} a_{ij} = \frac{1}{2}\sum_{i\in U} d'_i$. This implies that

$$\lambda_{\max}(A) \geq \frac{(Ax, x)}{\|x\|^2} \geq \frac{1}{2|U|}\sum_{i\in U} d'_i. \tag{9}$$

By (8)–(9), $\lambda_{\max}(A) \geq \frac{1}{2}\chi_{\mathcal{D}}(A)\,\tau(\mathcal{D})$. This completes the proof. $\square$

In the case of distance spaces $\mathcal{S}$ and $\mathcal{I}$, Theorem 4.2 yields the following bounds:

$$\chi_{\mathcal{S}}(A) \leq 2\lambda_{\max}(A) \tag{10}$$

and

$$\chi_{\mathcal{I}}(A) \leq 2\lambda_{\max}(A). \tag{11}$$

These bounds can be improved in the special case of MTSP as follows.

Theorem 4.3 *Suppose that the entries of $A \in \mathbb{M}_0$ satisfy the triangular inequality. Then*

$$\chi_{\mathcal{S}}(A) \leq \frac{n}{n-1}\lambda_{\max}(A) \tag{12}$$

and

$$\chi_{\mathcal{I}}(A) \leq \lambda_{\max}(A). \tag{13}$$

Proof. For every cyclic ordering $\gamma = (i_1, i_2, \ldots, i_n)$ of $V(A) = \{1, \ldots, n\}$, there is the minimum real number p_γ such that A has a $p_\gamma\mathcal{S}$-coloring c_γ whose cyclic order of used colors on S^1 is equal to $(c_\gamma(i_1), c_\gamma(i_2), \ldots, c_\gamma(i_n))$. Clearly,

$$p_\gamma = \sum_{j=1}^{n} d^+(c_\gamma(i_j), c_\gamma(i_{j+1})) = \sum_{j=1}^{n} a_{i_j i_{j+1}} \geq \chi_{\mathcal{S}}(A). \tag{14}$$

The second equality in (14) follows from the triangular inequality. If we take all $(n-1)!$ possible cyclic orders γ and consider all corresponding equations (14), each distance a_{ij} appears in the sum on the left hand side of precisely $(n-2)!$ inequalities. Therefore,

$$(n-2)!\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} \geq (n-1)!\chi_{\mathcal{S}}(A). \tag{15}$$

On the other hand, if $x \in \mathbb{R}^V$ is the all-1 vector, then $(Ax, x)/(x, x) = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}$. Consequently,

$$\lambda_{\max}(A) \geq \frac{(Ax, x)}{(x, x)} = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} \geq \frac{n-1}{n}\chi_{\mathcal{S}}(A). \tag{16}$$

This completes the proof of (12). The proof of (13) is similar except that (14) changes to $p_\gamma = \sum_{j=1}^{n-1} a_{i_j i_{j+1}} \geq \chi_{\mathcal{I}}(A)$ and (15) is replaced by

$$(n-1)!\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} \geq n!\chi_{\mathcal{I}}(A). \tag{17}$$

Therefore, $\lambda_{\max}(A) \geq \chi_{\mathcal{I}}(A)$. □

Theorem 4.3 can be extended to the case when the triangular inequality is replaced by the following inequalities:

$$a_{uv} \leq t(a_{uw} + a_{wu})$$

for all distinct $u, v, w \in V(A)$. If $t \geq 1$, then (14) can be replaced by

$$t \sum_{i=1}^{n} \sum_{j=1}^{n} a_{i_j i_{j+1}} \geq \chi_{\mathcal{S}}(A).$$

This implies:

$$\chi_{\mathcal{S}}(A) \leq \frac{tn}{n-1} \lambda_{\max}(A) \qquad \text{and} \qquad \chi_{\mathcal{I}}(A) \leq t\lambda_{\max}(A). \tag{18}$$

5 Some directions for further research

(1) First of all, one should ask if there is a corresponding theory which would take into account also matrices with negative elements.

(2) Other special cases of distance spaces of particular interest (besides $\mathcal{I}$ and $\mathcal{S}$) are higher dimensional spheres and balls in Euclidean spaces and, more generally, in compact Riemannian manifolds. The coloring problems would generalize packing and covering problems for such metric spaces. Another interesting special case is the flat torus $T_{\alpha,\beta} = \alpha S^1 \times \beta S^1$, either with the geodesic distance or with the ℓ^1 metric. Colorings in $T_{\alpha,\beta}$ with the ℓ^1 distance may be related to the famous nowhere-zero flow conjectures of Tutte (see, e.g., [8]).

All these spaces have symmetric distance functions. In some cases it may be possible to define the distance function d (in analogy with $\mathcal{S}$) so that it would not be symmetric.

(3) It is easy to see that for every $A \in \mathbb{M}_0$ with integer entries, $\chi_{\mathcal{I}}(A)$ is an integer.

Problem 5.1 *Classify all distance spaces $\mathcal{D}$ for which $\chi_{\mathcal{D}}(A)$ is an integer for every $A \in \mathbb{M}_0$ with integer entries.*

It may be expected that such distance spaces can be described as being "composed" of $\mathcal{I}$ and some discrete spaces with unit distances, where each of these can also be multiplied with an integer fraction.

(4) There is an open problem related to Theorem 4.2. The bound of that result can be improved in some special cases as shown by Theorem 4.3. Is this possible in general? Is this possible for the distance spaces $\mathcal{I}$ and $\mathcal{S}$?

Problem 5.2 *Is there an $\varepsilon > 0$ such that for every nonnegative symmetric matrix A with zero diagonal, $\chi_{\mathcal{S}} \leq (2-\varepsilon)\lambda_{\max}(A) + \max\{a_{ij} \mid i,j \in V(A)\}$?*

It may be that $\varepsilon = 1$ is the right answer for Problem 5.2, but some results in [6] suggest that the answer to Problem 5.2 could be negative.

(5) Hoffman [2] and Cvetković [1], respectively, found lower bounds on the chromatic number of a graph expressed in terms of the eigenvalues of its adjacency matrix. Is there a similar result for the generalized chromatic number of a matrix?

(6) As a function of nondiagonal entries of A, the circular span $\chi_{\mathcal{S}}(A)$ is a piecewise linear function with finitely many regions of linearity (for a fixed dimension of A). This was proved in [6]. There are two sets of questions related to this fact. First, how could one describe the regions of linearity? And second, is the same true for any other "interesting" distance space?

Problem 5.3 *For which distance spaces $\mathcal{D}$ is $\chi_{\mathcal{D}}(A)$ a piecewise smooth function of the off-diagonal entries of A?*

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