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BLOCKING  
NONORIENTABILITY OF A  
SURFACE

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# Blocking nonorientability of a surface

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## Abstract

Let  $\mathbb{S}$  be a nonorientable surface. A collection of pairwise noncrossing simple closed curves in  $\mathbb{S}$  is a *blockage* if every one-sided simple closed curve in  $\mathbb{S}$  crosses at least one of them. Robertson and Thomas [9] conjectured that the orientable genus of any graph  $G$  embedded in  $\mathbb{S}$  with sufficiently large face-width is “roughly” equal to one half of the minimum number of intersections of a blockage with the graph. The conjecture was disproved by Mohar [7] and replaced by a similar one. In this paper, it is proved that the conjectures in [7, 9] hold up to a constant error term: For any graph  $G$  embedded in  $\mathbb{S}$ , the orientable genus of  $G$  differs from the conjectured value at most by  $O(g^2)$ , where  $g$  is the genus of  $\mathbb{S}$ .

## 1 Introduction

We follow standard graph theory terminology [2]. By a *surface* we mean a compact connected PL 2-manifold without boundary. The *genus*  $\mathbf{g}(G)$

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of a graph  $G$  is the smallest integer  $g$  such that  $G$  has an embedding in the orientable surface  $\mathbb{S}_g$  of genus  $g$ . The nonorientable surface of genus  $g$  will be denoted by  $\mathbb{N}_g$ . So,  $\mathbb{N}_1$  is the projective plane and  $\mathbb{N}_2$  is the Klein bottle. The *nonorientable genus* of  $G$  is the smallest  $g$  such that  $G$  admits an embedding in  $\mathbb{N}_g$ .

All embeddings of graphs in surfaces considered in this paper are *2-cell embeddings* in which every face is homeomorphic to an open disk in the plane. If  $\Pi$  is an embedding of a connected graph  $G$  in some surface, the *Euler genus* of  $\Pi$  is defined as the number  $\mathbf{eg}(G, \Pi) = 2 - |V(G)| + |E(G)| - f$ , where  $f$  is the number of  $\Pi$ -facial walks. We refer to [8] for additional information on embeddings of graphs in surfaces.

A *closed curve* on a surface  $\mathbb{S}$  is a continuous PL mapping  $\gamma : S^1 \rightarrow \mathbb{S}$ , and we sometimes identify  $\gamma$  with its image  $\gamma(S^1)$  in  $\mathbb{S}$ . If a graph  $G$  is embedded in  $\mathbb{S}$ , then  $\text{cr}(\gamma, G)$  denotes the number of points  $z \in S^1$  such that  $\gamma(z)$  is a point of  $G$  in  $\mathbb{S}$ . The curve  $\gamma$  is *onesided* if every neighborhood of  $\gamma$  on  $\mathbb{S}$  contains a Möbius strip, and *twosided* otherwise.

## 2 The orientable genus of graphs with a given nonorientable embedding

Let  $\Pi$  be a (2-cell) embedding of a graph  $G$  into a *nonplanar* surface  $\mathbb{S}$ , i.e. a surface distinct from the 2-sphere. Then we define the *face-width*  $\mathbf{fw}(G, \Pi)$  (also called the *representativity*) of the embedding  $\Pi$  as the minimum number of facial walks of  $G$  whose union contains a noncontractible curve. Alternatively,  $\mathbf{fw}(G, \Pi)$  is the minimum  $\text{cr}(\gamma, G)$  taken over all noncontractible closed curves  $\gamma$  on  $\mathbb{S}$ .

It is easy to see that the nonorientable genus of every graph  $G$  is bounded by a linear function of the genus  $\mathbf{g}(G)$ . On the other hand, Auslander, Brown, and Youngs [1] proved that there are graphs embeddable in the projective plane whose orientable genus is arbitrarily large. This phenomenon is now appropriately understood after Fiedler, Huneke, Richter, and Robertson [3] proved that the genus  $\mathbf{g}(G)$  of a graph  $G$  that is  $\Pi$ -embedded in the projective plane equals

$$\mathbf{g}(G) = \left\lfloor \frac{1}{2} \mathbf{fw}(\Pi) \right\rfloor \quad (1)$$

if  $\mathbf{fw}(\Pi) \neq 2$ . If  $\mathbf{fw}(\Pi) = 2$ , then  $\mathbf{g}(G)$  is either 0 or 1.

This result has been generalized to the Klein bottle by Robertson and Thomas [9] as follows. Let  $\Pi$  be an embedding of  $G$  in  $\mathbb{N}_2$ . Denote by

$\text{ord}_2(G, \Pi)$  the minimum of  $\lceil \text{cr}(\gamma, G)/2 \rceil$  taken over all noncontractible and nonseparating twosided simple closed curves  $\gamma$ . Similarly, let  $\text{ord}_1(G, \Pi)$  denote the minimum of  $\lfloor \text{cr}(\gamma_1, G)/2 \rfloor + \lfloor \text{cr}(\gamma_2, G)/2 \rfloor$  taken over all pairs  $\gamma_1, \gamma_2$  of nonhomotopic onesided simple closed curves. The latter minimum restricted to all noncrossing pairs  $\gamma_1, \gamma_2$  of onesided simple closed curves is denoted by  $\text{ord}'_1(G, \Pi)$ . Let

$$g = \min\{\text{ord}_1(G, \Pi), \text{ord}_2(G, \Pi)\} \quad (2)$$

and

$$g' = \min\{\text{ord}'_1(G, \Pi), \text{ord}_2(G, \Pi)\}. \quad (3)$$

Robertson and Thomas [9] proved that if  $g \geq 4$ , then  $\mathbf{g}(G) = g = g'$ . Equations (1) and (2) imply that the genus of graphs that can be embedded in the projective plane or the Klein bottle can be computed in polynomial time.

By [11], genus testing is **NP**-complete for general graphs. Therefore, it is interesting that the classes of projective planar graphs and graphs embeddable in the Klein bottle admit a polynomial time genus testing algorithm. Very likely the genus problem for graphs with bounded nonorientable genus is solvable in polynomial time as suggested in [9].

Robertson and Thomas [9] conjectured that (1) and (2) can be generalized as follows. Suppose that  $\Gamma = \{\gamma_1, \dots, \gamma_p\}$  is a set of closed curves in the surface  $\mathbb{N}_k$ . Then  $\Gamma$  is *crossing-free* if the following holds:

- (a) No  $\gamma_i$  crosses itself.
- (b) For  $1 \leq i < j \leq p$ , the curves  $\gamma_i$  and  $\gamma_j$  do not cross each other.

If there exist simple closed curves  $\gamma'_1, \dots, \gamma'_p$  with pairwise disjoint images in  $\mathbb{N}_k$  such that  $\gamma'_i$  is homotopic to  $\gamma_i$  ( $i = 1, \dots, p$ ) and such that every onesided closed curve in  $\mathbb{N}_k$  crosses at least one of the curves  $\gamma'_1, \dots, \gamma'_p$ , then we say that the family  $\Gamma$  is a *blockage* and that  $\Gamma$  *blocks onesided curves* in the surface.

Suppose that a graph  $G$  is embedded in  $\mathbb{N}_k$ . Robertson and Thomas [9] define the *order* of a blockage  $\Gamma = \{\gamma_1, \dots, \gamma_p\}$  as

$$\text{ord}(\Gamma, G) = \frac{1}{2}(k - 2p + s) + \sum_{i=1}^p \text{ord}(\gamma_i, G) \quad (4)$$

where  $s$  is the number of onesided closed curves in  $\Gamma$  and

$$\text{ord}(\gamma_i, G) = \begin{cases} \lfloor \text{cr}(\gamma_i, G)/2 \rfloor, & \text{if } \gamma_i \text{ is onesided} \\ \lceil \text{cr}(\gamma_i, G)/2 \rceil, & \text{if } \gamma_i \text{ is twosided.} \end{cases}$$

Let us observe that the term  $\frac{1}{2}(k - 2p + s)$  in (4) is an integer and that it is equal to the genus of the (bordered) orientable surface obtained by cutting  $\mathbb{N}_k$  along the curves in  $\Gamma$ . It is easy to prove [9]:

**Lemma 2.1.** *Let  $G$  be a graph embedded in  $\mathbb{N}_k$ , and let  $\Gamma$  be a blockage in  $\mathbb{N}_k$ . Then  $\mathbf{g}(G) \leq \text{ord}(\Gamma, G)$ .*

Based on (1)–(3) and Lemma 2.1, Robertson and Thomas proposed the following

**Conjecture 2.2 (Robertson and Thomas [9]).** *Suppose that  $G$  is embedded in  $\mathbb{N}_k$  with sufficiently large face-width. Let  $g$  (respectively  $g'$ ) be the minimum order of a blockage (crossing-free blockage) in  $\mathbb{N}_k$ . Then  $\mathbf{g}(G) = g = g'$ .*

Mohar [7] disproved this conjecture and posed a related conjecture what the correct expression for  $\mathbf{g}(G)$  might be (Conjecture 2.3 below). The value for the orientable genus of  $G$  conjectured in [7] can differ only by a constant (depending on  $k$ ) from the conjectured value of Robertson and Thomas.

Suppose that  $G$  is embedded in  $\mathbb{N}_k$ . Consider a crossing-free blockage  $\Gamma = \{\gamma_1, \dots, \gamma_p\}$  and cut the surface  $\mathbb{N}_k$  along  $\gamma_1, \dots, \gamma_p$ . This results in a graph  $\overline{G}$  embedded in an orientable surface. If a vertex  $a \in V(G)$  lies on at least one of the curves  $\gamma_i$  ( $1 \leq i \leq p$ ), then  $a$  gives rise to two or more vertices in  $\overline{G}$  (called *copies* of  $a$ ). Add a new vertex  $v_a$  and join it to all copies of  $a$  in  $\overline{G}$ . Call the resulting graph  $G'$  and note that contraction of the new edges results in the original graph  $G$ . Now, the orientable embedding of  $\overline{G}$  defines local rotations of all vertices of  $G'$  except for the new vertices  $v_a$ . The minimum genus of an orientable embedding of  $G'$  extending this partial embedding is called the *genus order* of the blockage  $\Gamma$ . It is easy to see that in the case when no vertex of  $G$  is split into more than two vertices of  $\overline{G}$ , the genus order coincides with (4), and that in general it is majorized by (4).

**Conjecture 2.3 (Mohar [7]).** *If  $G$  is embedded in a nonorientable surface with sufficiently large face-width, then the orientable genus of  $G$  is equal to the minimum genus order of a crossing-free blockage.*

In this paper it is proved that Conjectures 2.2 and 2.3 hold up to a constant error term, even without the assumption on large face-width. It is shown that for any graph  $G$  embedded in  $\mathbb{N}_g$ , the orientable genus of  $G$  differs from the minimum (genus) order of a crossing-free blockage for less than  $(64g)^2$ . See Theorem 4.7.

### 3 Blocking onesided curves

Suppose that  $G$  is a graph that is  $\Pi$ -embedded in some surface  $\mathbb{S}$ . We denote by  $\Gamma = \Gamma(G, \Pi)$  the corresponding *vertex-face graph*. Its vertices are the union of vertices of  $G$  and the vertices of the geometric dual  $G^*$  of  $G$ , i.e., the  $\Pi$ -facial walks. The edges of  $\Gamma$  correspond to the incidence of vertices and faces, with multiple edges if a vertex appears more than once on a  $\Pi$ -facial walk. The graph  $\Gamma$  has a natural quadrilateral embedding in  $\mathbb{S}$ . The geometric dual of  $\Gamma$ , the graph which we shall denote by  $M = M(G, \Pi)$ , is known as the *medial graph* of  $G$ .

A set  $B \subseteq E(M)$  is an *edge-blockage* in  $M$  if every onesided cycle of  $M$  contains an edge of  $B$ . If  $B \subseteq E(M)$ , let  $B^* \subseteq E(\Gamma)$  be the set of dual edges, and let  $\Gamma(B^*)$  be the subgraph of  $\Gamma$  generated by  $B^*$ .

**Lemma 3.1.** *Suppose that  $G$  is  $\Pi$ -embedded in  $\mathbb{N}_g$  and that  $B \subseteq E(M)$  is an edge-blockage in  $M$  that is minimal (with respect to inclusion). Then*

- (a)  $\Gamma(B^*)$  is a bipartite Eulerian graph (possibly disconnected).
- (b) The edge set  $B^*$  of  $\Gamma(B^*)$  can be partitioned into a set of edge-disjoint crossing-free closed walks. Any such partition into crossing-free closed walks is a crossing-free blockage in the surface.
- (c)  $\mathbb{N}_g \setminus \Gamma(B^*)$  is connected.
- (d) Let  $n_i$  be the number of vertices of degree  $2i + 2$  in  $\Gamma(B^*)$ . Then

$$\sum_{i=0}^{\infty} i n_i \leq g - 1. \quad (5)$$

*Proof.* To prove claim (a), suppose that  $\Gamma(B^*)$  contains a vertex  $x$  of odd degree  $d$ . Let  $e_1, \dots, e_d$  be the edges in  $B$  dual to the edges of  $\Gamma(B^*)$  that are incident with  $x$ . By the minimality of  $B$ , there exist  $\Pi$ -onesided cycles  $C_i \subseteq E(M) \setminus (B \setminus e_i)$ ,  $i = 1, \dots, d$ . Let  $C_0$  be the facial walk in  $M$  that corresponds to the vertex  $x$  of  $\Gamma$ . It is easy to see that the symmetric difference of the edges of these cycles,  $C = C_0 + C_1 + \dots + C_d$ , contains a onesided cycle in  $M$ . This yields a contradiction since  $C$  is disjoint from  $B$ .

(b) Any partition of  $B^*$  into closed walks is obtained as follows. For each vertex  $x \in V(\Gamma(B^*))$ , partition the edges incident with  $x$  into pairs and then join the paired edges to form a collection  $\mathcal{C}$  of closed walks in  $\Gamma$  (which may be viewed as closed curves in  $\mathbb{N}_g$ ). By choosing the pairs so that they are not crossing with any other chosen pair of edges incident with the

same vertex, none of the curves in  $\mathcal{C}$  crosses itself and no two of them cross each other.

Suppose that there is a one-sided simple closed curve  $\gamma$  in  $\mathbb{N}_g$  that crosses no member of  $\mathcal{C}$ . By elementary topology, it may be assumed that  $\gamma$  does not intersect any edge of  $\Gamma$  in its internal point, i.e.,  $\gamma$  passes through faces and vertices of  $\Gamma$ . Then  $\gamma$  is determined (up to homotopy) by a cyclic sequence  $v_1 f_1 v_2 f_2 \dots v_k f_k v_1$  of vertices  $v_i \in V(\Gamma)$  and faces  $f_i$  of  $\Gamma$  that are traversed by  $\gamma$ . Note that  $f_1, \dots, f_k \in V(M)$ . For  $i = 1, \dots, k$ , let  $S_i$  be a walk in  $M$  that starts with the vertex  $f_{i-1}$  of  $M$ , traverses a segment of the facial walk in  $M$  which corresponds to  $v_i$ , and ends at  $f_i$ . Clearly, the closed walk  $W$  in  $M$  which is composed of  $S_1, \dots, S_k$  is homotopic to  $\gamma$  (in  $\mathbb{N}_g$ ), so it is one-sided. Since  $\gamma$  crosses no curve from  $\mathcal{C}$ , each  $S_i$  contains an even number of edges of  $B$ . Let  $e_1, \dots, e_{2d}$  be the edges of  $B$  that are traversed by  $W$  an odd number of times and let  $C_1, \dots, C_{2d}$  be as in the proof of part (a). Then  $W + C_1 + \dots + C_{2d}$  contains a one-sided cycle that is disjoint from  $B$ , a contradiction.

(c) Suppose that  $\mathbb{N}_g \setminus B^*$  is disconnected. Then there is an edge  $e^* \in B^*$  such that on each side of  $e^*$  there is a different component of  $\mathbb{N}_g \setminus B^*$ . Let  $e \in B$  be the edge which is dual to  $e^*$ . Let  $C$  be a  $\Pi$ -one-sided cycle in  $M \setminus (B \setminus e)$ . Since  $C$  contains  $e$ , it intersects two components of  $\mathbb{N}_g \setminus B^*$ . Therefore,  $C$  crosses  $B^*$  at least twice, a contradiction.

(d)  $\Gamma(B^*)$  is a graph in  $\mathbb{N}_g$  having  $n = \sum_i n_i$  vertices and  $m = \sum_i (i+1)n_i$  edges. It may be disconnected, and its embedding in  $\mathbb{N}_g$  may not be 2-cell. But Euler's inequality still holds:  $n - m + f \geq \chi(\mathbb{N}_g) = 2 - g$ . By (c), the number  $f$  of connected components of  $\mathbb{N}_g \setminus B^*$  is 1, hence  $\sum_i i n_i = m - n \leq g - 2 + f = g - 1$ .  $\square$

A vertex set  $U \subseteq V(G)$  of a  $\Pi$ -embedded graph  $G$  is a *vertex-blockage* if every  $\Pi$ -one-sided cycle of  $G$  contains a vertex in  $U$ . Similarly, a set  $U^* \subseteq V(G^*)$  of  $\Pi$ -faces is a *face-blockage* if every one-sided cycle in the dual graph  $G^*$  contains a vertex in  $U^*$ . It is easy to see that  $U^*$  is a face-blockage if and only if every one-sided closed curve which does not contain vertices of  $G$  intersects a face in  $U^*$ .

**Lemma 3.2.** *Suppose that  $G$  is  $\Pi$ -embedded in  $\mathbb{N}_g$  and that  $B \subseteq E(M)$  is an edge-blockage in  $M$  that is minimal (with respect to inclusion). Let*

$$U = V(\Gamma(B^*)) \cap V(G) \quad \text{and} \quad U^* = V(\Gamma(B^*)) \cap V(G^*). \quad (6)$$

*Then  $U$  is a vertex-blockage,  $U^*$  is a face-blockage in  $G$ , and the following*

inequalities hold:

$$2|U| \leq |B| \leq 2|U| + 2g - 2, \quad (7)$$

$$2|U^*| \leq |B| \leq 2|U^*| + 2g - 2, \quad (8)$$

$$|U| + |U^*| \leq |B| \leq |U| + |U^*| + g - 1. \quad (9)$$

*Proof.* Let  $C$  be a  $\Pi$ -onesided cycle of  $G$ . By Lemma 3.1(b),  $C$  intersects  $\Gamma(B^*)$ . Hence it intersects  $U$ . This proves that  $U$  is a vertex-blockage. By duality,  $U^*$  is a face-blockage.

To prove the first inequality of (7), observe that the minimum degree in  $\Gamma(B^*)$  is  $\geq 2$  (by Lemma 3.1(a)) and that  $|B|$  equals the sum of degrees of vertices in  $U$ . To verify the second inequality, we shall apply Lemma 3.1(d) and denote by  $n'_i$  the number of vertices in  $U$  whose degree in  $\Gamma(B^*)$  is  $2i+2$ . Then

$$\begin{aligned} |B| &= \sum_{u \in U} \deg(u) = 2|U| + 2 \sum_i i n'_i \\ &\leq 2|U| + 2 \sum_i i n_i \leq 2|U| + 2(g-1). \end{aligned}$$

Similar proofs yield (8) and (9).  $\square$

Let  $\beta = \beta(G, \Pi)$  denote the *vertex-blockage number*, i.e. the minimum number of vertices in a vertex-blockage. Similarly, let  $\beta^* = \beta^*(G, \Pi)$  be the *face-blockage number* (the minimum number of faces in a face-blockage), and  $\beta' = \beta'(G, \Pi)$  the *edge-blockage number* (the minimum number of edges in an edge-blockage in  $M$ ).

**Corollary 3.3.** *Suppose that  $G$  is  $\Pi$ -embedded in the nonorientable surface  $\mathbb{N}_g$ . Then the following inequalities hold:*

$$2\beta \leq \beta' \leq 2\beta + 2g - 2, \quad (10)$$

$$2\beta^* \leq \beta' \leq 2\beta^* + 2g - 2, \quad (11)$$

$$\beta + \beta^* \leq \beta' \leq \beta + \beta^* + g - 1. \quad (12)$$

*Proof.* Let  $B$  be a minimum edge-blockage, i.e.  $|B| = \beta'$ . Then  $U = \Gamma(B^*) \cap V(G)$  is a vertex-blockage by Lemma 3.2. This easily implies that  $2\beta \leq \beta'$ .

Suppose now that  $U$  is a minimum vertex-blockage in  $G$ . If  $u \in U$  is a vertex of degree  $d$ , split  $u$  into  $d$  new vertices  $u_1, \dots, u_d$ , with  $u_i$  joined only to the  $i$ th neighbor of  $u$  ( $i = 1, \dots, d$ ). This operation can be performed on the surface for all vertices in  $U$  simultaneously. Since  $U$  is a blockage, the



resulting graph contains no onesided cycles. We now start identifying some of the new vertices corresponding to  $\Pi$ -consecutive neighbors of  $u$ , say  $u_i$  and  $u_{i+1}$ . We perform such identifications on the surface as long as possible so that the resulting graph  $G'$  contains no onesided cycles.

Let  $B \subseteq E(M)$  be the set of those edges of  $M$  that correspond to those  $\Pi$ -consecutive pairs  $u_i, u_{i+1}$  that have not been identified. Since  $G'$  contains no onesided cycles,  $B$  is an edge-blockage. Moreover, since every further identification gives rise to a onesided cycle,  $B$  is a minimal edge-blockage (with respect to inclusion). It is also obvious that  $V(\Gamma(B^*)) \cap V(G) \subseteq U$ . By Lemma 3.2 we thus have:

$$\begin{aligned} |B| &\leq 2|V(\Gamma(B^*)) \cap V(G)| + 2g - 2 \\ &\leq 2|U| + 2g - 2 = 2\beta + 2g - 2. \end{aligned}$$

This implies the second inequality in (10).

Relation (11) follows by duality, while (12) is proved analogously.  $\square$

**Corollary 3.4.** *Let  $G$  be a graph that is  $\Pi$ -embedded in  $\mathbb{N}_g$ . Then*

$$\begin{aligned} \mathbf{g}(G) &\leq \frac{1}{4}\beta'(G, \Pi) + \frac{g}{4}, \\ \mathbf{g}(G) &\leq \frac{1}{2}\beta(G, \Pi) + \frac{3g-2}{4}, \text{ and} \\ \mathbf{g}(G) &\leq \frac{1}{2}\beta^*(G, \Pi) + \frac{3g-2}{4}. \end{aligned}$$

*Proof.* Let  $B$  be a minimum edge-blockage. By Lemma 3.1(b),  $\Gamma(B^*)$  defines a crossing-free blockage  $\Gamma = \{\gamma_1, \dots, \gamma_p\}$ . Clearly,  $\sum_i \text{cr}(\gamma_i, G) = \frac{1}{2}|B^*| = \frac{1}{2}|B| = \frac{1}{2}\beta'(G, \Pi)$ . By Lemma 3.1(c), it follows that  $\Gamma$  contains at most  $\frac{g}{2}$  twosided curves. Consequently,  $\text{ord}(\Gamma, G) \leq \frac{1}{2}\sum_i \text{cr}(\gamma_i, G) + \frac{1}{2}|\{i \mid \gamma_i \text{ is twosided}\}| \leq \frac{1}{4}\beta'(G, \Pi) + \frac{g}{4}$ . By Lemma 2.1,  $\mathbf{g}(G) \leq \text{ord}(\Gamma, G)$ . This proves the first inequality. The second and the third inequality follow from the first one by (10) and (11), respectively.  $\square$

## 4 Unstable faces and blockages

Let  $\Pi_0$  be an embedding of a graph  $G$ . Suppose that there is a facial walk  $F$  in which some vertex  $v$  appears twice. Then there is a simple closed curve  $\gamma$  in the surface which is contained in the face bounded by  $F$  such that  $\gamma \cap G = \{v\}$  and  $\gamma$  intersects  $F$  in two distinct appearances of  $v$  in  $F$ . If  $\gamma$  is contractible and its interior contains a vertex or an edge of  $G$ , then

we delete the vertices and edges of  $G$  in the interior of  $\gamma$ . This operation is called an *elementary reduction of type I*.

Suppose now that there are facial walks  $F$  and  $F'$  such that there exist distinct vertices  $v, v' \in V(F) \cap V(F')$ . Then there is a simple closed curve  $\gamma$  in the surface which is composed of two segments  $\alpha, \beta$  joining  $v$  and  $v'$  in the faces bounded by  $F$  and  $F'$ , respectively. If  $\gamma$  is contractible and its interior contains at least two edges of  $G$ , then we replace all edges and vertices in its interior by a single edge joining  $v$  and  $v'$ . Such an operation is called an *elementary reduction of type II*.

The embedded graph  $G$  is *essentially 3-connected* if no elementary reductions of type I or II are possible. See also [5]. An obvious property of elementary reductions is the following:

**Lemma 4.1.** *Let  $\Pi$  be an embedding of a graph  $G$ . If the  $\Pi'$ -embedded graph  $G'$  is obtained from  $G$  by a sequence of elementary reductions, then  $\mathbf{g}(G') = \mathbf{g}(G)$  and  $\beta(G, \Pi) = \beta(G', \Pi')$ ,  $\beta'(G, \Pi) = \beta'(G', \Pi')$ , and  $\beta^*(G, \Pi) = \beta^*(G', \Pi')$ .*

By Lemma 4.1, we shall be able to restrict ourselves to essentially 3-connected embeddings.

Suppose now that we have two embeddings,  $\Pi$  and  $\Pi'$ , of a graph  $G$ . Let  $F = v_0 e_1 v_1 \dots v_{k-1} e_k v_0$  be a  $\Pi$ -facial walk. A subsequence  $e_i v_i e_{i+1}$  (indices modulo  $k$ ),  $i \in \{1, \dots, k\}$ , is called an *angle* of  $F$ . The angle  $e_i v_i e_{i+1}$  is identified with the angle  $e_{i+1} v_i e_i$  obtained by traversing the facial walk  $F$  in the reverse direction. The angle  $e_i v_i e_{i+1}$  is  $(\Pi, \Pi')$ -*unstable* if it is not an angle of the embedding  $\Pi'$ . If two consecutive angles  $e_i v_i e_{i+1}$  and  $e_{i+1} v_{i+1} e_{i+2}$  of the facial walk  $F$  are  $(\Pi, \Pi')$ -stable but  $e_i v_i e_{i+1} v_{i+1} e_{i+2}$  is not a subwalk of a  $\Pi'$ -facial walk, then the angles  $e_i v_i e_{i+1}$  and  $e_{i+1} v_{i+1} e_{i+2}$  are said to be *weakly  $(\Pi, \Pi')$ -unstable*.

Suppose that  $W = \dots e_1 v e_2 \dots$  and  $W' = \dots e_3 v e_4 \dots$  are walks in a  $\Pi'$ -embedded graph  $G$ . If the edges  $e_1, \dots, e_4$  are distinct and their  $\Pi'$ -clockwise order around  $v$  is  $e_1 e_3 e_2 e_4$  or  $e_1 e_4 e_2 e_3$ , then we say that  $W$  and  $W'$   $\Pi'$ -*cross* at  $v$ . Similarly we define  $\Pi'$ -*crossing* of two walks at a common edge  $e$ .

**Lemma 4.2.** *Let  $\Pi$  and  $\Pi'$  be embeddings of a graph  $G$ .*

- (a) *If  $e v f$  is a  $(\Pi, \Pi')$ -unstable angle of a  $\Pi$ -facial walk  $F$ , then there is a  $\Pi$ -facial walk  $F'$  with an angle  $e' v f'$  such that  $F$  and  $F'$   $\Pi'$ -cross each other at  $v$ .*
- (b) *Suppose that  $d v e$  and  $e u f$  are weakly  $(\Pi, \Pi')$ -unstable angles of a  $\Pi$ -facial walk  $F$ . Let  $F' = \dots d' v e u f' \dots$  be the second  $\Pi$ -facial walk*

containing the edge  $e$ . Then  $F$  and  $F'$   $\Pi'$ -cross each other at  $e$ . Moreover, either one of the angles  $d've$  or  $eu'f'$  is  $(\Pi, \Pi')$ -unstable, or these two angles of  $F'$  are weakly  $(\Pi, \Pi')$ -unstable.

*Proof.* To prove (a), consider the local  $\Pi'$ -clockwise ordering  $e, e_1, \dots, e_s, f, f_1, \dots, f_t$  of edges around  $v$ . Since  $e$  and  $f$  are not  $\Pi'$ -consecutive, we have  $s \geq 1$  and  $t \geq 1$ . It is easy to see that there are  $\Pi$ -consecutive edges  $e', f'$  such that  $e' = e_i$  for some  $i$  ( $1 \leq i \leq s$ ), and  $f' = f_j$  for some  $j$  ( $1 \leq j \leq t$ ), or vice versa. This implies (a).

Claim (b) is obvious and we leave the details for the reader.  $\square$

A collection of cycles  $C_1, \dots, C_k$  is called a *collection of bouquets* if there exist vertices  $x_1, \dots, x_p$  such that every cycle  $C_i$  ( $1 \leq i \leq k$ ) contains precisely one of these vertices and such that for any two distinct cycles  $C_i, C_j$  ( $1 \leq i < j \leq k$ ), the intersection  $C_i \cap C_j$  is either empty, one of the vertices  $x_1, \dots, x_p$ , or an edge incident to one of these vertices.

Part (a) of the following lemma is proved in [4], while part (b) is easy to see (cf., e.g., [6]).

**Lemma 4.3.** *Let  $G$  be a graph embedded in a surface of Euler genus  $g$ , and let  $C_1, \dots, C_k$  be a collection of bouquets of cycles of  $G$ .*

- (a) *If  $C_1, \dots, C_k$  are noncontractible and pairwise nonhomotopic then  $k \leq 3g$ .*
- (b) *If no subset of  $C_1, \dots, C_k$  separates the surface then  $k \leq g$ .*

The proof of the next lemma is essentially contained in [6].

**Lemma 4.4.** *Let  $G$  be a  $\Pi'$ -embedded graph and let  $\{(C_i, C'_i) \mid i = 1, \dots, k\}$  be a collection of pairs of closed walks of  $G$  with the following properties:*

- (a)  *$C_1, \dots, C_k$  are distinct cycles of  $G$  and no two of them are  $\Pi'$ -crossing.*
- (b) *If  $1 \leq i < j \leq k$  then  $C_i$  does not  $\Pi'$ -cross with  $C'_j$ .*
- (c) *For  $i = 1, \dots, k$ ,  $C_i \cap C'_i$  is either a vertex or an edge.*
- (d) *For  $i = 1, \dots, k$ ,  $C_i$  and  $C'_i$  are  $\Pi'$ -crossing at their intersection.*

*Then the genus  $\mathbf{g}(G, \Pi')$  of  $\Pi'$  is at least  $k$ .*

**Lemma 4.5.** *Let  $B_U \subseteq E(M)$  be the set of the edges of the medial graph  $M(G, \Pi)$  which correspond to the  $(\Pi, \Pi')$ -unstable and to the weakly  $(\Pi, \Pi')$ -unstable angles. If  $\Pi'$  is an orientable embedding, then  $B_U$  is an edge-blockage for  $\Pi$ .*

*Proof.* Let  $C$  be a onesided cycle in  $M$ . An open (normal) neighborhood of  $C$  is homeomorphic to the Möbius band. If  $E(C) \cap B_U = \emptyset$ , then it is easy to see that the same neighborhood would be a neighborhood of  $C$  in the embedding  $\Pi'$ . Since  $\Pi'$  is orientable, we conclude that  $E(C) \cap B_U \neq \emptyset$ .  $\square$

Suppose that the set  $\{1, 2, \dots, 2p\}$  is partitioned into pairs  $A_i = \{a_i, b_i\}$ , where  $a_i < b_i$ ,  $i = 1, \dots, p$ . Suppose that  $1 \leq i \leq p$  and  $1 \leq j \leq p$  and that  $b_i \geq a_j$  and  $b_j \geq a_i$ . Then the pair  $A_i, A_j$  is called a *canonical pair*. An integer  $l \in \{1, 2, \dots, 2p\}$  is *covered* by this canonical pair if either

- (a)  $i = j$  and  $a_i \leq l \leq b_i$ , or
- (b)  $i \neq j$  and  $l$  is either between  $a_i$  and  $a_j$  or between  $b_i$  and  $b_j$ .

**Lemma 4.6.** *Under the assumptions given above, there is a set of at least  $\lceil \sqrt{p/20} \rceil$  canonical pairs such that every  $l \in \{1, 2, \dots, 2p\}$  is covered by at most one of these pairs.*

*Proof.* The proof is by induction on  $p$ . The proof is obvious for  $p \leq 20$  and easy for  $21 \leq p \leq 80$  (where we need only two canonical pairs).

Suppose now that  $p \geq 81$ . Let  $q = \lfloor p/2 \rfloor$ . Let us first consider the case when at least  $\lceil p/3 \rceil$  pairs  $A_i$  satisfy  $a_i \leq 2q$  and  $b_i > 2q$ . Let  $Z$  be the set of all such pairs. Define a partial order  $\preceq$  on  $Z$  by  $A_i \preceq A_j$  if  $a_i \leq a_j$  and  $b_i \leq b_j$ . By the Dilworth Theorem, this partial order either contains a chain or an antichain of cardinality  $z = \lceil \sqrt{|Z|} \rceil \geq \lceil \sqrt{p/3} \rceil$ . If  $A_{i_1}, \dots, A_{i_z}$  is a chain or an antichain, where  $a_{i_1} < a_{i_2} < \dots < a_{i_z}$ , then consecutive pairs in this order are canonical pairs that cover pairwise disjoint subsets of  $\{1, \dots, 2p\}$ . This gives rise to at least  $\lfloor z/2 \rfloor$  canonical pairs. Since  $p \geq 81$ ,  $\lfloor z/2 \rfloor \geq \frac{1}{2}\sqrt{p/3} - \frac{1}{2} \geq \sqrt{p/20}$ . This completes the proof in this case.

Suppose now that there are less than  $\lceil p/3 \rceil$  such pairs. The remaining subset of at least  $\lceil 2p/3 \rceil$  pairs  $A_i$  gives rise to two subsets containing  $p_1$  and  $p_2$  pairs, respectively, such that the pairs in the first set are contained in  $\{1, \dots, 2q\}$ , and the pairs from the second set are contained in  $\{2q + 1, \dots, 2p\}$ . Note that  $p_1 + p_2 \geq \lceil 2p/3 \rceil$  and that  $p_1 \leq \lfloor p/2 \rfloor$  and  $p_2 \leq \lceil p/2 \rceil$ . In fact, we may assume that  $p_1, p_2 \leq p/2$ . (If  $p_2 > p/2$ , then we take  $q = \lceil p/2 \rceil$  and repeat the above proof.)

By the induction hypothesis, these sets of pairs contain at least  $\rho = \lceil \sqrt{p_1/20} \rceil + \lceil \sqrt{p_2/20} \rceil$  canonical pairs that cover disjoint sets. The above conditions on  $p_1, p_2$  imply that  $\rho \geq \sqrt{(p/2)/20} + \sqrt{(p/6)/20} > \sqrt{p/20}$ . This completes the proof.  $\square$

**Theorem 4.7.** *Let  $G$  be a graph that is  $\Pi$ -embedded in the nonorientable surface  $\mathbb{N}_g$ . Then*

$$\frac{1}{2}\beta^*(G, \Pi) - (64g)^2 \leq \mathbf{g}(G) \leq \frac{1}{2}\beta^*(G, \Pi) + \frac{3g-2}{4}. \quad (13)$$

*Proof.* The second inequality in (13) holds by Corollary 3.4. To prove the first one, it suffices to verify that the bound  $\mathbf{g}(G) \geq \frac{1}{2}|\mathcal{F}| - (64g)^2$  holds for some face-blockage  $\mathcal{F}$  (not necessarily a minimum one). By Lemma 4.1, we may assume that the  $\Pi$ -embedded graph  $G$  is essentially 3-connected.

Let  $\Pi'$  be an orientable embedding of  $G$  with genus  $\mathbf{g}(G)$ . Let  $B_U \subseteq E(M)$  be the set of those edges of the medial graph  $M(G, \Pi)$  which correspond to the  $(\Pi, \Pi')$ -unstable and to the weakly  $(\Pi, \Pi')$ -unstable angles. By Lemma 4.5, the set  $B_U$  is an edge-blockage.

If a vertex  $v$  appears more than once on a facial walk  $F$ , then we say that the angles of  $F$  at the appearances of  $v$  are *1-singular*. If there are distinct facial walks  $F, F'$  such that there exist distinct vertices  $v, v' \in V(F) \cap V(F')$  which are not consecutive on (at least) one of these facial walks, then we say that the angles of  $F$  and of  $F'$  at  $v$  and  $v'$  are *2-singular*. For  $i = 1, 2$ , let  $B_i \subseteq E(M)$  be the set of the edges which correspond to the  $i$ -singular angles. Since  $G$  is essentially 3-connected, the edges in  $B_i^*$  correspond to the edges in noncontractible cycles of length  $2i$  in the vertex-face graph  $\Gamma$ .

Let  $B$  be an edge-blockage contained in  $B_U \cup B_1 \cup B_2$  of minimum cardinality. Let  $\Lambda = \Gamma(B^*)$  be the subgraph of  $\Gamma$  generated by the edges dual to  $B$ .

Consider the connected components of  $\Lambda$  which are cycles. On each of these cycles, select a vertex, and let  $A_0$  be the set of all selected vertices. By Lemma 3.1(c), no subset of these cycles separates the surface and hence, by Lemma 4.3(b),  $|A_0| \leq g$ .

Denote by  $A_3$  the set of vertices of  $\Lambda$  containing  $A_0$  and all vertices of degree  $> 2$  in  $\Lambda$ . Let  $A_4$  be the set of all vertices of  $\Lambda$  whose distance in  $\Lambda$  from  $A_3$  is 1 or 2. By (5) we have

$$|A_3 \cup A_4| \leq 5|A_0| + \sum_{i \geq 1} (1 + 2 \cdot (2i + 2))n_i \leq 5g + 9 \sum_{i \geq 1} i n_i \leq 14g - 9.$$

Similar arguments as used above imply that the graph  $\Lambda - (A_3 \cup A_4)$  is the union of  $r \leq 3g - 2$  disjoint paths  $P_1, \dots, P_r$ . Choose arbitrarily an orientation of each of the paths  $P_1, \dots, P_r$ . If  $C$  is a  $\Pi$ -facial walk corresponding to a vertex of  $P_i$  ( $1 \leq i \leq r$ ), let  $v_C \in V(G)$  be the vertex of  $G$  that follows  $C$  in  $\Lambda$  in the chosen direction of  $P_i$ . If the edge of  $\Lambda$  joining  $C$

and  $v_C$  belongs to  $B_U^*$ , then Lemma 4.2 implies that there is a  $\Pi$ -facial walk  $C'$  such that  $C$  and  $C'$   $\Pi'$ -cross at  $v_C$  or  $\Pi'$ -cross at a common edge incident with  $v_C$ . We say that  $C'$  is a *mate* of  $C$ . If the edge joining  $C$  and  $v_C$  is in  $B_2^*$ , then we let the *mate*  $C'$  of  $C$  be a face such that  $C$  and  $C'$  intersect at  $v_C$  and at another vertex that is not adjacent with  $v_C$ .

Let  $A_1$  be the set of vertices of  $\Lambda$  which correspond to  $\Pi$ -facial walks that are not cycles of  $G$ . For  $x \in A_1$ , let  $F$  be the corresponding facial walk, and let  $v \in V(F)$  be a vertex of  $G$  that appears twice in  $F$ . Since  $G$  is essentially 3-connected,  $v$  and  $F$  determine a noncontractible cycle of length 2 in  $\Gamma$ . (Possibly, the edges of that cycle are not contained in  $\Lambda$ .) Choose one such 2-cycle for every  $x \in A_1$ , and let  $C_1, \dots, C_k$  ( $k = |A_1|$ ) be the resulting collection of cycles of  $\Gamma$ . Clearly,  $C_1, \dots, C_k$  form a bouquet collection in  $\Gamma$ . If  $k > 9g$ , then Lemma 4.3(a) implies that four of the cycles are homotopic to each other, say  $Q_1, Q_2, Q_3, Q_4$ . These cycles of length 2 in  $\Gamma$  may intersect but their vertices corresponding to faces of  $G$  are distinct. We may assume that  $C_1$  and  $C_4$  bound a cylinder (or a disk) that contains  $C_2$  and  $C_3$ . Now, we add to  $B^*$  the edges of  $Q_1$  and  $Q_4$ . This gives rise to a new edge-blockage contained in  $B_U \cup B_1 \cup B_2$  whose cardinality is  $\leq |B| + 4$ . Since  $Q_2$  and  $Q_3$  are contained in the cylinder (disk) bounded by  $Q_1$  and  $Q_4$ , we may remove the edges of  $B^*$  incident with the vertices of  $A_1$  that are on  $Q_2$  and  $Q_3$  and also remove the edges of  $Q_4$  and still have a blockage  $B' \subset B^* \cup E(C_1)$ . Clearly,  $|B'| < |B^*|$ , a contradiction. Consequently,  $|A_1| \leq 9g$ .

Let  $A_2$  be the set of vertices of  $P_1, \dots, P_r$  that are not in  $A_1$  and correspond to  $\Pi$ -facial cycles which intersect their mate in more than just a vertex or an edge. Let  $C$  be such facial cycle, and let  $C'$  be its mate. Since  $G$  is essentially 3-connected, there is a noncontractible 4-cycle  $Q$  in  $\Gamma$  whose vertices are  $C, C', v_C$  and another vertex  $y \in V(C \cap C')$ . Let  $Z$  be the set of all such 4-cycles of  $\Gamma$ . For  $Q \in Z$ , we denote its vertices by  $C(Q), C'(Q), v_C(Q)$ , and  $y(Q)$ .

It is a simple exercise to prove that there is a subset  $Z_1 \subseteq Z$  of cardinality  $\geq \frac{1}{9}|A_2|$  such that for any  $Q_1, Q_2 \in Z_1$ ,  $v_C(Q_1) \neq y(Q_2)$  and  $C(Q_1) \neq C'(Q_2)$ . (Hint: Consider the directed graph on all  $v_C$  and  $y$ -vertices, with an edge from  $v_C(Q)$  to  $y(Q)$  for each  $Q \in Z$ , and observe that the outdegree of this digraph is at most 1.) Clearly,  $V(Q_1) \cap V(Q_2) \subseteq \{C'(Q_1), y(Q_1)\}$ . If  $Q_1$  and  $Q_2$  intersect in two vertices, then we may assume that their intersection is the edge  $C'(Q_1)y(Q_1) = C'(Q_2)y(Q_2)$ .

Let  $z = \sqrt{|Z_1|}$ . If there is a vertex  $y$  such that  $y = y(Q)$  for at least  $z$  members of  $Z_1$ , then those 4-cycles in  $Z_1$  that contain  $y$  form a collection of bouquets of cardinality at least  $z$ . Otherwise, there is a subset of  $Z_1$  of

cardinality  $\geq z$  such that no two cycles in this subset have their  $y$ -vertex in common. Again, this subset forms a collection of bouquets. If  $z > 9g$ , then four of the cycles in that collection of bouquets are homotopic, and a proof similar to the above proof of the fact that  $|A_1| \leq 9g$  yields a contradiction to the minimality of  $B$ . This shows that  $z \leq 9g$  and, therefore,  $|A_2| \leq 729g^2$ .

Let  $F_1, \dots, F_N$  be the facial cycles corresponding to the vertices on  $P_1, \dots, P_r$  which are not in  $A_1 \cup A_2$ , enumerated in the order of the paths  $P_1, \dots, P_r$  and with respect to their selected orientation. Let  $F'_1, \dots, F'_N$  be their mates. Since the facial cycles corresponding to the vertices in  $\Lambda$  form a face-blockage, we have

$$\begin{aligned} N &\geq \beta^*(G, \Pi) - |A_1 \cup A_2| - |A_3 \cup A_4| \\ &\geq \beta^*(G, \Pi) - (27g + 1)^2. \end{aligned} \tag{14}$$

If  $i, j \in \{1, \dots, N\}$  and  $j - i \geq 2$ , then we say that  $\{i, j\}$  is a *bad pair* if either  $F_i$  and  $F_j$ , or  $F_i$  and  $F'_j$  intersect and  $\Pi'$ -cross each other. Let  $M$  be a set of bad pairs of maximum cardinality such that no two members of  $M$  have an element in common. Our goal is to prove that  $|M| = O(g^2)$ .

Each bad pair  $\{i, j\}$  determines a path  $Q_{ij}$  joining two vertices of  $\Lambda$ : If  $F_i$  and  $F_j$   $\Pi'$ -cross at vertex  $x$ , then  $Q_{ij}$  is the path of length 2 connecting  $F_i$  and  $F_j$  through  $x$ . If  $F_i$  and  $F'_j$   $\Pi'$ -cross at vertex  $x$ , then  $Q_{ij}$  is the path of length 3 connecting  $F_i$  and a vertex in  $F_j \cap F'_j$  through  $x$ . Clearly,  $E(Q_{ij}) \subseteq B_U^*$ .

If  $\{i, j\}$  is a bad pair and  $F_i$  and  $F_j$  are in the same path  $P_a$  ( $1 \leq a \leq r$ ), then  $Q_{ij}$  and the edges of  $P_a$  determine a cycle  $R_{ij}$  which is called the *canonical cycle* of the bad pair  $\{i, j\}$ . Observe that

$$|E(Q_{ij})| < \frac{1}{2}|E(R_{ij})|. \tag{15}$$

Suppose that  $\{i, j\}$  and  $\{i', j'\}$  ( $i < j, i' < j'$ ) are disjoint bad pairs such that  $i' < j$  and  $i < j'$ . If  $F_j$  and  $F_{j'}$  are in the same path  $P_a$  ( $1 \leq a \leq r$ ) and  $F_i$  and  $F_{i'}$  are in the same path  $P_b$  ( $1 \leq b \leq a$ ), then there is a cycle  $R_{ij, i'j'}$  in  $\Lambda$  that is composed of  $Q_{ij}$ ,  $Q_{i'j'}$  and two paths  $P_{jj'} \subseteq P_a$  and  $P_{ii'} \subseteq P_b$  joining the “upper” and “lower” ends of  $Q_{ij}$  and  $Q_{i'j'}$ , respectively. The cycle  $R_{ij, i'j'}$  is called the *canonical cycle* of bad pairs  $\{i, j\}$  and  $\{i', j'\}$ . We shall need an analogy of (15). That is not automatic, but if  $|j - j'| \geq 4$ , then the length of the segment  $P_{jj'}$  is at least 7. Consequently,

$$|E(Q_{ij})| + |E(Q_{i'j'})| < |E(P_{jj'})|. \tag{16}$$

We can view  $P_1 \cup \dots \cup P_r$  as being a single path by adding auxiliary edges joining the end of  $P_l$  with the beginning of  $P_{l+1}$ ,  $l = 1, \dots, r-1$ . Then we can define canonical cycles for bad pairs (or pairs of bad pairs) also when the ends of  $Q_{ij}$  (and  $Q_{i'j'}$ ) are not in the same path(s)  $P_a$  (and  $P_b$ ). The canonical cycles that use the auxiliary edges are called *fake canonical cycles*; the others are said to be *genuine*.

In order to meet the condition  $|j - j'| \geq 4$  needed for (16), we order the bad pairs in  $M$  according to their larger members  $j$ , and let  $M_1$  be the subset consisting of every fourth bad pair in this order.

Lemma 4.6 shows that there is a set of at least  $\lceil \sqrt{|M_1|/20} \rceil$  canonical cycles (using only bad pairs in  $M_1$ ) whose intersections with  $P_1 \cup \dots \cup P_r$  are pairwise disjoint. Since  $r \leq 3g - 2$ , at most  $3g - 3$  of these canonical cycles are fake. Let  $R_1, \dots, R_s$  ( $s \geq \lceil \sqrt{|M_1|/20} \rceil - 3g + 3$ ) be the subset of genuine canonical cycles. These canonical cycles are disjoint except that they may have a vertex in common if the mate  $F'_j$  of  $F_j$  is the same as the mate  $F'_l$  of  $F_l$ , and  $F_j, F_l$  are in distinct canonical cycles. Therefore,  $R_1, \dots, R_s$  form a collection of bouquets.

If one of these cycles, say  $R_l = R_{ij}$  (or  $R_l = R_{ij,i'j'}$ ) would be contractible, then the replacement in  $B^*$  of  $E(R_l) \cap P_a$  (or  $E(P_{jj'})$ ) with  $Q_{ij}$  (or  $Q_{ij} \cup Q_{i'j'}$ ) would give rise to another blockage. By (15) (or (16)), this blockage would contradict minimality of  $B$ . Therefore,  $R_l$  is noncontractible. Similar conclusion holds if two of these genuine canonical cycles are homotopic (in which case we can add to  $B^*$  the missing edges of one of them and remove the edges of the second one). Lemma 4.3 implies that  $s \leq 3g$ . Consequently,  $|M| \leq 4|M_1| \leq 3240g^2$ .

Let  $A$  be the set of facial cycles  $F_l$  such that  $l$  is contained in some bad pair in  $M$ . As proved above,  $|A| \leq 2 \cdot 3240g^2$ . Let  $C_1, \dots, C_t$  be a maximum subsequence of  $F_1, \dots, F_N$  such that none of  $C_i$  is in  $A$  and such that, for  $i = 1, \dots, t-1$ , if  $C_i = F_j$ , then  $C_{i+1} \neq F_{j+1}$ . Clearly,

$$t \geq \frac{1}{2}(N - |A|) \geq \frac{1}{2}N - 3240g^2. \quad (17)$$

For  $j = 1, \dots, t$ , let  $C'_j$  be the mate of  $C_j$ . Let us consider the collection of pairs

$$\mathcal{C} = \{(C_j, C'_j) \mid j = 1, \dots, t\}.$$

We claim that  $\mathcal{C}$  satisfies conditions (a)–(d) of Lemma 4.4. No facial walk  $C_i$  is in  $A_1 \cup A_2$ . Therefore, every  $C_i$  is a cycle and (c) holds. Since the cycles in  $A$  do not participate in the sequence  $C_1, \dots, C_t$ , the pairs in  $\mathcal{C}$  satisfy (a) and (b). Clearly, (d) is also satisfied.



By Lemma 4.4 and inequalities (14) and (17),

$$\mathbf{g}(G) = \mathbf{g}(G, \Pi') \geq |\mathcal{C}| \geq \frac{1}{2}N - 3240g^2 \geq \frac{1}{2}\beta^*(G, \Pi) - (64g)^2.$$

The proof is complete.  $\square$

The “error” term  $(64g)^2$  in (13) is not best possible. There are examples which show that such term of order  $\Omega(g)$  is necessary, and we conjecture that (13) can be improved to

$$\mathbf{g}(G) \geq \frac{1}{2}\beta^*(G, \Pi) - O(g).$$

Lemma 2.1, Theorem 4.7, and Corollary 3.3 imply:

**Corollary 4.8.** *Let  $G$  be a graph embedded in  $\mathbb{N}_g$ , and let  $r$  be the minimum order of a crossing-free blockage. Then*

$$r - (64g)^2 \leq \mathbf{g}(G) \leq r.$$

Finally, let us observe that Corollary 3.3 implies that  $\beta(G, \Pi)$  and  $\beta^*(G, \Pi)$  cannot differ too much. Therefore,  $\mathbf{g}(G)$  is also approximately equal to  $\beta(G, \Pi)$ , up to a term which depends on  $g$  only.

It is not clear if there is an efficient algorithm for finding a minimum (crossing-free) blockage or its approximation for a graph embedded in  $\mathbb{N}_g$ . For every fixed  $g$ , this task is solvable in polynomial time since there is only a bounded number of possibilities for homotopies of curves in an optimum crossing-free blockage. However, this approach seems complicated, and we restrain of describing further details. The case when  $g = 2$  is described in [9].

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