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BICIRCULAR PROJECTIONS AND CHARACTERIZATION OF HILBERT SPACES

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Abstract. We prove that every JB* triple with rank one bicircular projection is a direct sum of two ideals, one of which is isometrically isomorphic to a Hilbert space.

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1. Introduction

The topic, which we study in the present paper in a completely algebraic way, was motivated by our previous investigation of continuous Reinhardt domains. These are open subsets of $C_0(\Omega)$, satisfying some additional properties (cf. [14]). In [14] we were led, at some technical point, to study projections with one dimensional range and additional norm property which is called bicircularity. What we needed to show was that their kernels are Hilbert spaces.

The definition of bicircularity is however a general Banach space notion, which is not necessarily connected to $C_0(\Omega)$. We thus expand, what was a technical point in [14], into an independent study in its proper algebraic setting, which is that of JB* triples. Our main result and its corollary can be interpreted as a characterization of Hilbert spaces among JB* triples.

2. Bicircular projections and JB* triples

Let W be a complex Banach space and $P : W \to W$ a projection. We denote by $\overline{P}$ the projection $1 - P$. Clearly $P\overline{P} = \overline{P}P = 0$ holds.

Definition 1. *The projection P is called bicircular if for all $\alpha, \beta \in R$ the operator $e^{i\alpha}P + e^{i\beta}\overline{P}$ is an isometry.*

Every bicircular projection is automatically bicontractive, i.e. $||P|| \leq 1$ and $||\overline{P}|| \leq 1$. This follows from the above definition by taking $e^{i\alpha} = 1$ and $e^{i\beta} = -1$. Then $\phi = P - (1 - P) = 2P - 1$ is isometric and so

$$||P|| = \frac{1}{2}\,||1 + \phi|| \leq \frac{1}{2}(1 + ||\phi||) = 1.$$

In a similar way we can prove that $||\overline{P}|| \leq 1$.

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EXAMPLE 1. *Let H be a complex Hilbert space and $a \in H$ a unit vector. Then the minimal projection, generated by a, is given by*

$$Px = \langle x, a \rangle\, a : x \in H.$$

Obviously P has onedimensional range. Moreover, for every real α and β we have, since the ranges of P and $\overline{P}$ are orthogonal,

$$\left|\left|e^{i\alpha}Px + e^{i\beta}\overline{P}x\right|\right|^2 = \left|\left|Px\right|\right|^2 + \left|\left|\overline{P}x\right|\right|^2 =$$
$$\left|\left|\langle x, a \rangle\, x\right|\right|^2 + \left|\left|x - \langle x, a \rangle\, x\right|\right|^2 = \left|\left|x\right|\right|^2$$

for all $x \in H$, which shows that P is bicircular.

EXAMPLE 2. *Let $P : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ be defined by*

$$P \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & \delta \end{bmatrix}$$

and consider the matrix algebra equipped with the operator norm. Then, since

$$\left|\left|\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}\right|\right| \geq \max\left\{\left|\alpha\right|, \left|\beta\right|, \left|\gamma\right|, \left|\delta\right|\right\} \geq$$
$$\max\left\{\left|\alpha\right|, \left|\delta\right|\right\} = \left|\left|\begin{bmatrix} \alpha & 0 \\ 0 & \delta \end{bmatrix}\right|\right|,$$

the projection P is contractive. In a similar way we get that $\overline{P}$ is also contractive and so P is bicontractive. We can see however, that P is not bicircular. If

$$x = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

then $||x|| = 2$ but for $\phi = P + i\overline{P}$ we have

$$\phi x = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}.$$

Since $||\phi x|| = \sqrt{2}$, the projection P is not bicircular.

Historically JB* triples arose in the study of bounded symmetric domains in complex holomorphy. They can be defined as those complex Banach spaces W for which there exists a continuous triple product

$$\{...\} : W \times W \times W \to W$$

such that

(JBi) $\{xyz\}$ is symmetric and linear in x, z and conjugate linear in y

(JBii) $\{xy\{abc\}\} = \{\{xya\}bc\} - \{a\{yxb\}c\} + \{ab\{xyc\}\}$

(JBiii) the box operator $x\Box x$, defined by $(x\Box x)(y) = \{xxy\}$ is hermitian with nonnegative spectrum

(JBiv) $||\{xxx\}|| = ||x||^3$

This notion can be regarded as a simultaneous generalization of Hilbert spaces and C* algebras. If H is a complex Hilbert space, we can define its JB* triple product by

$$\{xyz\} = \frac{1}{2}\langle x, y\rangle z + \frac{1}{2}\langle z, y\rangle x.$$

If A is a C* algebra, we can define its JB* triple product by

$$\{xyz\} = \frac{1}{2}xy^*z + \frac{1}{2}zy^*x$$

JB* triples were studied from algebraic (see for example [7], [8] and [11]), holomorphic (see for example [4], [10], [9] and [15]), operator theoretic (see for example [6] and [12]) and Banach geometric (see for example [2], [3] and [5]) viewpoint. For a recent survey see [13].

In our first example we saw that every Hilbert space is a JB* triple with many rank one bicircular projections. This can easily be extended in the following way. Let H be a complex Hilbert space and $P : H \to H$ a bicircular projection of rank one. Let J be any JB* triple and define

$$Q : J \oplus H \to J \oplus H$$

by $Q(j, h) = (0, Ph)$. Then Q is also a bicircular projection of rank one. Our main result shows that this in fact the only possibility. More precisely, we shall prove

THEOREM 1. *Let W be a JB* triple and $P : W \to W$ a bicircular projection of rank one. Then there exist two closed ideals J and H of W such that $W = J \oplus H$, where H is isomorphic to a Hilbert space, $P(W) \subset H$ and $J \subset Ker\ (P)$.*

COROLLARY 1. *The only prime JB* triples admitting bicircular projections of rank one are Hilbert spaces.*

Before we start our proof, we note that the assumption of rank one cannot be omitted from the above results. If $W = M_2(\mathbb{C})$ then $P : W \to W$, defined by

$$P\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ 0 & 0 \end{bmatrix},$$

is bicircular. Namely, for every real t and s, we have

$$(e^{it}P + e^{is}\overline{P})\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} e^{it}\alpha & e^{it}\beta \\ e^{is}\gamma & e^{is}\delta \end{bmatrix}$$

and it only take some elementary computations of eigenvalues to verify that the operator norms of the matrices

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \text{ and } \begin{bmatrix} e^{it}\alpha & e^{it}\beta \\ e^{is}\gamma & e^{is}\delta \end{bmatrix}$$

are the same, which means that P is bicircular.

3. Proof of the main result

Since the range of P is onedimensional, we have $P(x) = f(x)a$ for some functional $f \in W'$ and $a \in W$, both of norm one.

First we want to deduce some algebraic consequences of the bicircular property. The crucial fact we use (see [10]) is that every surjective linear isometry of a JB* triple is automatically an algebraic automorphism. Since

$$(P + e^{i\alpha}\overline{P})(P + e^{-i\alpha}\overline{P}) = P + \overline{P} = 1,$$

isometries $P + e^{i\alpha}\overline{P}$ are surjective. This implies that for every $x, y, z \in W$ we have

$$(P + e^{i\alpha}\overline{P})(\{xyz\}) = \left\{(P + e^{i\alpha}\overline{P})(x)(P + e^{i\alpha}\overline{P})(y)(P + e^{i\alpha}\overline{P})(z)\right\}.$$

If we expand this equality and compare the corresponding $\alpha-$coefficients on both sides, we obtain four identities

(I1) $\left\{P(x)\overline{P}(y)P(z)\right\} = 0$

(I2) $\left\{\overline{P}(x)P(y)\overline{P}(z)\right\} = 0$

(I3) $P\{xyz\} = \{P(x)P(y)P(z)\} + \left\{P(x)\overline{P}(y)\overline{P}(z)\right\} + \left\{\overline{P}(x)\overline{P}(y)P(z)\right\}$

(I4) $\overline{P}\{xyz\} = \left\{\overline{P}(x)\overline{P}(y)\overline{P}(z)\right\} + \left\{\overline{P}(x)P(y)P(z)\right\} + \left\{P(x)P(y)\overline{P}(z)\right\}$

which will be used in the sequel. We divide the rest of the proof into several steps, to make its logic more clear.

Step 1. *$P(W) = \mathbb{C}a$ where a is an atom (also called minimal tripotent) of W. This means that $\{aaa\} = a$ and $\{aWa\} = \mathbb{C}a$.*

As we noted already, there exist $a \in W$ and $f \in W'$ such that $||f|| = ||a|| = 1$ and $Px = f(x)a$. Since $P^2 = P$, we have $f(a) = 1$. Since $\overline{P}(a) = 0$, it follows from (I3) that

$$f(\{aaa\}\,a = P\{aaa\} = \{P(a)P(a)P(a)\} + 0 + 0 = \{aaa\}.$$

By the C* -equality (JBiv) in the definition of a JB* triple, we have

$$1 = ||a||^3 = ||\{aaa\}|| = |f(\{aaa\})| \cdot ||a|| = |f(\{aaa\})|.$$

Consequenty $\{aaa\} = e^{i\varphi}a$ for some real φ. From (JBiv) it follows

$$e^{i\varphi} \in \sigma(a\Box a) \subset \mathbb{R}^+$$

and so $\{aaa\} = a$.

It remains to be proved that $\{aWa\} = \mathbb{C}a$. In fact a stronger fact is true. Let $x \in W$ be arbitrary. Decompose

$$x = P(x) + \overline{P}(x) = f(x)a + \overline{P}(x).$$

By (I1) we have

$$\left\{a\overline{P}(x)a\right\} = \left\{P(a)\overline{P}(x)P(a)\right\} = 0,$$

and so

$$\{axa\} = \overline{f(x)}\,\{aaa\} + \left\{a\overline{P}(x)a\right\} = \overline{f(x)}a.$$

Step 2. *To every atom $a \in W$ there corresponds a so called Peirce decomposition (cf. for example [11])*

$$W = W_1 \oplus W_{\frac{1}{2}} \oplus W_0 = \mathbb{C}a \oplus W_{\frac{1}{2}} \oplus W_0,$$

where $W_1 = \{x \in W : \{aax\} = x\} = \mathbb{C}a$, $W_0 = \{x \in W : \{aax\} = 0\}$ and $W_{\frac{1}{2}} = \left\{x \in W : \{aax\} = \frac{1}{2}x\right\}$. Now we claim that

$$\mathrm{Ker}(f) = \mathrm{Im}(\overline{P}) = W_0 \oplus W_{\frac{1}{2}}.$$

The arithmetic of the Peirce subspaces is well know (see for example [11]) and can be expressed in the following manner

$$\{W_1 W_0 W\} = \{W_0 W_1 W\} = 0,$$

$$\{W_i W_j W_k\} \subset W_{i-j+k} \text{ if } i - j + k \in \left\{0, \frac{1}{2}, 1\right\},$$

$$\{W_i W_j W_k\} = 0 \text{ if } i - j + k \notin \left\{0, \frac{1}{2}, 1\right\}.$$

This implies, in our case,

$$\{aW_0a\} = \left\{aW_{\frac{1}{2}}a\right\} = 0.$$

Since we already know that $\{axa\} = \overline{f(x)}a$, it follows $W_0 \oplus W_{\frac{1}{2}} \subset \mathrm{Ker}(f)$. Since both spaces have codimension one, they are in fact equal.

Step 3. *We have* $\{aWW_0\} = 0$.

Before we proceed with our argument, we note that in general JB* triples such a relation is far from being true. In our case it is however a consequence of (I1) - (I4).

By the general rules of Peirce arithmetics we have

$$\{aW_0W_0\} = \{aW_1W_0\} = 0,$$

so it remains to be proved that $\left\{aW_{\frac{1}{2}}W_0\right\}$ is also trivial. Take $y \in W_{\frac{1}{2}}$ and $z \in W_0$. We know already that $y, z \in \mathrm{Im}(\overline{P})$ so that $Py = Pz = 0$. Then $\{ayz\} = \left\{P(a)\overline{P}(y)\overline{P}(z)\right\}$ and using (I3) we have

$$\begin{aligned}\{ayz\} &= \left\{P(a)\overline{P}(y)\overline{P}(z)\right\} = \\ &= P\{ayz\} - \{P(a)P(y)P(z)\} - \left\{\overline{P}(a)\overline{P}(y)P(z)\right\} \\ &= P\{ayz\} - 0 - 0.\end{aligned}$$

By the general rules of Peirce arithmetics, in every JB* triple

$$\{ayz\} \in \left\{W_1W_{\frac{1}{2}}W_0\right\} \subset W_{1-\frac{1}{2}+0} = W_{\frac{1}{2}},$$

so Step 2 implies $P\{ayz\} = 0$.

Step 4. *The subspaces* $J = W_0$ *and* $H = \mathbb{C}a \oplus W_{\frac{1}{2}}$ *are ideals of* W *and so* $W = J \oplus H$, *where* $P(J) = 0$ *and* $a \in H$.

At this point we recall [2, Proposition 2.1], which states that given any sets $A, B \subset W$ with $\{AWB\} = 0$, there exist two norm closed ideals J, H of W, such that

$$\begin{aligned} J &= \{x \in W : \{AWx\} = 0\} \supset B, \\ H &= \{x \in W : \{xWJ\} = 0\} \supset A \text{ and} \\ \{JWH\} &= J \cap H = 0.\end{aligned}$$

We use this for $A = \{a\}$ and $B = W_0$.

From the above we know that $W_0 \subset J$, so we must prove the converse. Take any $j \in J$ and decompose it in the Peirce sum $j = \alpha a + x_{\frac{1}{2}} + x_0$. Since $\{aWj\} = 0$, we have in particular $\{aaj\} = 0$. By the definition of Peirce subspaces we have

$$0 = \{aaj\} = \alpha a + \frac{1}{2}x_{\frac{1}{2}} + 0.$$

Since both factors on the right hand side belong to different components W_1 and W_0 of a direct sum, they must both be zero, so $j = x_0 \in W_0$. Thus $W_0 = J$ is a closed ideal of W.

For the second ideal H we know, in particular, that $a \in H$. Given $x \in W_{\frac{1}{2}}$, we have $\{aax\} \in H$. Since $\{aax\} = \frac{1}{2}x$, it follows $W_{\frac{1}{2}} \subset H$, and so $\mathbb{C}a \oplus W_{\frac{1}{2}} \subset H$. As

$$\begin{gathered} W = \mathbb{C}a \oplus W_{\frac{1}{2}} \oplus J, \\ \mathbb{C}a \oplus W_{\frac{1}{2}} \subset H \text{ and } H \cap J = 0,\end{gathered}$$

it follows $H = \mathbb{C}a \oplus W_{\frac{1}{2}}$. Now it only remains to prove that H is isomorphic to a Hilbert space. In the special case of the so called JBW* triples this is a well know classical result. We also feel that it should be known for the general case, but were not able to find such a statement in the literature. Since the argument is short, we include it for the sake of completeness.

Step 5. *The ideal H is isomorphic to a Hilbert space, with the triple product being the same as in the previous section.*

It is sufficient to construct an inner product on $W_{\frac{1}{2}}$. Take any x, y in this subspace. Since

$$\{ayx\} \in \left\{W_1 W_{\frac{1}{2}} W_{\frac{1}{2}}\right\} \subset W_{1-\frac{1}{2}+\frac{1}{2}} = \mathbb{C}a,$$

there exists a complex number $\langle x, y\rangle$ such that

$$\{ayx\} = \frac{1}{2}\langle x, y\rangle\, a.$$

From (JBi) it follows that this mapping is linear in x and conjugate linear in y. Since

$$\langle x, x\rangle \in \sigma(x\Box x),$$

the axiom (JBiii) implies $\langle x, x\rangle \geq 0$.

If $\langle x, x\rangle = 0$, we have $\{axx\} = 0$. Since on the one hand

$$\{xax\} \in \left\{W_{\frac{1}{2}} W_1 W_{\frac{1}{2}}\right\} \subset W_{\frac{1}{2}-1+\frac{1}{2}} = W_0 = J$$

and on the other, as H is an ideal of W, we have $\{xax\} \in H$, it follows $\{xax\} = 0$. Now we can use (JBii) to obtain

$$-\frac{1}{2}\{xxx\} = -\left\{x(\frac{1}{2}x)x\right\} = -\{x\{xaa\}x\} = \{ax\{xax\}\} - 2\{\{axx\}ax\} = 0.$$

Using (JBiv), we get $x = 0$, so that the mapping $x, y \longmapsto \langle x, y\rangle$ is in fact an inner product on H. Now it remains to verify the relation between this inner product and the triple product restricted to $W_{\frac{1}{2}}$.

Take any $x, y, z, w \in W_{\frac{1}{2}}$. Since we know the formula for the triple product of (pre)Hilbert spaces, we are led to consider the expression

$$\begin{aligned}
e &= \langle 2\{xyz\} - \langle x, y\rangle z - \langle z, y\rangle x, w\rangle\, a \\
&= 2\langle\{xyz\}, w\rangle a - \langle x, y\rangle\langle z, w\rangle a - \langle z, y\rangle\langle x, w\rangle a \\
&= \{\{xyz\}wa\} - \frac{1}{2}\langle x, y\rangle\{zwa\} - \frac{1}{2}\langle z, y\rangle\{xwa\} \\
&= \{\{xyz\}wa\} - \left\{zw\left(\frac{1}{2}\langle x, y\rangle a\right)\right\} - \left\{xw\left(\frac{1}{2}\langle z, y\rangle a\right)\right\} \\
&= \{\{xyz\}wa\} - \{zw\{xya\}\} - \{xw\{zya\}\}.
\end{aligned}$$

Using (JBii) we have

$$e = -\{x\{waz\}y\}$$

and since

$$\{waz\} \in H \cap W_0 = 0$$

it follows $e = 0$. As w was arbitrary, we obtain

$$\{xyz\} = \frac{1}{2}\langle x, y\rangle z + \frac{1}{2}\langle z, y\rangle x : x, y, z \in W_{\frac{1}{2}}.$$

From this it follows also that

$$\{xxx\} = \langle x, x\rangle\, x$$

and using (JBiv) we obtain

$$||x||^2 = \langle x, x\rangle : x \in W_{\frac{1}{2}}$$

which means that the norm of W, when restricted to $W_{\frac{1}{2}}$, is a preHilbert norm. Since $W_{\frac{1}{2}}$ is closed in W, the completeness of $(W_{\frac{1}{2}}, \langle ., .\rangle)$ also follows.

Now it is not difficult to verify that the inner product on H, given by

$$\langle \alpha a + x, \beta a + y\rangle = \alpha\overline{\beta} + \langle x, y\rangle$$

satisfies the same equalities, so in fact all of H is a Hilbert triple.

With this the proof of the main theorem is concluded. The corollary is now immediate. Since the ideal H is always nonzero, its complement J must be zero if W is assumed to be prime. We conclude the paper with the following

CONJECTURE 1. *Let W be a prime JB* triple and $P : W \to W$ a nonzero bicircular projection whose range is finite dimensional. Then either W itself is finite dimensional or W is isomorphic to a Hilbert triple.*

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