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&0. Introduction.

Let X be a complex surface, i.e., a connected two-dimensional complex manifold. A domain $\Omega \subset X$ is called *Stein* if it is biholomorphic to a closed complex submanifold of a Euclidean space $\mathbb{C}^N$ (we can take $N = 4$ according to [EG] and [Sch]). By Lefschetz's theorem Ω is homotopically equivalent to a CW-complex of real dimension at most two, and it is natural to ask which such complexes can be realized by Stein domains in a given complex surface. Our main interest is to find 'large' Stein domains with interesting global properties.

In this paper we obtain several results in this direction by considering the existence of regular Stein neighborhoods of embedded and immersed compact real surfaces S in complex surfaces X. To each immersion $\pi: S \to X$ one associates the index $I(\pi) \in \mathbf{Z}$, as well as the positive and the negative index $I_\pm(\pi)$ if S is oriented, which count the number of complex points with algebraic multiplicities. If $I_\pm(\pi) \leq 0$ (resp. $I(\pi) \leq 0$ when S is unorientable) then π can be $\mathcal{C}^0$-approximated by a regularly homotopic immersion whose image has a basis of Stein neighborhoods homotopically equivalent to the immersed surface (Theorem 1.1). These neighborhoods are sublevel sets of a smooth nonnegative weakly plurisubharmonic function in a neighborhood of the surface which vanishes quadratically on the surface and has no other critical points nearby (Theorem 2.2). We give several explicit results for surfaces in the projective plane $\mathbb{CP}^2$ (Proposition 1.4 and Theorems 1.6 and 4.3).

For embedded oriented surfaces $S \subset X$ the index conditions $I_\pm(S) \leq 0$ are equivalent to a lower estimate on the genus $g(S)$ which follows in many cases from the theory of Seiberg-Witten invariants (Nemirovski [N]; see the Appendix). This holds in particular if X is Stein and $S \subset X$ is not an embedded null-homologous sphere. We also obtain results for immersed surfaces. The existence of a Stein neighborhood of an immersed oriented surface $\pi(S) \subset X$ with nontrivial homology class implies that the sum $g(S) + \delta_+$ (where δ_+ is the number of positive double points of π) is bounded below by a quantity which only depends on the homology class of $\pi(S)$ in X. For $X = \mathbb{CP}^2$ we find immersions which realize this lower bound (Theorem 4.3).

Our examples, together with a symplectic approximation theorem of Gromov, imply the existence of an immersed symplectic sphere $\pi: S \to \mathbb{CP}^2$ with a Stein neighborhood (Corollary 4.4). A theorem of Ivashkovich and Shevchishin [IS] shows that such π must have some negative double points since otherwise the envelope of meromorphy of any neighborhood of $\pi(S)$ would contain a compact complex rational curve and hence $\pi(S)$ could not have a Stein neighborhood. The question about the minimal number of double points of immersed symplectic spheres with a Stein neighborhood in $\mathbb{CP}^2$ remains open.

Very little seems to be known about these questions in dimensions ≥ 3. Constructions of n-dimensional Stein manifolds homotopic to a given CW-complex of dimension $\leq n$ were given by Eliashberg [E1] for $n > 2$ and Gompf [Go] for $n = 2$. However, it is difficult to decide whether a given Stein manifold can be realized as a domain in another complex manifold. The *Oka principle* fails in this problem, for Stout and Zame [SZ] showed that the complex seven-sphere $\{\sum z_j^2 = 1\} \subset \mathbb{C}^8$ is real-analytically equivalent to a domain in $\mathbb{C}^7$ but is not holomorphically equivalent to a domain in $\mathbb{C}^7$.

&1. Regular Stein neighborhoods of embedded surfaces.

Let S be a connected compact real surface without boundary embedded in a complex surface X. A point $p \in S$ is said to be *complex* if the tangent plane T_pS is a complex line in T_pX. The other points of S are called *totally real*. In local holomorphic coordinates (z, w) on X near a complex point $p \in S$ we have $p = 0$ and S is the graph $w = f(z)$ of a smooth function over a domain in $\mathbb{C}$. It is easily seen that $(z, f(z)) \in S$ is a complex point of S if and only if $\frac{\partial f}{\partial \bar z}(z) = \frac{1}{2}(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y})(z) = 0$. If $p = 0$ is an isolated complex point then for sufficiently small $\delta > 0$ we have $\frac{\partial f}{\partial \bar z}(z) \neq 0$ for $0 < |z| \leq \delta$, and we define the *index* $I(p; S)$ as the winding number of $\frac{\partial f}{\partial \bar z}(z)$ on the circle $|z| = \delta$. The index is independent of the choice of local coordinates and is related to the *Maslov index* [F1, F2]. A totally real point has index zero.

A generic compact real surface S in a complex surface X has finitely many complex points and one defines its total index $I(S) = \sum_{p\in S} I(p; S)$. These notions extend to immersions $\pi\colon S \to X$ and we shall write $I(\pi) = \sum_{p\in S} I(p; \pi)$. If S is oriented, one defines $I_+(S)$ (resp. $I_-(S)$) as the sum of indices over all positive (resp. negative) complex points of S. (A complex point of $S \subset X$ is *positive* (resp. negative) if the orientation on the complex line T_pS induced by the complex structure agrees (resp. disagrees) with the chosen orientation of S.) The indices $I_\pm$ and $I = I_+ + I_-$ depend only on the regular homotopy class of the immersion $S \to X$ (see (1.1) below) and hence they can be defined for any immersion by considering a generic perturbation.

Definition. *A family of open sets $\{\Omega_\epsilon\colon \epsilon \in (0, \epsilon_0)\}$ in X is said to be a regular basis of neighborhoods of $S \subset X$ if for each $\epsilon \in (0, \epsilon_0)$ we have $\Omega_\epsilon = \cup_{t<\epsilon}\Omega_t$, $\overline{\Omega}_\epsilon = \cap_{t>\epsilon}\Omega_t$, and $S = \bigcap_{0<t<\epsilon_0} \Omega_t$ is a strong deformation retraction of Ω_ϵ (and hence each Ω_ϵ is homotopically equivalent to S).*

1.1 Theorem. *Let S be an embedded compact real surface in a complex surface X. Assume either that S is orientable and $I_\pm(S) \leq 0$ or that S is unorientable and $I(S) \leq 0$. Then S can be $\mathcal{C}^0$-approximated by an isotopic embedding $S' \subset X$ with a regular basis of smoothly bounded Stein neighborhoods. The analogous result holds for immersions.*

Theorem 1.1 is proved in section 2. The special case $X = \mathbb{C}^2$ was considered in [F2] where it was proved that *every real surface except the two-sphere admits an embedding in $\mathbb{C}^2$ with a regular Stein neighborhood basis.*

Nemirovski [N] observed that the inequalities $I_\pm(S) \le 0$ in many cases follow from the theory of *Seiberg-Witten invariants* (see the Appendix). His chief contribution was to transfer the results which were obtained for compact Kähler surfaces to the corresponding results for Stein surfaces via the algebraic approximation theorem of Stout [St]. In particular, *an oriented embedded real surface S in a Stein surface X satisfies $I_\pm(S) \le 0$ unless S is a null-homologous two-sphere, and hence the conditions $I_\pm(S) \le 0$ in Theorem 1.1 are necessary* [N, Theorem 9] (see part (c) of Theorem I in the Appendix).

We recall the *index formulas* which expres the indices by other topological invariants of the immersion. Isolated complex points of real surfaces in complex surfaces were first investigated by Chern and Spanier [CS] and Bishop [B] who classified them into *elliptic* (these have index +1), *hyperbolic* (with index −1), and *parabolic* (these are degenerate and may have any index). If $S \subset X$ has e elliptic points and h hyperbolic points then $I(S) = e - h$ (this happens for a generic S). Bishop [B] proved that for every oriented embedded surface $S \subset \mathbb{C}^2$ we have $I(S) = \chi(S) = 2 - 2g(S)$ and $I_\pm(S) = \frac{1}{2} I(S) = 1 - g(S)$. The most general index formula is the following ([La], [We], [E1]):

$$I(\pi) = \chi(S) + \chi_n(\pi), \quad I_\pm(\pi) = \frac{1}{2}\big(I(\pi) \pm c_1(X)\cdot \pi(S)\big). \tag{1.1}$$

The first of these formulas also holds if S is unorientable. Here $\chi_n(\pi)$ denotes the Euler number of the normal bundle $\nu_\pi = \pi^*(TX)/TS$ of the immersion π (the self-intersection number of the zero section in ν_π), and $c_1(X)\cdot \pi(S)$ is the value of the first Chern class $c_1(X) = c_1(TX) \in H^2(X;\mathbf{Z})$ on the homology class $[\pi(S)] \in H_2(X;\mathbf{Z})$. The normal Euler number $\chi_n(\pi)$ is defined also for unorientable surfaces since any local orientation of S coorients ν_π by the condition that the two orientations add up to the orientation of X by the complex structure, and $\chi_n(\pi)$ is independent of these choices.

The normal bundle of an embedding $\pi: S \hookrightarrow X$ is diffeomorphic to a neighborhood of S in X and hence $\chi_n(\pi) = S^2$ is the *self-intersection number* of the homology class $[S] \in H_2(X;\mathbf{Z})$. Hence the indices of a compact oriented embedded surface $S \subset X$ satisfy

$$I_\pm(S) = 1 - g(S) + \frac{1}{2}\left(S^2 \pm c_1(X)\cdot S\right), \tag{1.2}$$

and the conditions $I_\pm(S) \le 0$ are equivalent to the *genus inequality*

$$g(S) \ge 1 + \frac{1}{2}\left(S^2 + |c_1(X)\cdot S|\right). \tag{1.3}$$

The number on the right hand side depends only on the homology class of S in X. Theorem 1.1, together with [N, Theorem 9], implies

1.2 Corollary. *A compact oriented embedded real surface S in a complex surface X which is not a null-homologous two-sphere is isotopic to an embedded surface with a regular Stein neighborhood basis if and only if (1.3) holds.*

A list of known cases when the genus inequality (1.3) holds can be found in the Appendix. For immersions see Corollary 4.1 below.

An important special case are oriented surfaces $S \subset X$ without negative complex points. These include complex curves and, more generally, surfaces which are *symplectic* with respect to a positive symplectic form on X. (If J denotes the almost complex structure on X then a symplectic form ω on X is *J-positive*, or J is *tamed* by ω, if $\omega(v, Jv) > 0$ for any $0 \neq v \in TX$. An immersion $\pi\colon S \to X$ is ω-symplectic if $\pi^*\omega > 0$.) In this case (1.2) gives

$$0 = 2I_-(S) = \chi(S) + S^2 - c_1(X)\cdot S, \quad I_+(S) = c_1(X)\cdot S = \chi(S) + S^2. \quad (1.4)$$

The identity $I_-(S) = 0$ is equivalent to the classical *genus formula* $g(S) = 1 + \frac{1}{2}(S^2 - c_1(X)\cdot S)$. For complex (or symplectic) curves this also follows from $\Lambda^2 TX|_S = TS \otimes \nu_S$ which gives $c_1(TX)\cdot S = c_1(TS)\cdot S + c_1(\nu_S)\cdot S = \chi(S) + S^2$.

1.3 Corollary. *A compact oriented embedded real surface S without negative complex points in a complex surface X is isotopic to an embedded surface with a regular basis of Stein neighborhoods if and only if $c_1(X)\cdot S \le 0$. This applies to embedded complex and symplectic curves.*

Example 1. If S is a compact Riemann surface and $p\colon E \to S$ is a holomorphic line bundle over S then the zero section $S \subset E$ satisfies $I_+(S) = \chi(S) + S^2 = 2 - 2g(S) + c_1(E)$. Thus S is isotopic to a surface with a regular Stein neighborhood basis in E if and only if $c_1(E) \le 2g(S) - 2$. If S is the Riemann sphere, this holds for bundles of degree ≤ -2. (If $E = [D]$ for a divisor D on S then $c_1(E) = deg(D)$.)

Example 2. Let S be a smooth compact oriented surface of degree $d \ge 1$ in $\mathbb{CP}^2$, i.e., $[S] = d[H] \in H_2(\mathbb{CP}^2; \mathbf{Z}) = \mathbf{Z}$, where $H \simeq \mathbb{CP}^1$ is the projective line. The Chern class $c_1(\mathbb{CP}^2)$ is represented by $3\omega_{FS}$ where ω_{FS} is the Fubini-Study form [GH, p. 409]. Thus $c_1(\mathbb{CP}^2)\cdot [H] = \int_H 3\omega_{FS} = 3$ and $c_1(\mathbb{CP}^2)\cdot [S] = c_1(\mathbb{CP}^2)\cdot d[H] = 3d$. By Corollary 1.2 S *is isotopic to an embedded surface in* $\mathbb{CP}^2$ *with a regular Stein neighborhood basis if and only if*

$$g(S) \ge 1 + \frac{1}{2}\left(S^2 + |c_1(\mathbb{CP}^2)\cdot S|\right) = 1 + \frac{1}{2}(d^2 + 3d) = \frac{1}{2}(d+1)(d+2). \quad (1.5)$$

In particular we must have $g(S) \ge 3$. If S is isotopic to a complex or symplectic curve in $\mathbb{CP}^2$ then $g(S) = \frac{1}{2}(d-1)(d-2)$ and hence S does not admit any Stein neighborhood. (In fact, there are no nonconstant holomorphic functions in any neighborhood of S according to [N, Theorem 10].) We show that (1.5) is sharp.

1.4 Proposition. *For any $d \geq 1$ and $g \geq \frac{1}{2}(d+1)(d+2)$ there is an embedded oriented surface $S \subset \mathbb{CP}^2$ of genus g and degree d with a regular Stein neighborhood basis.*

For immersions in $\mathbb{CP}^2$ see Theorem 4.3 below. Proposition 1.4 is proved in section 3 and is a special case of the following.

1.5 Theorem. *Let $S \subset X$ be a compact real surface embedded in a complex surface X.*

(a) If S is oriented and $k = \max\{I_+(S), I_-(S), 0\}$ then S is homologous in X to an embedded surface $S' \subset X$ of genus $g(S') = g(S) + k$ with a regular Stein neighborhood basis.

(b) If S is unorientable and k is the smallest integer with $3k \geq \max\{I(S), 0\}$ then S is $\mathbf{Z}_2$-homologous in X to an embedded unorientable surface $S' \subset X$ of genus $g(S') = g(S) + k$ with a regular Stein neighborhood basis.

The next result shows that there are no genus restrictions for the existence of Stein neighborhoods of embedded unorientable surfaces in $\mathbb{CP}^2$.

1.6 Theorem. *Every compact unorientable real surface embeds in $\mathbb{CP}^2$ with a regular basis of Stein neighborhoods intersecting every projective line.*

Remark. By Theorem 1.8 in [F2] every compact unorientable surface S also embeds in $\mathbb{C}^2$ with a regular Stein neighborhood basis. If $\chi = 2 - g(S)$ denotes the Euler number of S, the set of indices $I(\pi)$ of embeddings $\pi \colon S \hookrightarrow \mathbb{C}^2$ equals $\{3\chi - 4, 3\chi, 3\chi + 4, \ldots, 4 - \chi\}$ [F2, p.358]. This set contains both positive and negative numbers for any value of χ. ♠

Further results on *Stein neighborhoods of immersed surfaces* are given in section 4. We mention a couple of related open problems.

Problem 1. Does there exist a Stein domain $\Omega \subset \mathbb{C}^2$ which is homotopically equivalent to the two-sphere ? This question, which was raised in [F2], is apparently still open. According to Nemirovski [N, Theorem 15] any embedded two-sphere $S \subset \Omega$ is null-homologous in Ω. There exists a precise description of the envelope of holomorphy of smooth two-spheres contained in closed strongly pseudoconvex hypersurfaces in $\mathbb{C}^2$ [BK, Kr].

Problem 2. Let $S \subset X$ be an embedded real surface with isolated complex points which admits a regular basis of Stein neighborhoods. Is $I(p; S) \leq 0$ for every $p \in S$? Clearly S cannot have elliptic complex points (of index $+1$) due to the family of Bishop discs [B, KW]. Further results on the existence of families of small analytic discs at certain complex points of higher index were obtained by Wiegerinck [Wi]. The problem seems open in general.

&2. Construction of regular Stein neighborhoods.

In this section S denotes a compact connected real surface without boundary. We first summarize the properties of special immersions to which Theorem 1.1 applies.

2.1 Theorem. *Any immersion $\pi_0: S \to X$ of a closed real surface S into a complex surface X can be $\mathcal{C}^0$-approximated by a regularly homotopic smooth immersion $\pi: S \to X$ satisfying the following properties:*

(a) *At every complex point $p \in S$ of π there are open neighborhoods $p \in U \subset S$, $q = \pi(p) \in V \subset X$ and local holomorphic coordinates (z, w) on V such that $z(q) = w(q) = 0$ and $\pi(U) \subset V$ is given either by $w = z\overline{z}$ (a* **special elliptic point***) or by $w = \overline{z}^2$ (a* **special hyperbolic point***). If $I_\pm(\pi_0) \le 0$ (resp. $I(\pi_0) \le 0$ if S is unorientable) then we can choose π as above to be without elliptic points.*

(b) *The immersion π only has transverse double points (and no multiple points), and in a neighborhood of each double point there exist local holomorphic coordinates $z = x + iy$, $w = u + iv$ on X such that $\pi(S)$ is given locally by $\{y = 0,\ v = 0\} \cup \{x = 0,\ u = 0\} = (\mathbb{R}^2 \times i\{0\}^2) \cup (\{0\}^2 \times i\mathbb{R}^2)$.*

(c) *Any embedding $S \hookrightarrow X$ can be $\mathcal{C}^0$-approximated by an isotopic embedding satisfying (a).*

Theorem 2.1 is proved by cancellation of pairs of complex points due to Eliashberg and Harlamov (see the reference in [E2]). It seems that their paper is not available in a standard source and we refer instead to Theorem 1.1 in [F2] where all details can be found. (We apologize to the readers of [F2] for the misleading reference [24].) The result falls within the scope of Gromov's h-principle (sec. 2.4.5 in [Gro]). We recall the main steps.

First we modify the immersion near each double point so that the new immersion satisfies property (b) (this is completely elementary). Suppose now that $p, q \in S$ are distinct isolated complex points of π with $I(p;\pi) + I(q;\pi) = 0$. If S is oriented we assume that either both points are positive or both are negative. Choose a simple smooth arc $\gamma \subset S$ with endpoints p and q which does not contain any other complex point or double point of π. There exist holomorphic coordinates in an open set $U \supset \pi(\gamma)$ in X which embed U onto a domain in $\mathbb{C}^2$ such that $\pi(S) \cap U$ is mapped onto a graph $w = f(z)$ over a disc $D \subset \mathbb{C}$ (see section 5 in [F2] for the details). The winding number of $\frac{\partial}{\partial \overline{z}} f$ around bD equals $I(p;\pi) + I(q;\pi)$ which is assumed to be zero. Such f can be uniformly approximated by a smooth function g on D which equals f near bD and satisfies $\frac{\partial}{\partial \overline{z}} g \ne 0$ on D (Lemma 4.1 in [F2] or Lemma 1 in [N, p.735]). This gives a $\mathcal{C}^0$-approximation of π by a regularly homotopic immersion which equals π outside a small neighborhood $V \supset \gamma$ in S and is totally real on V.

Using this procedure repeatedly one obtains an immersion $\pi': S \to X$ such that each orientation class contains only elliptic or only hyperbolic points of π', depending on the sign of $I_\pm(\pi)$. If $I_\pm(\pi) \le 0$ then all complex points of π' are hyperbolic. It is completely elementary to modify each elliptic resp. hyperbolic point to a complex point of special type as in Theorem 2.1 (a). If π is an embedding, these deformations are carried out by an isotopy of embeddings. ♠

2.2 Theorem. *If $\pi: S \to X$ is an immersion satisfying Theorem 2.1 and containing only hyperbolic complex points then there are an open set $\Omega \subset X$ containing $M = \pi(S)$ and a smooth plurisubharmonic function $\rho: \Omega \to [0,1]$ satisfying $M = \{x \in \Omega: \rho(x) = 0\}$, $d\rho \neq 0$ on $\Omega \backslash M$, and such that for every $\epsilon \in (0,1)$ the set $\Omega_\epsilon = \{x \in \Omega: \rho(x) < \epsilon\}$ is a smoothly bounded Stein domain which admits a strong deformation retraction onto M.*

Remarks. 1. An embedded real surface in a complex surface is locally holomorphically convex in a neighborhood of any totally real point and hyperbolic complex point [FSt]. It is easy to show (by patching local nonnegative plurisubharmonic functions which vanish on the surface) that a surface $S \subset X$ with only these types of points has a basis of Stein neighborhoods (see [F2] and [N, Theorem 4]). However, *it is not clear whether there exist Stein neighborhoods homotopically equivalent to S* since the plurisubharmonic function constructed by the patching argument may have critical points off S which accumulate at a complex point of S. Here we find *regular Stein neighborhoods of immersed surfaces with special hyperbolic complex points and special double points.*

2. Theorems 2.1 and 2.2 also hold for immersions of surfaces with finitely many connected components $S = S_1 \cup S_2 \cup \ldots \cup S_k$. The index formulas of section 1 can be extended to this more general situation. Of course we can only remove pairs of complex points within the same connected component, and hence we must assume that the index of the immersion is nonpositive on each S_j in order to obtain an immersion to which Theorem 2.2 applies. ♠

Proof. We first define ρ near the complex and the self-intersection points. Let $p_1, \ldots, p_m \in M = \pi(S) \subset X$ be the complex points and $q_1, \ldots, q_k \in M$ the double points of π. By hypothesis there exist a neighborhood $V_j \subset X$ of p_j and holomorphic coordinates $\phi_j(p) = (z(p), w(p))$ on V_j such that $z(p_j) = w(p_j) = 0$, $\phi_j(V_j) = r_j \triangle \times \triangle \subset \mathbb{C}^2$ for some $r_j \in (0,1)$ (where $\triangle = \{\zeta \in \mathbb{C}: |\zeta| < 1\}$), and $M \cap V_j = \{w = \overline{z}^2\}$. The nonnegative function

$$\rho(z,w) = |w - \overline{z}^2|^2 = |w|^2 + |z|^4 - 2\Re(wz^2)$$

has all required properties in V_j. Indeed it is plurisubharmonic (since $\Re(wz^2)$ is pluriharmonic), strongly plurisubharmonic outside the complex disc $\Lambda_j = \{p \in V_j: z(p) = 0\}$, and $d\rho$ vanishes precisely on $M \cap V_j = \{w = \overline{z}^2\}$.

Each double point $q_j \in M$ of π has a neighborhood $W_j \subset X$ and holomorphic coordinates $\psi_j = (z,w) = (x+iy, u+iv)$ on W_j such that in these coordinates $q_j = 0$ and $M \cap W_j = \{y = 0,\ v = 0\} \cup \{x = 0,\ u = 0\}$. Set

$$\rho(x+iy, u+iv) = (x^2+u^2)(y^2+v^2).$$

Its differential

$$d\rho = 2\big(x(y^2+v^2), y(x^2+u^2), u(y^2+v^2), v(x^2+u^2)\big)$$

vanishes precisely on $\{\rho = 0\} = M \cap W_j$. We have

$$\rho_{z\overline{z}} = \rho_{w\overline{w}} = \frac{1}{2}(x^2+y^2+u^2+v^2), \quad \rho_{z\overline{w}} = i(xv-yu) = \overline{\rho_{w\overline{z}}}\,.$$

Since $\rho_{z\overline{z}} > 0$ except at the origin, its complex Hessian $H_\rho(z,w)$ has at least one positive eigenvalue there. By Cauchy-Schwarz we have

$$\begin{aligned} 4\det H_\rho(z,w) &= (x^2+y^2+u^2+v^2)^2 - 4(xv-yu)^2 \\ &\geq (x^2+y^2+u^2+v^2)^2 - 4(x^2+y^2)(u^2+v^2) \\ &= |z|^4 + |w|^4 - 2|z|^2|w|^2 \\ &= \big(|z|^2 - |w|^2\big)^2. \end{aligned}$$

Thus $\det H_\rho(z,w) \geq 0$ which shows that both eigenvalues are nonnegative and hence ρ is plurisubharmonic. The equality $\det H_\rho(z,w) = 0$ holds precisely when $(x,y) = \lambda(v,-u)$ for some $\lambda \in \mathbb{R}$ and $|z|^2 = |w|^2$, and this gives $w = \pm iz$. Let $L_j = \{p \in W_j \colon w(p) = \pm iz(p)\}$. Thus ρ satisfies all required properties on W_j and is strongly plurisubharmonic on $W_j \backslash L_j$.

A function ρ with the required properties has been defined on neighborhoods $p_j \in V_j$ and $q_j \in W_j$. We now extend ρ to a smooth nonnegative function in a neighborhood $V \supset M$ such that the extension vanishes precisely on M and its real Hessian is nondegenerate in any normal direction to M at all points of $M_0 = M \backslash \{p_j, q_j\}$. More precisely, if we denote by $\nu \to M_0$ the normal bundle to M_0 in X and realize it as a subbundle of $TX|_{M_0}$ such that $TX|_{M_0} = TM_0 \oplus \nu$, we require that the second order derivatives of ρ in the fiber directions ν_x at any $x \in M_0$ give a nondegenerate quadratic form on ν_x (which is necessarily positive definite since ρ has a local minimum at $0_x \in \nu_x$). One can obtain such an extension by taking a suitable second order jet along M with the required properties and applying Whitney's theorem to find a function which matches this jet.

A more explicit construction is the following. Choose a Riemannian metric $h(x;\xi,\eta)$ on the normal bundle $\nu = \{(x,\xi) \colon x \in M_0, \xi \in \nu_x\}$. The function $\widetilde{\rho}(x,\xi) \to h(x;\xi,\xi) \in \mathbb{R}_+$ has suitable properties on ν (with M_0 corresponding to the zero section of ν). Let ψ be a diffeomorphism of ν onto a tubular neighborhood $U \subset X$ of M_0 such that $\psi(x,0) = x$ and $d\psi(x,0)$ is the identity for each $x \in M_0$. Then $\rho_0 = \psi \circ \widetilde{\rho} \circ \psi^{-1} \colon U \to \mathbb{R}_+$ satisfies the required properties near M_0. Within each coordinate neighborhood V_j and W_j as above we patch ρ_0 with the previously chosen function $\rho = \rho_j$ on this neighborhood by taking $\rho = \chi_j \rho_j + (1-\chi_j)\rho_0$ on V_j, where χ_j is a smooth cut-off function which is supported in V_j (resp. in W_j) and equals one in a smaller neighborhood of p_j resp. q_j. At points $x \in M_0 \cap V_j$ the real Hessian of ρ equals

$$H^{\mathbb{R}}_{\rho}(x) = \chi_j(x) H^{\mathbb{R}}_{\rho_j}(x) + (1 - \chi_j(x)) H^{\mathbb{R}}_{\rho_0}(x).$$

This is nonnegative on $T_x X$ and strongly positive on ν_x (since it is a convex combination of two forms with these properties).

We claim that the function ρ obtained in this way is plurisubharmonic in a neighborhood of M, it has no critical points near M (except on M), and the sublevel sets $\{\rho < \epsilon\}$ for sufficiently small $\epsilon > 0$ are Stein. We first show plurisubharmonicity. By construction ρ is such near each p_j and q_j. Let $p \in M_0 = M \backslash \{p_j, q_j\}$. Since M_0 is totally real, there exist local holomorphic coordinates $(z, w) = (x + iy, u + iv)$ near p such that in these coordinates $p = 0$ and $T_p M_0 = \mathbb{R}^2 \oplus i\{0\}^2 = \{y = 0,\ v = 0\}$. Since $\rho = 0$ and $d\rho = 0$ on M_0, those second order derivatives of ρ at 0 which contain at least one differentiation on x or u vanish at 0 and hence

$$4\rho_{z\overline{z}}(0) = \rho_{yy}(0), \quad 4\rho_{w\overline{w}}(0) = \rho_{vv}(0), \quad 4\rho_{z\overline{w}}(0) = \rho_{yv}(0).$$

Thus the complex Hessian of ρ at 0 equals one quarter of the real Hessian of the function $(y, v) \to \rho(0 + iy, 0 + iv)$ at $y = 0, v = 0$. By construction this is positive definite and hence ρ is strongly plurisubharmonic at every $p \in M_0$. Thus ρ is plurisubharmonic in a small neighborhood $\Omega \supset M$ and strongly plurisubharmonic in $\Omega' = \Omega \backslash \big((\cup \Lambda_j) \cup (\cup L_j)\big)$. We may choose $\Omega = \{\rho < \epsilon_0\}$ for some $\epsilon_0 > 0$.

Next we show that $d\rho \neq 0$ on $\Omega \backslash M$ if $\epsilon_0 > 0$ (and hence Ω) are chosen sufficiently small. By construction this holds in small neighborhoods of the points p_j and q_j. Over $M_0 = M \backslash \{p_j, q_j\}$ we consider the conjugate function $\widetilde{\rho} = \psi^{-1} \circ \rho \circ \psi$ on the normal bundle $\nu \to M_0$. Suppose that $d\widetilde{\rho}(x, \xi) \cdot \xi = 0$ for some $(x, \xi) \in \nu$ with $x \in M_0$ and $\xi \neq 0$. Consider the function $t \in \mathbb{R} \to \widetilde{\rho}(x, t\xi)$. By hypothesis its derivative vanishes at $t = 0$ and $t = 1$ and hence its second derivative vanishes at some $t_0 \in (0, 1)$. This means that the Hessian of $\widetilde{\rho}(x, \cdot)$ vanishes at the fiber point $t_0 \xi \in \nu_x$ in the direction of the vector $\xi \in \nu_x$. We have seen that this does not happen in a sufficiently small neighborhood of the zero section of ν which establishes our claim.

Thus for any $\epsilon \in (0, \epsilon_0)$ the set $\Omega_\epsilon = \{x \in \Omega : \rho(x) < \epsilon\}$ is a relatively compact smoothly bounded (weakly) pseudoconvex domain in X. We obtain a strong deformation retraction of Ω_ϵ onto M by integrating the flow of the negative gradient of ρ from $t = 0$ to $t = +\infty$ and rescaling the time interval to $[0, 1]$. (The gradient of ρ is defined by the equation $\nabla \rho \rfloor h = d\rho$ where h is a Riemannian metric on X and $\rfloor$ denotes the contraction.)

It remains to show that Ω_ϵ is Stein for each $\epsilon \in (0, \epsilon_0)$. Let $h_\epsilon : (-\infty, \epsilon) \to \mathbb{R}$ be an increasing strongly convex function with $\lim_{t \to \epsilon} h(t) = +\infty$. If there exists a strongly plurisubharmonic function ψ in a neighborhood of $\overline{\Omega}_\epsilon$ in X

(this is the case for instance if X is Stein) then $\psi + h_\epsilon \circ \rho$ is a strongly plurisubharmonic exhaustion function on Ω_ϵ and hence Ω_ϵ is Stein according to [Gra].

In general a weakly pseudoconvex domain in a non-Stein manifold need not be Stein. In our situation we proceed as follows. Recall that the Levi form of ρ and hence of $\rho_\epsilon = h_\epsilon \circ \rho$ is degenerate only on the complex curves $\Lambda_j \subset \Omega$ and $L_j \subset \Omega$. These curves intersect M transversely at the points p_j resp. q_j. We take $\phi = \rho_\epsilon + \delta\tau$ where $\delta > 0$ and τ is a smooth function in a neighborhood of $\overline{\Omega}_\epsilon$ which is strongly plurisubharmonic on the curves Λ_j and L_j and which vanishes outside the coordinate neighborhood V_j of p_j (resp. q_j). Since the complex Hessian of ρ_ϵ is bounded away from zero on the set in Ω_ϵ where τ fails to be plurisubharmonic, ϕ is a strongly plurisubharmonic exhaustion function on Ω_ϵ provided that $\delta > 0$ is chosen sufficiently small.

We take τ to be given in local coordinates (z, w) near p_j resp. q_j by $\tau(z, w) = \chi(z, w)(|z|^2 + |w|^2)$ where χ is a suitable cut-off function. At p_j we have $\Lambda_j = \{z = 0\}$, the coordinate neighborhood $V_j \subset X$ of p_j is mapped onto $\{|z| < r,\ |w| < 1\} \subset \mathbb{C}^2$, and a suitable cut-off function is $\chi(|z|)$ where $\chi(t) = 1$ for $t \le r/2$ and $\chi(t) = 0$ for $t \ge 3r/4$. At q_j we may assume that the coordinate neighborhood $W_j \subset X$ is mapped onto $\{|z| < 1,\ |w| < 1\} \subset \mathbb{C}^2$ and in this case a suitable cut-off function is $\chi(|z|^2 + |w|^2)$ where $\chi(t) = 1$ for $t \le 1/4$ and $\chi(t) = 0$ for $t \ge 3/4$. In both cases the support of the differential $d\chi$ intersects Ω_ϵ for all sufficiently small $\epsilon > 0$ in a set whose closure (in X) is compact and does not meet any of the curves Λ_j and L_j. On this set the eigenvalues of the Levi form of $\rho_\epsilon = h_\epsilon \circ \rho$ are bounded away from zero and hence for $\delta > 0$ sufficiently small $\rho_\epsilon + \delta\tau$ is a strongly plurisubharmonic exhaustion function in Ω_ϵ. Hence Ω_ϵ is Stein by Grauert's theorem [Gra].

Remarks. 1. It is easy to perturb each domain Ω_ϵ constructed above to a strongly pseudoconvex domain in X by adding to ρ a small function which is strongly plurisubharmonic on the sets $\Lambda_j \cap b\Omega_\epsilon$ and $L_j \cap b\Omega_\epsilon$.

2. We don't know whether Theorem 2.2 holds for embedded surfaces $S \subset X$ with hyperbolic complex points of arbitrary type. According to Bishop [B] a hyperbolic point is given in local holomorphic coordinates by

$$w = |z|^2 + \lambda(z^2 + \overline{z}^2) + o(z^2), \quad \lambda > 1/2.$$

At each hyperbolic point S is locally holomorphically convex [FSt] and admits a local nonnegative plurisubharmonic function which vanishes precisely on S and is strongly plurisubharmonic away from S. We don't know whether ρ can be chosen without critical points in some neighborhood of S. The analogous problem is open at totally real double points (again S is locally holomorphically convex near such points [FSt]).

Proof of Theorem 1.1. Let $S \subset X$ be an embedded surface with $I_\pm(S) \le 0$ (resp. $I(S) \le 0$ if S is unorientable). Let $S' \subset X$ be an isotopic embedding satisfying the conclusion of Theorem 2.1. Theorem 2.2 then gives a regular basis of Stein neighborhoods of S'. The analogous conclusions hold for immersions.

&3. The connected sum of embedded and immersed surfaces.

Let $S_1,\ S_2 \subset X$ be embedded or immersed real surfaces in a complex surface X. Their *connected sum* $S_1\#S_2 \subset X$ is an immersed surface in X obtained by removing a small totally real disc D_j from S_j for $j = 1, 2$ such that $D_1 \cap D_2 = \emptyset$ and connecting the resulting surfaces with boundary $S_j\backslash D_j \subset X$ by an embedded cylinder $\Sigma \simeq S^1 \times (0, 1) \subset X\backslash(S_1 \cup S_2)$ which is attached along its two boundary circles to the curves bD_j for $j = 1, 2$. The cylinder can be chosen such that it has two complex points which are both hyperbolic, one positively and one negatively oriented, and it preserves the orientations when both S_1 and S_2 are oriented. Thus $I(S_1\#S_2) = I(S_1) + I(S_2) - 2$, and if both surfaces are oriented we also have

$$I_+(S_1\#S_2) = I_+(S_1) + I_+(S_2) - 1, \quad I_-(S_1\#S_2) = I_-(S_1) + I_-(S_2) - 1.$$

The surface S parametrizing the connected sum $S_1\#S_2$ has genus $g(S) = g(S_1) + g(S_2)$. We also have $[S_1\#S_2] = [S_1] + [S_2] \in H_2(X; \mathbf{Z})$. If S_1 and S_2 are disjoint embedded surfaces in X then is also an embedded surface. For details see section 3 in [F2].

Proof of Theorem 1.5. Let $T \subset X$ be an embedded null-homologous totally real torus. For any embedded oriented surface $S \subset X$ we have

$$g(S\#T) = g(S) + 1, \quad I_\pm(S\#T) = I_\pm(S) - 1, \quad [S] = [S\#T] \in H_2(X; \mathbf{Z}).$$

Attaching $k = \max\{I_+(S), I_-(S), 0\}$ torus handles we obtain an embedded surface $S\#kT \subset X$ with $g(S\#kT) = g(S) + k$ and $I_\pm(S\#kT) = I_\pm(S) - k \leq 0$. Part (a) now follows from Theorem 1.1. The same argument applies to immersed surfaces.

Suppose now that $S \subset X$ is an embedded (or immersed) unorientable surface. Let $M \subset X$ be $\mathbb{RP}^2$ embedded in a coordinate chart in X with $I(M) = -1$ [F2, p. 367]. Then $g(S\#M) = g(S) + 1$ and $I(S\#M) = I(S) - 3$. Thus $g(S\#kM) = g(S) + k$, $I(S\#kM) = I(S) - 3k \leq 0$, and hence (b) follows from Theorem 1.1. We can use other types of handles such as a totally real torus T or a Klein bottle K (for an explicit totally real Klein bottle in $\mathbb{C}^2$ see [Ru]). The surfaces $S\#T$ and $S\#K$ are diffeomorphic and satisfy $I(S\#T) = I(S) - 2$ and $g(S\#T) = g(S) + 2$; hence these handles are less effective in lowering the index than $\mathbb{RP}^2$. ♠

Proof of Proposition 1.4. Let $C \subset \mathbb{CP}^2$ be a smooth complex curve of degree d. By (1.4) we have $I_-(C) = 0$, $I_+(C) = 3d$, $g(C) = g_{\mathbb{C}}(d) = \frac{1}{2}(d-1)(d-2)$.

Attaching $k \geq 3d$ torus handles we obtain an embedded surface $S = C\#kT \subset \mathbb{CP}^2$ homologous to C, with $I_\pm(S) \leq 0$ and

$$g(S) = g(C) + k = \frac{1}{2}(d-1)(d-2) + k \geq \frac{1}{2}(d+1)(d+2).$$

The result now follows from Theorem 1.1. Note that, by the Thom conjecture proved in [KM], a smooth complex curve of degree d in $\mathbb{CP}^2$ has the smallest genus $g_{\mathbb{C}}(d) = \frac{1}{2}(d-1)(d-2)$ among all smooth oriented real surfaces of degree d in $\mathbb{CP}^2$. For the symplectic case see [OS] and [MS]. ♠

Proof of Theorem 1.6. Let $\mathbb{RP}^2 \simeq M \hookrightarrow \mathbb{C}^2 \subset \mathbb{CP}^2$ be an embedding with index $I(M) = -1$ mentioned above [F2, p. 367]. Let $S = \{[x\colon y\colon z] \in \mathbb{CP}^2\colon x, y, z \in \mathbb{R}\} \simeq \mathbb{RP}^2$. Clearly S is totally real and intersects every projective line in $\mathbb{CP}^2$; in fact their (mod 2) intersection number equals one. The connected sum $S\#kM \subset \mathbb{CP}^2$ is an embedded unoriented surface with genus $1+k$ and index $-3k \leq 0$, and hence Theorem 1.1 applies.

&4. Stein neighborhoods of immersed surfaces.

In this section S denotes a compact connected oriented real surface of genus $g(S)$. We shall consider immersions $\pi\colon S \to X$ into a complex surface X with simple (transverse) double points and with no multiple points. Furthermore we shall assume that both tangent planes at any double point are totally real. At each double point π has self-intersection index ± 1 which is independent of the choice of the orientation on S. Double points with index $+1$ will be called *positive* and those with index -1 will be called *negative*. If π has δ_+ positive and δ_- negative double points then $\delta(\pi) = \delta_+ - \delta_-$ is the (geometrical) *self-intersection index* of π which only depends on its regular homotopy class.

Recall that $\chi_n(\pi)$ denotes the normal Euler number of π. It is easily seen that each double point of π contributes ± 2 (the sign depending on its self-intersection index) to the homological self-intersection number of the image $M = \pi(S)$ in X. This gives $\chi_n(\pi) + 2\delta(\pi) = M^2$. ¿From (1.1) we get

$$I_\pm(\pi) = 1 - g(S) - \delta(\pi) + \frac{1}{2}\left(M^2 \pm c_1(X)\cdot M\right), \qquad (4.1)$$

and the condition $I_\pm(\pi) \leq 0$ is equivalent to

$$g(S) + \delta(\pi) \geq 1 + \frac{1}{2}(M^2 + |c_1(X)\cdot M|\,). \qquad (4.2)$$

When π has no negative complex points (for instance if π is complex or symplectic) then $I_-(\pi) = 0$ which gives the well known *genus formula* (see e.g. [IS, p. 576])

$$g(S) = 1 - \delta(\pi) + \frac{1}{2}\left(M^2 - c_1(X)\cdot M\right).$$

4.1 Corollary. *If (4.2) holds then π is regularly homotopic to an immersion $S \to X$ whose image has a regular Stein neighborhood basis in X. The regular homotopy can be chosen such that it preserves the location (and number) of double points. Conversely, if $M = \pi(S)$ has a Stein neighborhood $\Omega \subset X$ and if $[M] \neq 0$ in $H_2(\Omega; \mathbf{Z})$ then*

$$g(S) + \delta_+ \geq 1 + \frac{1}{2}(M^2 + |c_1(X) \cdot M|). \tag{4.3}$$

The first statement follows from Theorems 2.1 and 2.2 and the converse is due to Nemirovski [N] (Theorem II in the Appendix). If π has no negative double points then (4.2) and (4.3) are equivalent.

Using a procedure for replacing double points of an immersion by embedded handles we shall prove the following.

4.2 Theorem. *Let $\pi: S \to X$ be an immersion with k double points. If*

$$g(S) + k \geq 1 + \frac{1}{2}(M^2 + |c_1(X) \cdot M|) \tag{4.4}$$

then any open neighborhood $\Omega \subset X$ of $M = \pi(S)$ contains an embedded oriented surface $M' \subset \Omega$ of genus $g(M') = g(S) + k$ such that $[M'] = [M] \in H_2(\Omega; \mathbf{Z})$ and M' admits a regular basis of Stein neighborhoods.

Proof. At each double point of $\pi(S)$ we replace a pair of small intersecting discs by an embedded annulus $\Sigma \simeq S^1 \times [0,1]$. This well known procedure, which amounts to replacing double points by handles, can be done within Ω and it does not change the homology class of the image, but it increases the genus of the immersed surface by k. Here is a precise description.

By Theorem 2.1 we may assume that in local holomorphic coordinates $(z, w) = (x + iy, u + iv)$ on X near a double point the immersed surface is the union of discs in lagrangian planes $\Lambda_1 = \{y = 0,\ v = 0\} \subset \mathbb{C}^2$, $\Lambda_2 = \{x = 0,\ u = 0\} \subset \mathbb{C}^2$. Let Λ_1 be oriented by $\partial_x \wedge \partial_u$ and Λ_2 by $\kappa\, \partial_v \wedge \partial_y$ where $\kappa = \pm 1$ is the self-intersection index of the double point.

If $\kappa = +1$, an orientation preserving handle is $\Sigma_+ = \{(x + iu)(y - iv) = \epsilon\}$ for a small $\epsilon \neq 0$, intersected with a suitable ball around the origin. Outside the ball we can patch Σ_+ smoothly with Λ_1 resp. Λ_2 without introducing new complex points. A simple calculation shows that Σ_+ is smooth and totally real in $\mathbb{C}^2$ for every $\epsilon \neq 0$, and hence this handle does not change the indices $I_\pm$.

If $\kappa = -1$, an appropriate handle is $\Sigma_- = \{(x + iu)(y + iv) = \epsilon\}$ for small $\epsilon \neq 0$. It has four hyperbolic complex points at $x = y = \pm\sqrt{\epsilon/2}$, $v = -u = \pm\sqrt{\epsilon/2}$ (independent choices of signs), two positive and two negative. Hence $I_\pm(\Sigma_-) = -2$ and the indices of the immersion decrease by two. (For unorientable surfaces we may use any of these two handles, but in this case a torus handle increases the genus by two.)

After replacing all double points by handles we obtain an embedded surface $M' \subset \Omega$ of genus $g(M') = g(S) + k$ homologous to $\pi(S)$. Since the right hand side of (4.4) is a homological invariant, we see that M' satisfies $g(M') \ge 1 + \frac{1}{2}(M'^2 + |c_1(X)\cdot M'|)$ and hence Theorem 1.1 guaranties the existence of a regular Stein neighborhood basis of a nearby surface isotopic to M'.

Remarks. 1. Replacing only the negative double points by handles gives an immersed surface $\pi' : S' \to \Omega$ homologous to $\pi(S)$, of genus $g(S') = g(S) + \delta_-$ and with δ_+ positive double points. Corollary 4.1 gives a nearby immersed surface with a regular Stein neighborhood basis.

2. The replacement of double point by handles is easily performed at *complex* double points, given in local holomorphic coordinates $(z, w) = (x + iy, u + iv)$ by $zw = 0$. Let $\Lambda_1 = \{w = 0\}$ be oriented by $\partial_x \wedge \partial_y$ and $\Lambda_2 = \{z = 0\}$ by $\kappa\, \partial_u \wedge \partial_v$ where $\kappa = \pm 1$ is the intersection index. An appropriate handle is $zw = \epsilon$ when $\kappa = +1$ and $z\overline{w} = \epsilon$ when $\kappa = -1$. ♠

We now consider immersed surfaces in $\mathbb{CP}^2$. Recall that the degree of $\pi : S \to \mathbb{CP}^2$ is the integer d satisfying $[\pi(S)] = d[H] \in H_2(\mathbb{CP}^2; \mathbf{Z}) = \mathbf{Z}$ where H is the projective line. Reversing the orientation on S if necessary we may assume $d \ge 0$. The number on the right hand side of (4.2)–(4.4) equals $1 + \frac{1}{2}(d^2 + 3d) = \frac{1}{2}(d + 1)(d + 2)$.

4.3 Theorem. *Let S_g be a compact oriented surface of genus g and let $\pi : S_g \to \mathbb{CP}^2$ be an immersion of degree $d \ge 1$ with δ_+ positive double points. If $\pi(S_g)$ has a Stein neighborhood $\Omega \subset \mathbb{CP}^2$ then $g + \delta_+ \ge \frac{1}{2}(d + 1)(d + 2)$. Conversely, for any pair of integers $g, \delta_+ \ge 0$ satisfying this inequality there exists an immersion $\pi : S_g \to \mathbb{CP}^2$ of degree d with δ_+ positive double points (and no negative double points) whose image $\pi(S_g)$ admits a regular Stein neighborhood basis in $\mathbb{CP}^2$. In particular, there is an immersed sphere in $\mathbb{CP}^2$ in the homology class of the projective line with three double points and with a regular basis of Stein neighborhoods.*

Proof. The first part follows from Corollary 4.1. We prove the converse part by an explicit construction. Let $\pi_0 : S_0 \to \mathbb{C}^2$ be Weinstein's lagrangian (hence totally real) 'figure eight' immersion of the sphere [Wn]:

$$\pi_0(x, y, u) = \big(x(1 + 2iu), y(1 + 2iu)\big), \quad x^2 + y^2 + u^2 = 1.$$

It identifies the points $(0, 0, \pm 1)$ and the corresponding double point at $0 \in \mathbb{C}^2$ has self-intersection index $+1$.

Let $C_1, C_2, \ldots, C_d \subset \mathbb{CP}^2$ be projective lines in general position. There are $\frac{1}{2}d(d - 1)$ positive intersection points. For each $j \in \{2, \ldots, d\}$ we replace one of the intersection points of C_j with $C_1 \cup \cdots \cup C_{j-1}$ by a handle as in the proof of Theorem 4.2. This eliminates $d-1$ double points and changes the given system of lines to an immersed sphere $\rho: S_0 \to \mathbb{CP}^2$ of degree d with $I_+(\rho) = 3d$, $I_-(\rho) = 0$ and with $\frac{1}{2}(d-1)(d-2)$ positive double points.

Write $g + \delta_+ = l + \frac{1}{2}(d+1)(d+2)$, with $l \geq 0$, and set $k = l + 3d \geq 3d$. Take k pairwise disjoint copies $M_1, \ldots, M_k \subset \mathbb{CP}^2 \backslash \rho(S_0)$ of Weinstein's sphere $\pi_0(S_0)$. The connected sum $\rho(S_0) \# M_1 \# \cdots \# M_k \subset \mathbb{CP}^2$ is parametrized by a degree d immersion $\pi: S_0 \to \mathbb{CP}^2$ with $I_+(\pi) = 3d - k \leq 0$, $I_-(\pi) = -k < 0$, and with $k + \frac{1}{2}(d-1)(d-2) = g + \delta_+$ positive double points. Replacing g of the double points by handles we obtain an immersed surface of genus g with δ_+ positive double points and with non-positive indices $I_\pm$. The result now follows from Theorem 1.1. ♠

By a theorem of Gromov [Gro, p. 334, Theorem A] every immersion $\pi: S \to \mathbb{CP}^2$ of degree $d \geq 1$ can be $\mathcal{C}^0$-approximated by a symplectic immersion $\widetilde{\pi}: S \to \mathbb{CP}^2$, i.e., $\widetilde{\pi}^*(\omega_{FS}) > 0$ on S. (If we specify a two-form $\theta > 0$ on S with $\int_S \theta = 1$, we can even choose $\widetilde{\pi}$ such that $\widetilde{\pi}^* \omega_{FS} = d \cdot \theta$.) Hence Theorem 4.3 implies

4.4 Corollary. *For any $d \geq 1$ there exists a symplectically immersed sphere in $\mathbb{CP}^2$ of degree d with a Stein neighborhood.*

I thank S. Ivashkovich for telling me the question answered in part by Corollary 4.4 and for pointing out the relevance of Gromov's theorem. The results of [IS] imply that *there exist no immersed symplectic spheres in $\mathbb{CP}^2$ with a Stein neighborhood basis.*

Corollary 4.4 may seem somewhat surprising since symplectic curves share many properties with complex curves. The crucial difference (which is essential here) is that complex curves only have positive double points while symplectic curves may also have negative double points.

The symplectic approximation furnished by Gromov's theorem need not be regularly homotopic to the initial immersion and hence the number of double points may increase. The question remains about the minimal number of double points of immersed symplectic spheres in $\mathbb{CP}^2$ of a given degree $d \geq 1$ which admit Stein neighborhoods. For $d = 1$ the genus formula gives $\delta(\pi) = 1 + \frac{1}{2}(d^2 - 3d) = 0$ which shows that the number of positive double points equals the number of negative double points. By Theorem 4.3 we have $\delta_+ \geq \frac{1}{2}(d+1)(d+2) = 3$ and hence *there are at least* 6 *double points.*

Question: *Is there an immersed symplectic sphere in $\mathbb{CP}^2$ of degree one (in the homology class of the projective line) with six double points and with a Stein neighborhood ?*

Appendix: Seiberg-Witten theory and genus inequalities.

We summarize the results of the Seiberg-Witten theory which are relevant to our problem. For a compact oriented four-manifold X we denote by $b^+(X)$ the number of positive eigenvalues (counted with multiplicities) of the intersection form on $H^2(X;\mathbb{R})$. If X is a compact Kähler surface then $b^+(X) = 1 + 2\dim_{\mathbb{C}}\Omega^2(X)$ is an odd positive integer.

Theorem I. ([KM], [MS], [FiS], [OS], [N]) *Let $S \subset X$ be an embedded compact oriented real surface in a complex surface X. Assume that S is not a two-sphere with trivial homology class in X. Then $\chi(S) + S^2 + |c_1(X)\cdot S| \le 0$ (and hence Theorem 1.1 applies to S) in each of the following cases:*

(a) X is a compact Kähler surface with $b^+(X) > 1$ and $S^2 \ge 0$.

(b) X is a compact Kähler surface, $S^2 < 0$, and none of the homology classes $\pm[S] \in H_2(X;\mathbb{R})$ can be represented by a (possibly reducible) complex curve $C \subset X$.

(c) X is a Stein surface.

The inequality in Theorem I can be written in the form

$$g(S) \ge 1 + \frac{1}{2}\left(S^2 + |c_1(X)\cdot S|\right). \qquad (*)$$

The conclusion in part (a) fails for any complex curve $C \subset \mathbb{CP}^2$ since $S^2 = d^2 \ge 1$ and $I_+(C) = 3d \ge 3$, where d is the degree of C. Hence the condition $b^+(X) > 1$ in (a) cannot be omitted. (Note that $b^+(\mathbb{CP}^2) = 1$.)

Case (a) follows from results of Kronheimer and Mrowka [KM] and Morgan, Szabó and Taubes [MS] (see Theorem 6 in [N]), and case (b) from results of Fintushel and Stern [FiS] and Ozsváth and Szabó [OS] (see Theorem 7 in [N]). Case (c) was proved by Nemirovski [N, Theorem 9] by applying Stout's result [St] (see also [Le], [DL]) to the effect that any relatively compact domain in a Stein manifold is biholomorphic to a domain in a projective algebraic manifold. Thus one may assume that S is embedded in a smooth projective surface. This surface can be modified to a compact Kähler surface Y such that $S^2 \ge 0$ implies $b^+(Y) > 1$ and hence part (a) applies, while $S^2 < 0$ implies that $\pm[S]$ is not homologous to a complex curve in Y and hence (b) applies.

For immersed surfaces in Stein surfaces one has the following result also due to Nemirovski [N, p. 742, Remark 3]:

Theorem II. *Let $\pi\colon S \to X$ be an immersed compact oriented real surface with simple double points in a Stein surface X. If π has δ_+ positive double points and if $[\pi(S)] \ne 0$ in X then*

$$g(S) + \delta_+ \ge 1 + \frac{1}{2}\left(\pi(S)^2 + |c_1(X)\cdot \pi(S)|\right). \qquad (**)$$

The inequality (**) clearly fails for immersed null-homologous spheres and is trivial for null-homologous surfaces of genus at least one.

Since a proof of Theorem II does not seem to be available in print, we include a short sketch (I thank S. Ivashkovich for explaining it to me). One first replaces a neighborhood of $\pi(S)$ in X by a domain in an affine algebraic surface V by Stout's approximation theorem [St], insuring that $\pi(S)$ remains homologically nontrivial in V. Let $\overline{V}$ denote the projective closure of V and let $\phi: Y \to \overline{V}$ be the desingularization of $\overline{V}$ (Y is then a smooth compact Kähler surface). One reduces (**) to (*) by replacing positive double points of the immersion $\pi: S \to Y$ by handles as in Theorem 4.2 (this does not change the indices and the homological type of the image, but it increases the genus of the immersed surface by δ_+) and separating the negative double points by blowing up Y at those points (the immersed surface is first made complex near each double point in order to insure separation in the blow-up). The latter operation preserves both sides of (**) since the homological self-intersection number does not change, and the Chern number on the immersed surface changes only by the intersection number of the proper transform of $\pi(S)$ in the blow-up $\widetilde{Y}$ with the exceptional divisor over the double point. This equals zero when the double point is negative since the proper transform has one positive and one negative intersection point with the exceptional divisor. We thus obtain an embedded homologically nontrivial surface $S' \subset \widetilde{Y}$. The argument used in Theorem I (c) applies to S' (see the proof of Theorem 9 in [N]) and shows that (*) (and hence also (**)) is valid.

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