University of Ljubljana
Institute of Mathematics, Physics and Mechanics
Department of Mathematics
Jadranska 19, 1000 Ljubljana, Slovenia

Preprint series, Vol. 40 (2002), 809

AN INFINITE FAMILY OF CUBIC EDGE- BUT NOT VERTEX-TRANSITIVE GRAPHS

Aleksander Malnič   Dragan Marušič
Primož Potočnik   Changqun Wang

ISSN 1318-4865

February 28, 2002

Ljubljana, February 28, 2002

AN INFINITE FAMILY OF CUBIC EDGE- BUT NOT VERTEX-TRANSITIVE GRAPHS

ALEKSANDER MALNIČ [1]
IMFM, Oddelek za matematiko
Univerza v Ljubljani
Jadranska 19, 1111 Ljubljana
Slovenija

DRAGAN MARUŠIČ[1]
IMFM, Oddelek za matematiko
Univerza v Ljubljani
Jadranska 19, 1111 Ljubljana
Slovenija

PRIMOŽ POTOČNIK[1]
IMFM, Oddelek za matematiko
Univerza v Ljubljani
Jadranska 19, 1111 Ljubljana
Slovenija

CHANGQUN WANG[2]
Department of System Sciences and Mathematics
Zhengzou University
Henan Province, 450052
People's Republic of China

Abstract

An infinite family of cubic edge- but not vertex-transitive graphs is constructed. The graphs are obtained as regular $\mathbb{Z}_n$-covers of $K_{3,3}$ where $n = p_1 p_2 \cdots p_k$ is a product of distinct primes congruent to 1 modulo 3. Moreover, it is proved that the Gray graph (of order 54) is the smallest cubic edge- but not vertex-transitive graph.

1 Introductory remarks

Throughout this paper graphs are assumed to be finite, and, unless specified otherwise, simple, undirected and connected. For the group-theoretic concepts and notation not defined here we refer the reader to [5, 9, 22]. Given a positive integer n, we shall use the symbol $\mathbb{Z}_n$ to denote the ring of residues modulo n as well as the cyclic group of order n.

For a graph X we let $V(X)$, $E(X)$ and $\mathrm{Aut}X$ be the vertex set, the edge set and the automorphism group of X, respectively. For two adjacent vertices u and v we denote by $[u, v]$ or, alternatively, by uv the edge joining u to v. Similarly, (u, v) denotes the arc from u to v. If a subgroup G of

[1] Supported in part by "Ministrstvo za znanost in tehnologijo Slovenije", proj. no. J1-4965-99.

[2] Supported in part by the grant from Zhengzhou University. Wang is grateful to the Institute of Mathematics, Physics and Mechanics at the University of Ljubljana for hospitality and financial support during his visit that led to the completion of this work.

Aut$X$ acts transitively on $V(X)$ and $E(X)$, we say that X is *G-vertex-transitive* and *G-edge-transitive*, respectively. In the special case when $G =$ Aut$X$, we say that X is *vertex-transitive* and *edge-transitive*, respectively. It may be easily seen that a G-edge- but not G-vertex-transitive graph X is necessarily bipartite, where the two parts of the bipartition are orbits of G. In particular, if X is regular the two parts of bipartition have equal cardinality. Such a graph will be referred to as a *G-semisymmetric* graph. If $G =$ Aut$X$ the graph X is said to be *semisymmetric*.

The study of semisymmetric graphs was initiated by Folkman [6] who, among others, gave a construction of several infinite families of such graphs and posed a number of open problems which spurred the interest in this topic (see [1, 2, 12, 13, 14, 15, 18]).

This article deals with cubic semisymmetric graphs. A first written account of such a graph, the so called Gray graph of order 54, is due to Bouwer [1], thus answering an open problem from [6] about the existence of semisymmetric graphs of prime valency. According to [1], however, the discovery of the graph is due to Marion C. Gray in 1932. Following [19], the Gray graph is a regular $\mathbb{Z}_3^2$-cover of $K_{3,3}$ (see Figure 1).

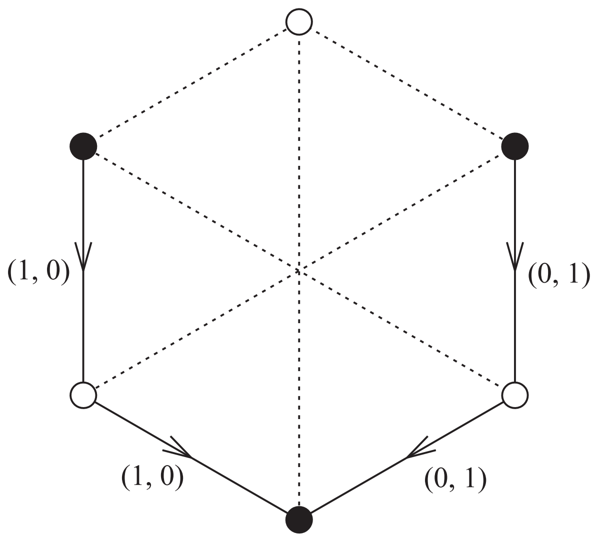

Figure 1: The Gray graph as a cover of $K_{3,3}$.

This fact motivates our hunt for new constructions of cubic semisymmetric graphs amongst regular covers of $K_{3,3}$. In order to reconstruct such covers by voltages valued in a given group (of covering transformations) N, we choose the tree carrying identity voltages as shown in Figure 2. Furthermore let $a, b, c, d \in N$ be the voltages of the remaining cotree arcs $(3,2)$, $(3,4)$, $(2,5)$ and $(4,5)$, respectively. This particular regular cover of $K_{3,3}$ will be referred to as a N-*cover* of $K_{3,3}$ with *voltage-quadruple* (a, b, c, d) and will be denoted by the symbol $X(N; a, b, c, d)$.

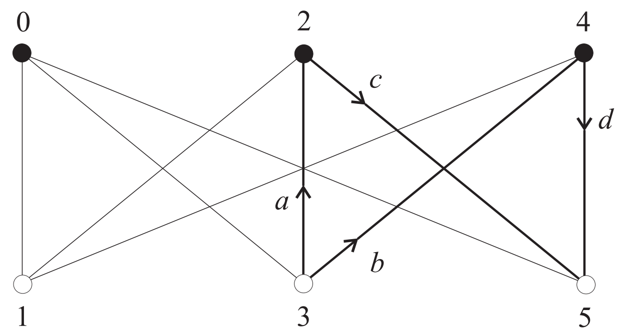

Figure 2: The voltage assignment of $K_{3,3}$.

In the special case when $N \cong \mathbb{Z}_n$, n a positive integer, we simplify the notation and set $X(n; a, b, c, d) = X(\mathbb{Z}_n; a, b, c, d)$. Moreover, assume that $\mathbb{Z}_n$ contains two elements r and s such that $r \neq s, s^{-1}$ and $r^2 + r + 1 = 0 = s^2 + s + 1$. (Consequently, $\mathbb{Z}_n^*$ contains as a subgroup an elementary abelian group of order 9 generated by r and s.) The following theorem is the main result of this article.

Theorem 1.1 *Let $n = p_1 p_2 \cdots p_k$ be a product of distinct primes congruent to 1 modulo 3. Then there are two solutions $r, s \in \mathbb{Z}_n^*$ of the equation*

$$x^2 + x + 1 = 0 \tag{1}$$

such that $r \neq s, s^{-1}$ and such that $X(n; 1, -r, s, -rs)$ is semisymmetric, with trivial edge stabilizers and thus vertex stabilizers isomorphic to $\mathbb{Z}_3$.

An analysis of semisymmetric properties of regular covers of $K_{3,3}$, which is the content of Section 2, lays out the basis for the proof of Theorem 1.1

which is carried out in Section 3. Finally, in Section 4, combining together results from Section 2, the original Folkman's work [6] and some recent work on semisymmetric graphs of particular orders [18], we prove the following result.

Theorem 1.2 *The Gray graph is the smallest cubic semisymmetric graph.*

In 1987 Ivanov [13] produced a computer-generated list of semisymmetric graphs on at most 41 vertices. Arguably, the minimality of the Gray graph could be easily deduced with the sophisticated computer technology we have at our hands today. We have however consciously opted for a purely combinatorial approach free of any use of computers. The tools developed along the way allow us to identify, among others, the importance of graph covers to structural and classification results on semisymmetric graphs.

2 Covers of $K_{3,3}$

An epimorphism $p : \widetilde{X} \to X$ of connected graphs is a *regular covering projection*, loosely speaking, if it arises essentially as a factorization $\widetilde{X} \to \widetilde{X}/N \cong X$, where the action of $N \leq \mathrm{Aut}\,\widetilde{X}$ is semiregular on both vertices and edges of $\widetilde{X}$. The graph $\widetilde{X}$ is called the *covering graph* and X is the *base graph*. The *vertex fibres* $p^{-1}(v)$, $v \in V(X)$, correspond to orbits of N on $V(\widetilde{X})$. Similarly, *edge fibres* correspond to orbits of N on $E(\widetilde{X})$.

It is well known that a regular covering projection $\widetilde{X} \to X \cong \widetilde{X}/N$ can be reconstructed in terms of *voltage assignments* valued in N [11]. Namely, label arbitrarily one of the vertices in each fibre by $1 \in N$, and then label all other vertices by the left regular action of $N \leq \mathrm{Aut}\,\widetilde{X}$ on fibres. The edges of $\widetilde{X}$ can now be recaptured from X by considering the right regular action of N induced by the above labelling. An arc uv of X is assigned the *voltage* $\mathrm{vol}(uv) = a \in N$ whenever the origins and termini of arcs in $p^{-1}(uv)$ are labeled, respectively, by g and ga, where $g \in N$. Clearly, inverse arcs receive inverse voltages, and the assignment on arcs naturally extends to all walks. It is well-known that an existing assignment can be modified to an equivalent one such that the arcs of an arbitrarily prescribed spanning tree receive the trivial voltage. Moreover, replacing the voltages of the cotree arcs by their images under an automorphism of N gives rise to an equivalent covering [20].

Let $p : \widetilde{X} \to X \cong \widetilde{X}/N$ be a regular covering projection. If $\phi \in \mathrm{Aut}\, X$ and $\tilde{\phi} \in \mathrm{Aut}\, \widetilde{X}$ satisfy $p\tilde{\phi} = \phi p$ we call $\tilde{\phi}$ the *lift* of ϕ, and ϕ the *projection* of $\tilde{\phi}$. Concepts such as the lift of a group of automorphisms and the projection of a group of automorphisms are self-explanatory. The lifts and the projections of groups are of course subgroups in $\mathrm{Aut}\, \widetilde{X}$ and $\mathrm{Aut}\, X$, respectively. In particular, the lift of the trivial group is the *group of covering transformations* $\mathrm{CT}(p)$. Obviously, $\mathrm{CT}(p)$ is isomorphic to N. Note that if N is normal in $\mathrm{Aut}\, \widetilde{X}$, then $\mathrm{Aut}\, \widetilde{X}$ does project (however, the projection need not be onto). The problem whether an automorphism ϕ lifts can be grasped in terms of voltages as follows. Define the mapping $\phi^{\#} : N \to N$ by $\mathrm{vol}(C) \mapsto= \mathrm{vol}(\phi(C))$, where C ranges over all fundamental closed cycles at some base point. Then ϕ lifts if and only if $\phi^{\#}$ extends to an automorphism of N [16]. If N is elementary abelian we regard $\mathrm{Aut}\, N$ as a group of linear transformations, and hence the calculations can be simplified considerably.

We are now ready to start exploring semisymmetric properties of cyclic covers of $K_{3,3}$.

Given a group N and $a, b, c, d \in N$, let us assume that $X = X(N; a, b, c, d)$ is a regular N-cover of $Y = K_{3,3}$ where N is normal in some edge-transitive subgroup H of $\mathrm{Aut}X$. Then H/N acts edge-transitively on Y. Clearly, an arbitrary edge-transitive group of automorphisms of Y must contain automorphisms $\varphi = (024)$ and $\psi = (135)$. Furthermore, an edge- and vertex-transitive subgroup of $\mathrm{Aut}Y$ is necessarily arc-transitive and must therefore contain an automorphism interchanging, say, vertices 0 and 1. It may be easily seen that in this case at least one of the three automorphisms $\tau_1 = (01)(23)(45)$, $\tau_2 = (01)(25)(34)$ and $\tau_3 = (01)(2345)$ belongs to this subgroup too. The converse is also true. Hence an edge-transitive subgroup of $\mathrm{Aut}Y$ is semisymmetric if and only if it contains none of τ_i, $i = 1, 2, 3$. Consequently, a necessary and sufficient condition for X to be H-semisymmetric is that both φ and ψ lift but none of τ_i, $i = 1, 2, 3$, lifts.

Let $\sigma_1 = (24)$ and $\sigma_2 = (35)$. The reader may verify, by checking the fundamental cycles 03210, 03410, 01250 and 01450 of Y, that $\varphi^{\#}$, $\psi^{\#}$, $\tau_1^{\#}$, $\tau_2^{\#}$, $\tau_3^{\#}$, $\sigma_1^{\#}$ and $\sigma_2^{\#}$ map the voltages a, b, c and d as follows (we use the additive notation for the operation in N):

TABLE 1: Voltages of the images of fundamental cycles in $K_{3,3}$.

	03210	03410	01250	01450
	a	b	c	d
$\varphi^{\#}$	$-a+b$	$-a$	$d-c$	$-c$
$\psi^{\#}$	$-c-a$	$-d-b$	a	b
$\tau_1^{\#}$	$-a$	c	b	$-d$
$\tau_2^{\#}$	d	$-b$	$-c$	a
$\tau_3^{\#}$	$-b$	d	a	$-c$
$\sigma_1^{\#}$	b	a	d	c
$\sigma_2^{\#}$	$-c$	$-d$	$-a$	$-b$

By the connectivity of X we have $N = \langle a, b, c, d\rangle$. Furthermore, from Table 1 we see that all of a, b, c, d are nonzero elements of the same order. Assume now that $N \cong \mathbb{Z}_n$ is cyclic. Then we have that $N = \langle a\rangle = \langle b\rangle = \langle c\rangle = \langle d\rangle$. Note that an automorphism of $\mathbb{Z}_n$ is of the form $x \mapsto kx$, $x \in \mathbb{Z}_n$, where k is coprime to n. The next lemma gives necessary and sufficient conditions for various automorphisms of $K_{3,3}$ to lift.

Lemma 2.1 *Let n be a positive integer, and let $X = X(n; a, b, c, d)$, where $a, b, c, d \in \mathbb{Z}_n$ be a connected regular $\mathbb{Z}_n$-cover of $K_{3,3}$. Then the following statements hold.*

(i) $\varphi = (024)$ *and* $\psi = (135)$ *lift if and only if there exist* $r, s \in \mathbb{Z}_n$ *solving equation (1) and such that* $b = -ar, c = as$ *and* $d = -ars$*;*

(ii) $\tau_1 = (01)(23)(45)$ *lifts if and only if* $b = -c$*;*

(iii) $\tau_2 = (01)(25)(34)$ *lifts if and only if* $a = -d$*;*

(iv) $\tau_3 = (01)(2345)$ *lifts if and only if* $c = -a^2b^{-1}$, $d = -a^{-1}b^2$ *and* $a^4 = b^4$*;*

(v) $\sigma_1 = (24)$ *lifts if and only if there exists* $k \in \mathbb{Z}_n^*$ *such that* $k^2 = 1$, *and* $b = ka$ *and* $d = kc$*;*

(vi) $\sigma_2 = (35)$ *lifts if and only if there exists* $k \in \mathbb{Z}_n^*$ *such that* $k^2 = 1$, *and* $c = -ka$ *and* $d = -kb$.

PROOF. Note that, by the comments preceding the statement of Lemma 2.1, we have that a, b, c and d are all coprime with n. We only prove (i) and (iv), proofs of (ii), (iii), (v) and (vi) are straightforward and done in an analogous way.

To prove (i), suppose first that there exist $r, s \in \mathbb{Z}_n^*$ such that $r^2+r+1 = 0 = s^2+s+1$. Letting $X = X(a, -ar, as, -ars)$, it is easily seen that $\varphi^\#$ and $\psi^\#$ extend to automorphisms of $\mathbb{Z}_n$, and so both φ and ψ lift.

Conversely, suppose now that $X = X(n; a, b, c, d)$ is a regular $\mathbb{Z}_n$-cover of $K_{3,3}$ such that both φ and ψ lift. Then $\varphi^\#$ and $\psi^\#$ extend to automorphisms of the cyclic group $\mathbb{Z}_n$. We may see from Table 1 that $\varphi^\# : x \mapsto kx$ for each $x \in \mathbb{Z}_n$, where $k = -d^{-1}c = c^{-1}(d-c) = -b^{-1}a = a^{-1}(b-a)$. Similarly, $\psi^\# : x \mapsto lx$ for each $x \in \mathbb{Z}_n$, where $l = d^{-1}b = -b^{-1}(b+d) = c^{-1}a = -a^{-1}(a+c)$. By computation, $a^2-ab+b^2 = c^2-cd+d^2 = a^2+ac+c^2 = b^2+bd+d^2 = 0$ and moreover $ad = bc$. Let $r = -a^{-1}b$ and $s = a^{-1}c$. Clearly, both r and s solve equation (1). Also, we have $b = -ar$, $c = as$ and $d = -ars$. Therefore $X = X(n; a, -ar, as, -ars)$, as required.

As for (iv), suppose first that $a, b \in \mathbb{Z}_n^*$ are such that $a^4 = b^4$ and let $X = X(n; a, b, -a^2b^{-1}, -a^{-1}b^2)$. Then it may be checked directly from Table 1 that τ_3 lifts.

Conversely, let us assume that $X = X(n; a, b, c, d)$ is a regular $\mathbb{Z}_n$-cover of $K_{3,3}$ such that τ_3 lifts. From Table 1 we have that τ_3 lifts if and only if $\tau_3^\#$ maps according to the rule $x \mapsto kx$, for each $x \in \mathbb{Z}_n$, where $k = -cd^{-1} = ac^{-1} = b^{-1}d = -a^{-1}b$. By computation, it may be seen that $a^4 = b^4$ and that c and d are of the desired form, that is, $X = X(n; a, b, -a^2b^{-1}, -a^{-1}b^2)$. ■

Let $n > 3$ be a positive integer such that $\mathbb{Z}_n$ contains two elements r and s satisfying $r \neq s, s^{-1}$ and $r^2+r+1 = 0 = s^2+s+1$. (Consequently, $\mathbb{Z}_n^*$ contains as a subgroup an elementary abelian group of order 9 generated by r and s.) Then we say that the ordered triple (n, r, s) (as well as the integer n) is *admissible*.

The following result, giving a necessary and sufficient condition for the largest projectable group of automorphisms of a connected regular cyclic cover to be semisymmetric, is a straightforward consequence of Lemma 2.1.

Corollary 2.2 *Let $X = X(n; a, b, c, d)$, $n > 3$, be a connected regular $\mathbb{Z}_n$-cover of $K_{3,3}$ and let H be the largest subgroup of* Aut$X$ *which projects. The following statements are equivalent.*

(i) X is H-semisymmetric;

(ii) $H = \langle\tilde{\varphi}, \tilde{\psi}\rangle$;

(iii) $X \cong X(1, -r, s, -rs)$ *where* $r, s \in \mathbb{Z}_n$ *and the triple* (n, r, s) *is admissible.*

Proof. Recall that X is H-semisymmetric if and only if both φ and ψ lift but neither of τ_i, $i = 1, 2, 3$, lifts. Combining together parts (i), (ii), (iii) and (iv) of Lemma 2.1 and noticing that $X(n; a, -ar, as, -ars) \cong X(n; 1, -r, s, -rs)$ we obtain the admissibilty of the triple (n, r, s). This proves the equivalence of parts (i) and (iii) of Corollary 2.2. Moreover, since $n > 3$, it may be seen that parts (v) and (vi) of Lemma 2.1 together force also the equivalence of parts (ii) and (iii) of Corollary 2.2. ■

In view of the above result a necessary condition for the existence of a regular $\mathbb{Z}_n$-cover of $K_{3,3}$ with a projectable semisymmetric group of automorphisms is that $\mathbb{Z}_n^*$ contains a subgroup isomorphic to an elementary abelian group of order 9. Consequently, there exist infinitely many n such that there are no regular $\mathbb{Z}_n$-covers of $K_{3,3}$ with a projectable semisymmetric group: just choose n so that 9 does not divide the Euler function $\phi(n)$ of n. However, searching for semisymmetric covers of $K_{3,3}$, our aim is quite the opposite. Corollary 2.2 will be used in the next section to construct a particular infinite family of semisymmetric regular cyclic covers of $K_{3,3}$ and thus prove Theorem 1.1.

3 Proof of Theorem 1.1

We start this section by showing that there exist infinitely many integers n satisfying the necessary and sufficient condition given in Corollary 2.2 for the semisymmetry of the largest projectable subgroup of automorphisms of a connected regular $\mathbb{Z}_n$-cover of $K_{3,3}$.

Proposition 3.1 *Let* $n = qt$ *be a positive integer such that* $q, t > 3$ *are coprime and such that the equation (1) has a solution in* $\mathbb{Z}_q$ *as well as in* $\mathbb{Z}_t$*. Then there are* r, s *in* $\mathbb{Z}_n$ *such that the triple* (n, r, s) *is admissible.*

Proof. Set $Q = \{kq \mid k \in \mathbb{Z}_n\}$ and $T = \{lt \mid l \in \mathbb{Z}_n\}$. We claim that

$$|(i + Q) \cap (j + T)| = 1 \text{ for any } i, j \in \mathbb{Z}_n. \tag{2}$$

Indeed, since q and t are coprime, there exist integers λ and μ such that $\lambda q + \mu t = 1$. Thus, for any i and j, we have $i - j = (i - j)(\lambda q + \mu t) =$

$(i-j)\lambda q+(i-j)\mu t$, and so $i+(j-i)\lambda q = j+(i-j)\mu t \in (i+Q)\cap(j+T)$. Hence $(i+Q)\cap(j+T)\neq\emptyset$. Moreover, suppose that $i+k_1q = j+l_1t$ and $i+k_2q = j+l_2t$ (in $\mathbb{Z}_n$), where $k_1,k_2,l_1,l_2\in\mathbb{Z}_n$. Then $(k_1-k_2)q=(l_1-l_2)t$ in $\mathbb{Z}_n$, implying that $k_1\equiv k_2 (\mathrm{mod}\ t)$ and $l_1\equiv l_2(\mathrm{mod}\ t)$. Hence $i+k_1p = i+k_2p = j+l_1q = j+l_2q$, completing the proof of (2).

Let $u\in\mathbb{Z}_q$ and $v\in\mathbb{Z}_t$ be two solutions of the equation (1) in $\mathbb{Z}_q$ and $\mathbb{Z}_t$, respectively. In view of (2) we may choose $r\in(u+Q)\cap(v+T)$ and $s\in(u+Q)\cap(v^2+T)$. Note that every element of $u+Q$ is a solution of the equation (1) in $\mathbb{Z}_q$. Similarly, every element of $v+T$ is a solution of the equation (1) in $\mathbb{Z}_t$. Consequently, r is a solution of the equation (1) in both $\mathbb{Z}_q$ and $\mathbb{Z}_t$, and thus q and t are coprime also in $\mathbb{Z}_n$. In an analogous way can prove that s is a solution of the equation (1) in $\mathbb{Z}_n$.

It remains to be seen that $s\neq r, r^{-1}$. By the choice of r and s we may write $r=k_1q+u=l_1t+v$ and $s=k_2q+u=l_2t+v^2$, for some $k_1,k_2,l_1,l_2\in\mathbb{Z}_n$. If $s=r$, then $l_1t+v=l_2t+v^2$, implying that $v^2-v=0$ in $\mathbb{Z}_n$. Thus, being a solution of the equation (1), v is necessarily invertible. Hence $v=1$ and so $0=v^2+v+1=3$, contradicting $t>3$. Therefore $s\neq r$. In a similar fashion, assuming that $s=r^{-1}=r^2$, we have $u^2-u=0$ in $\mathbb{Z}_q$, forcing $u=1$ and then $q=3$, a contradiction. This completes the proof of Proposition 3.1. ■

A straightforward induction argument gives us the following corollary, which implies that there are infinitely many admissible integers.

Corollary 3.2 *Let $n=p_1p_2\cdots p_k$, $k\geq 2$, where p_i, $i=1,2,\ldots,r$, are distinct primes congruent to 1 modulo 3. The n is admissible.*

PROOF. The result clearly holds for $k=2$ by Proposition 3.1. Let $a=n/p_k$ and $b=p_k$. By induction hypothesis we may assume that the equation (1) has a solution in $\mathbb{Z}_a$ and in $\mathbb{Z}_b$. The result follows by Proposition 3.1. ■

Theorem 3.3 *Let n be a positive odd integer not divisible by 3 and let $r,s\in\mathbb{Z}_n$ be such that the triple (n,r,s) is admissible. Then the graph $X(n;1,-r,s,-rs)$ is a cubic semisymmetric regular $\mathbb{Z}_n$-cover of $K_{3,3}$ with trivial edge stabilizers.*

PROOF. Let $X=X(n;1,-r,s,-rs)$ and let $H=\langle\varphi^\#,\psi^\#\rangle$ be the lifted group of $\langle\varphi,\psi\rangle$. Then $|H|=9n$ and X is H-semisymmetric with H acting regularly on edges of X. Hence the corresponding edge stabilizers are trivial.

Let $N \cong \mathbb{Z}_n$ be the group of covering transformations and let $A = \mathrm{Aut}X$. By Corolary 2.2 we have that $H = N_A(N)$. We need to show that

$$\mathrm{Aut}X = H. \tag{3}$$

A somewhat technical combinatorial argument involving an analysis of 12-cycles in X is needed to establish (3). Before that, however, we present a much simpler argument based on the use of the Sylow theorem. Unfortunately it does not work for those integers n which are divisible by 5, 7, 31 or 127.

Assume that $H \neq A$. Using Tutte's theorem [21] in the case when X is vertex-transitive, and Goldschmidt's theorem [8] in the case when X is semisymmetric, we have that vertex stabilizers A_v, $v \in V(X)$, have order $2^k \cdot 3$, where $0 \leq k \leq 7$. Hence $[A : H] = 2^k$. Let p be a prime divisor of n and P a corresponding Sylow p-subgroup in A (and H). Then the index $[A : N_A(P]$ is a divisor of 2^k and a number congruent to 1 modulo p. Since $k \leq 7$, it follows that $k = 1$. Consequently, P is normal in A unless $p \in \{5, 7, 31, 127\}$. It follows that N is normal in A unless n is divisible by 5, 7, 31 or 127. Now if N is normal then the whole of A projects, forcing $A = H$. Therefore X is semisymmetric with trivial edge stabilizers for all those integers n which are not divisible by 5, 7, 31 or 127.

We now turn to the aformentioned combinatorial approach which will complete the proof. The idea is to first prove that the fibres $\tilde{0}, \tilde{1}, \tilde{2}, \tilde{3}, \tilde{4}, \tilde{5}$ are blocks of imprimitivity of the automorphism group A of X. This will then give us a better upper bound on the order of A, and enable us to use the Sylow theorem for all primes dividing n, leading eventually to $A = H$. To this end we use an argument involving 12-cycles in X, singling out the following essential facts about them.

Claim 1. Each 12-cycle in X arises from a configuration in $K_{3,3}$ consisting of three 4-cycles missing out only vertices of one fibre (say, fibre $\tilde{4}$ in Figure 3 below).

The proof of this rather technical result is omitted.

Of course, there are precisely six such configurations in $K_{3,3}$, each one missing out precisely one of the six fibres, giving us a total of $6n$ cycles of length 12 in X. We call a 12-cycle *even* if it misses out an even fibre $\tilde{0}$, $\tilde{2}$ or $\tilde{4}$, and *odd* if it misses out an odd fibre $\tilde{1}$, $\tilde{3}$ or $\tilde{5}$. We shall use symbols $\mathcal{C}^+$ and $\mathcal{C}^-$ for the two respective sets of even and odd 12-cycles. Moreover, we shall say that two 12-cycles in X are of the same *type* provided

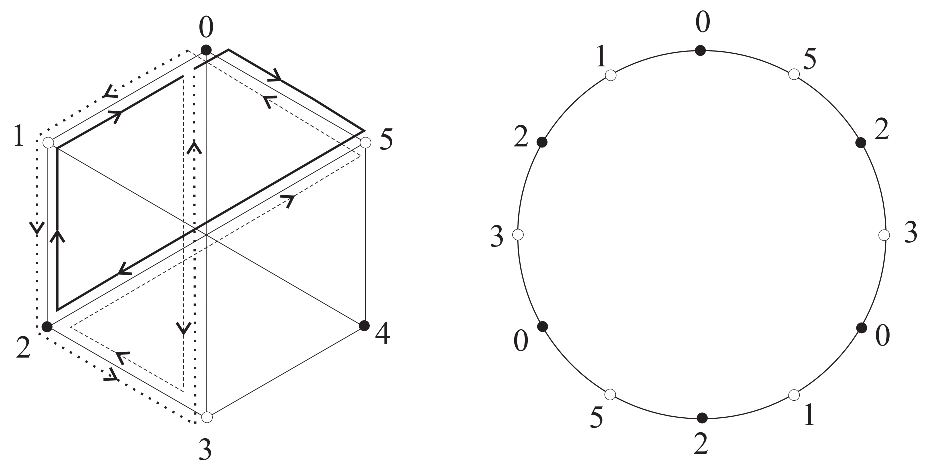

Figure 3: A configuration in $K_{3,3}$ giving rise to a 12-cycle in X.

they arise from the same configuration in $K_{3,3}$. Note that each 4-path in X defines a unique 12-cycle and that each 3-path in X defines precisely two 12-cycles, one even and one odd, which are glued together along a 3-path as is illustrated in Figure 4 below. We introduce an auxiliary graph, call it $Y = Y(n; r, s)$, whose vertices are the $6n$ cycles of length 12 in X and whose edges correspond to pairs of 12-cycles having a 3-path in common. Clearly, Y has valency 12.

Claim 2. Y is bipartite and connected.

Obviously, Y is bipartite with $\mathcal{C}^+$ and $\mathcal{C}^-$ as the bipartition sets. To see that X is connected, observe first that any two 12-cycles of the same type belong to the same connected component of Y. To illustrate this, consider 12-cycles C and D in Figure 4. Let C' be the 12-cycle of the same type as C which is adjacent to D along a 3-path $(2, 5, 0, 3)$, and having the 2-path $(2, 5, 0)$ in common with C. If ν denotes the generator of $N \cong \mathbb{Z}_n$ coresponding to the mapping $x \mapsto x + 1$ in $\mathbb{Z}_n$, we can easily see that $C' = \nu(C)$. Hence C and $\nu(C)$ (and so any 12-cycle of the same type as C) must be in the same connected component of Y. Of course, exactly the same holds for the 12-cycles arising from the other five configurations in $K_{3,3}$. But recall that adjacency in Y is defined in such a way that the quotient with respect to the six sets of 12-cycles of same type is isomorphic to $K_{3,3}$ (with

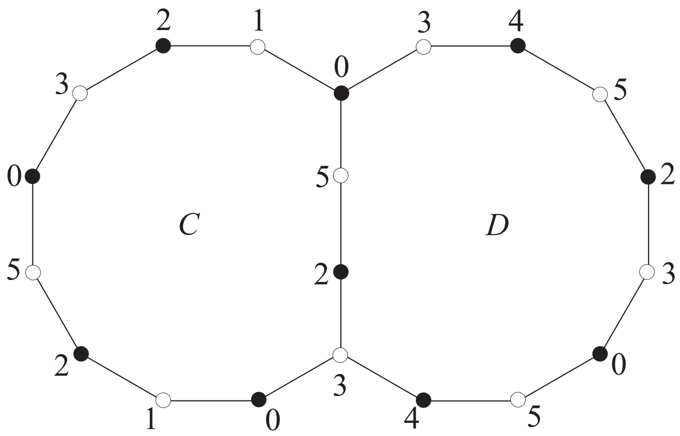

Figure 4: Two adjacent 12-cycles in X with a common 3-path.

multiple edges). Hence Y is connected, and Claim 2 is proved.

Next we discuss automorphisms of X in relation to their action on Y. Denote by G the largest subgroup of A^0 (the group that fixes the two parts of the bipartition of X) which fixes also the sets $\mathcal{C}^+$ and $\mathcal{C}^-$, the two parts of the bipartition of Y.

Claim 3. $G = H$.

To each $\alpha \in G$ and each 12-cycle $C \in \mathcal{C}$ we associate a permutation in S_6, call it α_C in the natural way as the "trace" of the image of α on C along the cycle $\alpha(C)$ (see Figure 5). It may be seen that $\alpha_C = \alpha_D$ for any two adjacent 12-cycles C and D (see Figure 5 where $\alpha_C = (024)(35) = \alpha_D$ for the adjacent cycles C and D). Since Y is connected by Claim 2, we have that $\alpha_C = \alpha_D$ for any two 12-cycles of X. The local assignment $\alpha \mapsto \alpha_C$ may thus be extended globally, and consequently the six fibres $\tilde{0}$, $\tilde{1}$, $\tilde{2}$, $\tilde{3}$, $\tilde{4}$ and $\tilde{5}$ are blocks of imprimitivity of G. In other words, G projects and hence $G = H$, as required. This concludes the proof of Claim 3.

We are now ready to show that $A = G$.

If every automorphism of X fixes $\mathcal{C}^+$ and $\mathcal{C}^-$, then $[A : H] \leq 2$ by Claim 3. Similarly, if there are automorphisms in A interchanging $\mathcal{C}^+$ and $\mathcal{C}^-$, then $[A : H] \leq 4$. This puts the upper bound for the order of A to $2^2 \cdot 3^2 \cdot n$. We are now in position to apply the argument based on the

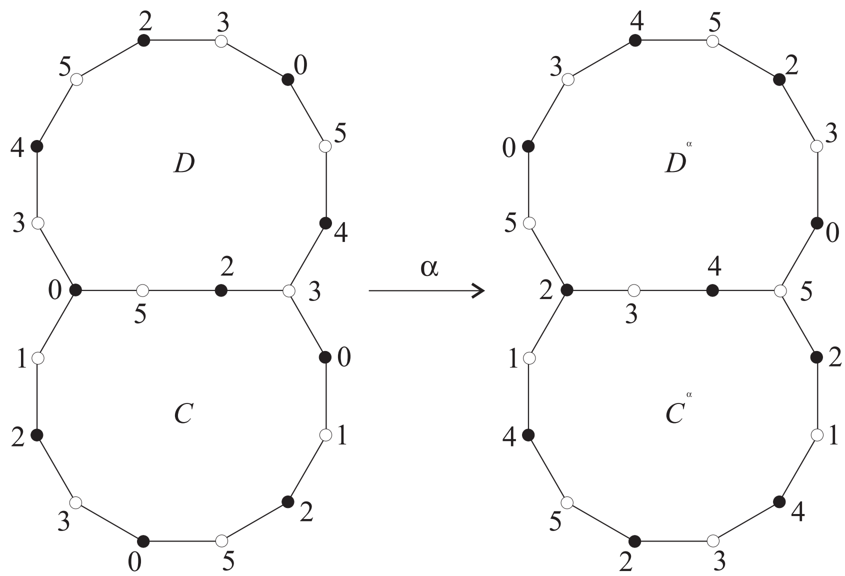

Figure 5: The permutation $\alpha_C \in S_6$ associated with an element of $\alpha \in G$ and a 12-cycle C of X.

Sylow theorem with no restriction on the prime divisors of n. Namely, by the above comments we have $[A : N_A(N)] \in \{1, 2, 4\}$. Also, by assumption, n is not divisible by 3 and therefore by the Sylow theorem, we have that for any prime divisor p of n , the corresponding Sylow p-subgroup of N is normal in A. Hence N is normal in A and so the whole of A projects, forcing $A = H$. Therefore X is semisymmetric with a trivial edge stabilizer (and vertex stabilizer isomorphic to $\mathbb{Z}_3$), as required. ∎

Proof of Theorem 1.1: Theorem 1.1 is a direct consequence of Theorem 3.3 and Corollary 3.2.

4 Minimality of the Gray graph

We start by giving a few lemmas that will be used in the proof of the main result of this section regarding minimality of the Gray graph among cubic semisymmetric graphs. We use the notation of Section 2 for the automorphisms of $K_{3,3}$.

Lemma 4.1 *Let X be a regular $\mathbb{Z}_2^2$-cover of $K_{3,3}$ such that the group $\langle \varphi, \psi \rangle$, where $\varphi = (024)$ and $\psi = (135)$, lifts. Then $\tau_1 = (01)(23)(45)$ lifts.*

PROOF. By Table 1 we see that a and b are distinct and thus linearly independent in $\mathbb{Z}_2^2$. Let $c = c_1 a + c_2 b$ and $d = d_1 a + d_2 b$. From Table 1, using the action of $\varphi^\#$ and $\psi^\#$, we get $c_1 = c_2 = d_1 = 1$ and $d_2 = 0$. Hence $c = a + b$ and $d = a$. Checking Table 1 again, it is easily seen that τ_1 lifts. ■

Lemma 4.2 *Let X be a regular $\mathbb{Z}_2^3$-cover of $K_{3,3}$. Then $\varphi = (024)$ does not lift.*

PROOF. Assume, by way of contradiction, that φ lifts. As in the proof of Lemma 4.1, we see that a and b are distinct and thus linearly independent in $\mathbb{Z}_2^3$. Then either $\{a, b, c\}$ or $\{a, b, d\}$ is a linearly independent set of vectors in $\mathbb{Z}_2^3$. Suppose the first case occurs, and let $d = d_1 a + d_2 b + d_3 c$. Then, using the action of $\varphi^\#$, we have that the image of c under $\varphi^\#$ equals $d_1(a + b) + d_2 a + d_3(c + d) = (d_1 + d_2 + d_1 d_3)a + (d_1 + d_2 d_3)b + (d_3 + d_3^2)c$. Consequently $1 + d_3 + d_3^2 = 0$ in $\mathbb{Z}_2$, a contradiction. ■

For $n \geq 3$ we let the symbol D_{2n} denote the dihedral group of order $2n$.

Lemma 4.3 *Let $n \geq 3$ be an integer and let X be a regular D_{2n}-cover of $K_{3,3}$. Then $\varphi = (024)$ does not lift.*

PROOF. Assuming that φ lifts we have form Table 1 that a and b are either both reflections or both rotations. (Note that we are using additive notations!) But if a and b are reflections then $\varphi^\#$ sends a to a rotation, which is not possible. So both a and b are rotations, and similarly c and d are rotations too. But then X is disconnected, a contradiction. ■

We now turn to covers of Q_3. In order to reconstruct such covers by voltages valued in a given group (of covering transformations) N, we choose the tree carrying identity voltages as shown in Figure 6. Furthermore let $a, b, c, d, e \in N$ be the voltages of the remaining cotree arcs $(3,6)$, $(1,4)$, $(4,7)$, $(7,2)$ and $(2,1)$, respectively. This particular cover of Q_3 will be referred to as a regular N-cover of Q_3 with *voltage-quintuple* (a, b, c, d, e) and will be denoted by the symbol $X(N; a, b, c, d, e)$

Given a group N, let us assume that $X = X(N; a, b, c, d, e)$ is a regular N-cover of $Y = Q_3$, where N is normal in some edge-transitive subgroup H of Aut$X$. Clearly, an arbitrary edge-transitive group of automorphisms of

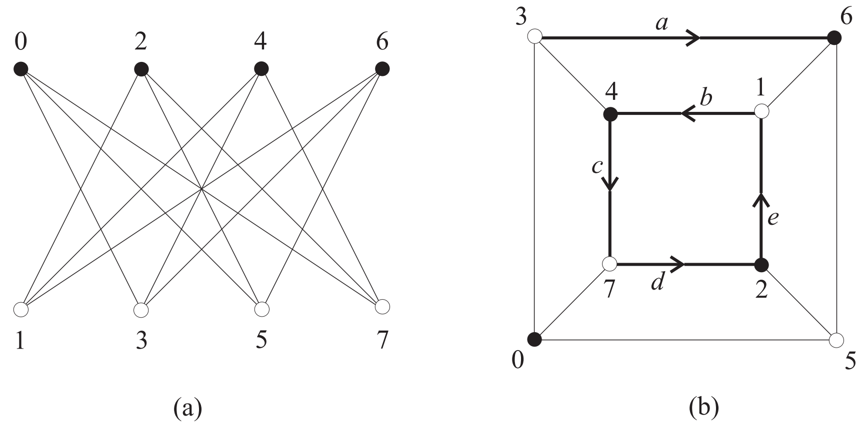

Figure 6: The voltage assignment in Q_3.

Y must contain automorphisms $\alpha = (246)(357)$ and $\beta = (02)(13)(46)(57)$. (Note that $\langle \alpha, \beta \rangle \cong A_4$.) Let $\gamma = \alpha\beta\alpha^{-1} = (040(15)(26)(37)$ and let $\tau = (01)(23)(45)(67)$. The reader may verify, by checking the fundamental cycles 36503, 1430561, 47034, 72507 and 21652 of Y, that $\alpha^{\#}$, $\beta^{\#}$, $\tau^{\#}$ and $\omega^{\#}$ map the voltages a, b, c, d and e as follows (we use the additive notation for the operation in N):

TABLE 2: Voltages of the images of fundamental cycles

	36503	1430561	47034	72507	21652
	a	b	c	d	e
$\alpha^{\#}$	$-d$	$d+e$	$-a$	c	$-b-e-d-c$
$\beta^{\#}$	$b+c+d+e$	$a-e-d-c$	e	d	c
$\gamma^{\#}$	$d+e+b+c$	$-c-b-e$	c	$a+b$	e
$\tau^{\#}$	$-d-c-b-e$	$e+b+c$	$-e$	$-a-b$	$-c$

Lemma 4.4 *Let N be abelian and let X be a connected regular N-cover of Q_3 such that the group $\langle \alpha, \beta \rangle$, where $\alpha = (024)(357)$ and $\beta = (02)(13)(46)(57)$, lifts. Then the automorphism $\tau = (01)(23)(45)(67)$ lifts and consequently X is vertex-transitive.*

PROOF. Since $\langle\alpha, \beta\rangle$ lifts it follows that $\gamma = \alpha\beta\alpha^{-1}$ lifts, too. From rows 5 and 6 of Table 2 we see that $\text{vol}(\tau(C)) = \text{vol}(C)$, for every fundamental cycle C of Q_3. As N is abelian, the mapping $x \mapsto -x$, for all $x \in N$, is a group automorphism. Hence $\tau^{\#} = -\gamma^{\#}$ extends to a group automorphism. Consequently, τ lifts and so X is vertex-transitive. ■

Lemma 4.5 *Let X be a conected regular D_6-cover of Q_3. Then the group $\langle\alpha, \beta\rangle$, where $\alpha = (024)(357)$ and $\beta = (02)(13)(46)(57)$, does not lift.*

PROOF. Assuming that $\langle\alpha, \beta\rangle$ lifts we have that γ lifts, too. Checking Table 2 we have immediately that a, c, d, e are reflections and b is a rotation. (Now this is true in general for any D_{2n}, $n \geq 3$.) Also, b is nontrivial, for otherwise using the action of $\alpha^{\#}$ (switching to multiplicative notation) we get $de = 1$, and so $e = d$ and $aedc = 1$. Therefore $c = a$. Applying the action of $\gamma^{\#}$ we have from column 3 and row 4 that $e = c$. Therefore $a = c = d = e$ and so $\langle a, c, d, e\rangle \cong \mathbb{Z}_2$, contradicting connectedness of X. We may therefore assume that $b \neq 1$. Then de is either b or b^{-1}. In the first case we derive a contradiction using the action of $\alpha^{\#}$, whereas in the second case, a contradicton is obtained using the action of $\gamma^{\#}$. ■

The next lemma concerns covers of the Heawood graph. Let $U = \{0, \ldots, 6\}$ and $U' = \{0', \ldots, 6'\}$ be two copies of $\mathbb{Z}_7$. The Heawood graph $\mathcal{H}$ has $V = U \cup U'$ as the vertex set, with $i \in U$ adjacent to $j' \in U'$ if and only if $j - i \in \{1, 2, 4\}$. Note that the unique minimal arc-transitive subgroup of Aut$\mathcal{H}$ (up to conjugation in Aut$\mathcal{H}$) is generated by the following permutations:

$$\begin{array}{lll} \rho: & i \mapsto i+1, & i' \mapsto (i+1)', \\ \sigma: & i \mapsto 2i, & i' \mapsto (2i)', \\ \tau: & i \mapsto -i, & i' \mapsto (-i)'. \end{array}$$

Moreover, the unique minimal edge-transitive subgroup of Aut$\mathcal{H}$ (up to conjugation in Aut$\mathcal{H}$) is the group $H = \langle\rho, \sigma\rangle$.

Lemma 4.6 *Let X be a regular $\mathbb{Z}_3$-cover of the Heawood graph such that the automorphism ρ lifts. Then the automorphism τ lifts.*

PROOF. Without loss of generality we can assume that the arcs of the Hamiltonian path $50'61'02'13'24'35'46'$ have the trivial voltage. Let e be the voltage of the arc $56'$ and x_i the voltage of the arc $i(i+4)'$, for $i \in \mathbb{Z}_7$. Considering the base cycles determined by the Hamiltonian path and their images under the automorphisms ρ and τ, we obtain that $\rho^\#$ and $\tau^\#$ map the voltages as follows:

	e	x_0	x_1	x_2	x_3	x_4	x_5	x_6
$\rho\#$	e	x_1	x_2	$e+x_3$	x_4	$-e+x_5$	x_6	x_0
$\tau^\#$	$-e$	$-e-x_3$	$-x_2$	$-x_1$	$e-x_0$	$e-x_6$	$-x_5$	$-e-x_4$

Since ρ lifts, there is $\lambda \in \mathbb{Z}_3^*$ such that $\rho^\#(x) = \lambda x$ for $x \in \{e, x_0, \ldots x_6\}$. Since ρ is of order 7, we have that $\lambda^7 = 1$ and hence $\lambda = 1$. This implies $x_0 = x_1 = x_2 = x_5 = x_6$ and $x_3 = x_4 = -e + x_0$. But then $\tau^\#(x) = -x$ for $x \in \{e, x_0, \ldots x_6\}$. This shows that τ lifts. ■

Let $d_1, d_2 \ldots, d_r$ be distinct positive integers. A graph X is said to be a $\{d_1, d_2, \ldots, d_r\}$-graph provided the valency of any vertex of X is one of $d_1, d_2, \ldots, d_r$. For a graph X and a normal subgroup of $\text{Aut}X$ we define the *quotient graph* X/N *of* X *relative to* N as the graph with vertex set coinciding with the set of orbits of N, where two orbits are adjacent in X/N if there is an edge in X joining the two orbits.

The proof of the next lemma is left to the reader.

Lemma 4.7 *Let X be a G-edge-transitive graph, where $G \leq \text{Aut}X$, of valency d and let N be a normal subgroup of* $\text{Aut}X$. *Then the quotient graph X/N is a $\{d_1, d_2, \ldots, d_r\}$-graph, where each d_i divides d.*

We are now ready to prove that the Gray graph is the minimal cubic semisymmetric graph.

Proof of Theorem 1.2. By [6], there are no semisymmetric graphs of order less than 20 or of orders $2p$ or $2p^2$, where p is a prime. Moreover, by [13, 18] there are no semisymmetric cubic graphs of order $4p$, for $p \geq 5$ a prime. Also, in view of [17] the Gray graph is the only cubic semisymmetric graph of order 54. Therefore it suffices to show that there are no semisymmetric cubic graphs of orders $24 = 2^3 \cdot 3$, $32 = 2^5$, $36 = 2^2 \cdot 3^2$, $40 = 2^3 \cdot 5$, $42 = 2 \cdot 3 \cdot 7$ or $48 = 2^4 \cdot 3$. Note that order 24 was excluded by Ivanov (see [13]). However, we include it in our discussion for the sake of completeness.

By way of contradiction, suppose that X is a semisymmetric cubic graph of one of the above orders. Now, a vertex stabilizer of a cubic semisymmetric graph has order $2^k \cdot 3$, for some $k \leq 7$ [8]. Therefore, with the exception of orders 40 and 42, the automorphism group of X is a $\{2,3\}$-group and so solvable by the well known Burnside $p^a q^b$-theorem. In view of this it seems natural to distinguish two separate cases depending on whether the automorphism group $A = \mathrm{Aut}X$ is solvable or not.

CASE 1: A is nonsolvable.

As mentioned above we have $|V(X)| \in \{40, 42\}$ and $|A| = 2^k \cdot 3 \cdot 5$ for some $k \leq 9$ or $|A| = 2^l \cdot 3^2 \cdot 7$ for some $l \leq 8$, respectively, by [8]). Then the minimal normal subgroup N of A is simple and hence, by the classification of finite simple $\{2,3,5\}$-groups and $\{2,3,7\}$-groups, we have $N \cong A_5$ or $N \cong PSL(2,8)$, respectively. Since the action of A is semisymmetric, it folows by Lemma 4.7 that the quotient graph X/N is a $\{1,3\}$-graph, admitting an edge- but not vertex-transitive action of A/N. In particular, X/N is bipartite. Hence X/N is either a cubic graph or isomorphic to $K_{1,3}$ or to K_2.

Observe that $[A : N]$ is a 2-group. Hence the first two possibility cannot occur. We are therefore left with K_2 as the only possibility for X/N, meaning that N acts semisymmetrically on X. Note that any two cyclic subgroups of order 3 in $N \cong A_5$ as well as in $N \cong PSL(2,8)$ are conjugate in $\mathrm{Aut}N$. It is then not hard to see that $\mathrm{Aut}\,X$ contains an element interchanging the two parts of the bipartition, contradicting semisymmetry of X.

CASE 2: A is solvable.

Let $N \cong \mathbb{Z}_p^r$ be the minimal normal elementary abelian subgroup of A. As in the previous case, we consider the action of the quotient group A/N on the quotient graph X/N, which is again either cubic, or isomorphic to $K_{1,3}$ or to K_2. Since N is abelian, the latter possibility would clearly imply the existence of an automorphism interchanging the two parts of the bipartition. Further, it is easily seen that the second possibility implies $|V(X)| = 2 \cdot 3^k$, and therefore it cannot occur as none of the orders in question is of such a form. We may therefore assume that X/N is a (smaller) cubic graph admitting a semisymmetric action of A/N. Let us now go over all the possibilities for X.

SUBCASE 2.1: $|V(X)| = 24$.

Since by [3] there are no edge-transitive graphs of order 12, we have that X/N has order 6 or 8. Hence X is either a regular $\mathbb{Z}_3$-cover of Q_3 or a regular $\mathbb{Z}_4$-cover or a regular $\mathbb{Z}_2^2$-cover of $K_{3,3}$. In both cases X is vertex-transitive in view of Lemmas 2.1, 4.1 and 4.4.

SUBCASE 2.2: $|V(X)| = 32$.

Now X/N is of order 4, 8 or 16 and so we have the following possibilities for X. In the first case it is the homological $\mathbb{Z}_2^3$-cover of K_4 and hence vertex-transitive. In the second case it is a regular $\mathbb{Z}_2^2$-cover of Q_3, and thus vertex-transitive by Lemma 4.4. In the second case it is a regular $\mathbb{Z}_2$-cover of the unique vertex- and edge-transitive graph on 16 vertices, the so called Moebius-Kantor graph. However, the latter is a regular $\mathbb{Z}_2$-cover of Q_3, and hence X is either a regular $\mathbb{Z}_4$-cover or a regular $\mathbb{Z}_2^2$-cover of Q_3. Thus X is vertex-transitive in view of Lemma 4.4, a contradiction.

SUBCASE 2.3: $|V(X)| = 36$.

Since X/N is of even order, it follows that X is a regular $\mathbb{Z}_2$-cover of a bipartite edge-transitive graph of order 18. But such a graph is necessarily also vertex-transitive and thus arc-transitive. By [3], the Pappus graph is the only such graph. But this graph is a regular $\mathbb{Z}_3$-cover of $K_{3,3}$. Hence X is either a regular $\mathbb{Z}_6$-cover or a regular D_6-cover of $K_{3,3}$. By Lemmas 2.1 and 4.3, we have that X vertex-transitive, a contradiction.

SUBCASE 2.4: $|V(X)| = 40$.

If $p = 5$ we have that X is a regular $\mathbb{Z}_5$-cover of Q_3, and thus vertex-transitive by Lemma 4.4. If $p = 2$, then X/N is a bipartite graph of order 10 or 20 admitting a semisymmetric group action. But the generalized Petersen graph $GP(10, 3)$, the Levi graph of the Desarguesian configuration, is the only graph meeting these requirements. However, its automorphism group does not contain a solvable semisymmetric group.

SUBCASE 2.5: $|V(X)| = 42$.

First, observe that $p = 2$ cannot occur for arithmetic reasons. If $p = 3$ it follows that X is a regular $\mathbb{Z}_3$-cover of the Heawood graph and therefore vertex-transitive by Lemma 4.6. Finally, if $p = 7$ we have that X is a regular $\mathbb{Z}_7$-cover of $K_{3,3}$, and thus vertex-transitive by Lemma 2.1.

SUBCASE 2.6: $|V(X)| = 48$.

If $p = 3$, then X is a regular $\mathbb{Z}_3$-cover of the Moebius-Kantor graph and as such it is either a regular $\mathbb{Z}_6$-cover or a regular D_6-cover of Q_3, and hence vertex-transitive by Lemmas 4.4 and 4.5. Suppose now that $p = 2$. Then the order of X/N is either 6 or 24. In the first case X is a regular $\mathbb{Z}_2^3$-cover of $K_{3,3}$, and thus vertex-transitive, or a regular $\mathbb{Z}_2$-cover of the unique arc-transitive graph of order 24, the generalized Petersen graph $GP(12, 5)$. However, the latter happens to be a regular $\mathbb{Z}_3$-cover of Q_3. Consequently, X is again either a regular $\mathbb{Z}_6$- or a regular D_6-cover of Q_3. In both cases X is vertex-transitive.

All of these contradictions complete the proof of Theorem 1.2. ■

References

[1] I. Z. Bouwer, An edge but not vertex transitive cubic graph, *Bull. Can. Math. Soc.* **11** (1968), 533–535.

[2] I. Z. Bouwer, On edge but not vertex transitive regular graphs, *J. Combin. Theory, B* **12** (1972), 32–40.

[3] I. Z. Bouwer (ed.), *The Foster Census*, Charles Babbage Research Centre, Winnipeg, 1988.

[4] P. Cameron, Finite permutation groups and finite simple groups, *Bull. London Math. Soc.* **13** (1981), 1–22.

[5] J. D. Dixon and B. Mortimer, "Permutation Groups", Springer-Verlag, New York, 1996.

[6] J. Folkman, Regular line-symmetric graphs, *J. Combin. Theory* **3** (1967), 215–232.

[7] C. Godsil, On the full automorphism group of Cayley graphs, *Combinatorica* **1** (1981), 143–156.

[8] D. Goldschmidt, Automorphisms of trivalent graphs, *Ann. Math.* **111** (1980), 377–406.

[9] D. Gorenstein, "Finite Groups", Harper and Row, New York, 1968.

[10] D. Gorenstein, "Finite Simple Groups, An Introdction To Their Classification", Plenum Press, New York, 1982.

[11] J. L. Gross and T. W. Tucker, "Topological Graph Theory", Wiley–Interscience, New York, 1987.

[12] M. E. Iofinova and A. A. Ivanov, Biprimitive cubic graphs, Investigations in Algebraic Theory of Combinatorial Objects, (Proceedings of the seminar, Institute for System Studies, Moscow, 1985) Kluwer Academic Publishers, London, 1994, pp 459–472.

[13] A. V. Ivanov, On edge but not vertex transitive regular graphs, *Ann. Discrete Math.* **34** (1987), 273–286.

[14] M. H. Klin, On edge but not vertex transitive graphs, *Coll. Math. Soc. J. Bolyai*, (25. Algebraic Methods in Graph Theory, Szeged, 1978), Budapest, 1981, 399–403.

[15] F. Lazebnik and R. Viglione, An infinite series of regular edge- but not vertex-transitive graphs, manuscript.

[16] A. Malnič, R. Nedela, M. Škoviera, Lifting graph automorphisms by voltage assignments, *Europ. J. Combin.* **21** (2000), 927–947.

[17] A. Malnič, D. Marušič and C. Q. Wang, Cubic edge-transitive graphs of order $2p^3$, submitted.

[18] D. Marušič and P. Potočnik, Semisymmetry of generalized Folkman graphs, *European J. Combin.* **22** (2001), 333–349.

[19] D. Marušič and T. Pisanski, The Gray graph revisited, *J. Graph Theory*, (2000).

[20] M. Škoviera, A contribution to the theory of voltage graphs, *Discrete Math.* **61** (1986), 281–292.

[21] W. T. Tutte, A family of cubical graphs, *Proc. Cambridge Phil. Soc.*, **43**(1948), 459–474.

[22] H. Wielandt, "Finite Permutation Groups", Academic Press, New York, 1964.

[23] S. Wilson, A worthy family of semisymmetric graphs, manuscript.