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Abstract

Using covering graph techniques, a structural result about connected cubic simple graphs admitting an edge-transitive solvable group of automorphisms is proved. This implies, among others, that every such graph can be obtained from either the 3-dipole Dip_3 or the complete graph K_4, by a sequence of elementary-abelian covers. Another consequence of the main structural result is that the action of an arc-transitive solvable group on a connected cubic simple graph is at most 3-arc-transitive, and similarly, the action of an edge- but not vertex-transitive solvable group is at most 5-path-transitive. Moreover, new infinite families of semisymmetric cubic graphs are constructed, arising as regular elementary abelian covering projections of $K_{3,3}$, the Heawood graph and the Moebius-Kantor graph.

1 Introduction

Throughout this paper graphs are assumed to be finite, and unless specified otherwise, simple, undirected and connected. It transpires that, when investigating edge-transitive cubic (simple) graphs the concept of graph coverings plays a central role. A correct treatment of this concept calls for a more general definition of a graph (see Section 2), with the class of simple graphs as a special case.

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The study of cubic arc-transitive graphs has its roots in the classical result of Tutte [19], who proved that cubic graphs are at most 5-arc-transitive. A number of articles on the subject followed over the years, some of them of purely combinatorial content, others linking this topic of research to group theory and to the theory of maps on surfaces [17]. On the other hand, regular edge- but not vertex-transitive graphs (cubic in particular) have also received considerable attention [2, 3, 7, 6, 9, 10, 11].

In this article we deal with cubic graphs admitting an edge-transitive solvable subgroup of automorphisms. Using covering graph techniques we prove a structural reduction theorem (see Theorem 4.4) which implies, among others, that every such graph can be obtained from either the 3-dipole Dip_3 or the complete graph K_4, by a sequence of elementary abelian covers (see Corollary 4.5). Another intersting consequence of Theorem 4.4 is that the action of an arc-transitive solvable group on a connected cubic simple graph is at most 3-arc-transitive, and similarly, the action of an edge- but not vertex-transitive solvable group is at most 5-path-transitive (see Corollaries 4.6 and 4.7).

This article is organized as follows. In Section 2 we introduce additional notation and formal definitions pertaining to graph coverings. In Section 3 we give a complete classification of edge-transitive elementary abelian covering projections onto Dip_3, K_4, and $K_{3,3}$. These results are then used in Section 4, where the proof of Theorem 4.4 is given. Finally, an application of the above is presented in Section 5, with a special emphasis to new constructions of cubic edge- but not vertex-transitive graphs, arising as elementary abelian covers of $K_{3,3}$, the Heawood graph, and the Moebius-Kantor graph.

2 Preliminaries

A *graph* is an ordered 4-tuple $(D, V; \mathrm{beg}, \mathrm{inv})$ where D and $V \neq \emptyset$ are disjoint finite sets of *darts* and *vertices*, respectively, $\mathrm{beg} : D \to V$ is a mapping which assigns to each dart x its *initial vertex* $\mathrm{beg}\, x$, and $\mathrm{inv} : D \to D$ is an involution which interchanges every dart x and its *inverse dart* x^{-1}. The orbits of inv are called *edges*. An edge is called a *semiedge* if $\mathrm{inv}\, x = x$, a *loop* if $\mathrm{inv}\, x \neq x$ while $\mathrm{beg}\,(x^{-1}) = \mathrm{beg}\, x$, and is called a *link* otherwise. The set of *boundary vertices of an edge* e, denoted by ∂e, comprises the initial vertices of the darts contained in the edge. Two edges are *parallel* if they have the same boundary vertices. The *valency* of a vertex v is the number of darts having v as their initial vertex. A graph with no semiedges, no loops and no parallel links is referred to as *simple*. A simple graph with vertex-set V and edge-set E is isomorphic to the graph $(D, V; \mathrm{beg}, \mathrm{inv})$, where $D = \{(u, v) \mid \{u, v\} = \partial e, e \in E\}$, $\mathrm{beg}(u, v) = u$, and $\mathrm{inv}(u, v) = (v, u)$. In view of this fact a simple graph can be defined as an ordered pair (V, E) with vertex-set V and edge-set $E \subseteq \{\{u, v\} \mid u, v \in V, u \neq v\}$. The symbol $u \to v$ is used to denote the dart (u, v). Formal definitions of *graph morphisms, mono-, epi-* and *automorhisms* is left to the reader. Note that all functions, unless explicitly stated otherwise, are composed on the left.

Let k be a nonnegative integer. A *walk* of length k in a graph X is a sequence $v_1, x_1, v_2, x_2, \ldots, v_k, x_k, v_{k+1}$ of vertices and darts such that $v_i = \operatorname{beg} x_i$ for each $i = 1, 2, \ldots, k$ and $v_{i+1} = \operatorname{beg} \operatorname{inv} x_i$ for each $i = 1, 2, \ldots, k$. A walk of length 0 is called *trivial* and contains a single vertex. A *reduced walk* is a walk such that no two consecutive darts are inverse to each other. A reduced walk of length k is also called a *k-arc*. A *k-path* corresponding to a given k-arc is the underlying subgraph of X, containing the darts and vertices of this k-arc. Let s be a nonnegative integer and let G be a subgroup of the automorphism group $\operatorname{Aut} X$ of a graph X. We say that X is *(G, s)-arc-transitive* if G acts transitively on the set of s-arcs of X, and that it is *(G, s)-path transitive* if G acts transitively on the set of s-paths of X. We use the terms *arc-transitive* and *edge-transitive* instead of 1-arc transitive and 1-path-transitive, respectively, and the term *vertex-transive* instead of 0-arc-transitive. A graph X is *G-semisymmetric* if all the vertices of X have constant valency and is G-edge-transitive but not G-vertex-transitive. If the group G in the above definitions is the full automorphism group, then the symbol G is omitted. Let G be an arc-transitive subgroup of $\operatorname{Aut} X$ and let $H \leq G$ be an edge-transitive subgroup of $\operatorname{Aut} X$. The pair (H, G) is a *minimal edge-arc-transitive pair of* X if H contains no proper edge-transitive subgroups and G contains no proper arc-transitive subgroups containing H.

Let $X = (D, V; \operatorname{beg}, \operatorname{inv})$ be a graph and $N \leq \operatorname{Aut} X$ a subgroup of automorphisms of X. Let D_N and V_N denote the sets of its orbits on darts and vertices of X, respectively, and let $\operatorname{beg}_N [x] = [\operatorname{beg} x]$ and $\operatorname{inv}_N [x] = [\operatorname{inv} x]$. This defines a graph $X_N = (D_N, V_N; \operatorname{beg}_N, \operatorname{inv}_N)$ together with a natural epimorphism $\wp_N \colon X \to X_N$ called the *quotient projection relative to* N. Moreover, any graph morphism isomorphic (in the corresponding category of pairs) to a quotient projection is itself called a *quotient projection.* A quotient projection $\wp_N$ is called a *regular N-covering projection* whenever it is valency preserving, that is, whenever N acts without fixed points on D. In this context the group N, also known as the *group of covering transformations*, acts regularly on each of its orbits of darts, and all of these orbits have the same size. *Trivial* covering projections, that is, those with the group of covering transformations being trivial, are excluded from our considerations unless explicitly stated otherwise. An *isomorphism of covering projections* $\wp \colon \tilde{X} \to X$ and $\wp' \colon \tilde{X}' \to X'$ is an ordered pair $(\alpha, \tilde{\alpha}) \colon \wp \to \wp'$ of graph isomorphisms $\alpha \colon X \to X'$ and $\tilde{\alpha} \colon \tilde{X} \to \tilde{X}'$ such that $\wp' \tilde{\alpha} = \alpha \wp$. The isomorphism $\tilde{\alpha}$ is called the *lift* of α and α is the *projection* of $\tilde{\alpha}$. An isomorphism of the form $(\tilde{\alpha}, \mathrm{id})$ is called an *equivalence.* In particular, the group of *selfequivalences* of the same covering projection constitutes the group of covering transformations. We shall be mainly interested with lifts of automorphisms of a graph X along a given regular covering projection $\wp \colon \tilde{X} \to X$. Let G be a subgroup of $\operatorname{Aut} X$ and let $\alpha \in \operatorname{Aut} X$. A covering projection $\wp$ is *G-admissible* if every automorphism in G has a lift, and is *α-admissible* if it is $\langle \alpha \rangle$-admissible. A regular G-admissible covering projection is *minimal* if it cannot be written as a composition of two regular covering projections such that the lifted group $\tilde{G}$ successively projects along this decomposition; equivalently, the group of covering transformations is a minimal normal subgroup in the lifted group $\tilde{G}$ [14]. A covering projection $\wp \colon \tilde{X} \to X$ is *edge-transitive, arc-transitive* or *semisymmetric* if the largest subgroup of Aut X that lifts along $\wp$ is edge-transitive, arc-transitive or semisymmetric, respectively. Observe that the covering graph $\tilde{X}$ can fail to be semisymmetric even if the corresponding covering projection is semisymmetric. For details on combinatorial treatment of covering projections and consequently, of the problem of liting automorphisms in terms of voltages, we refer the reader to [8, 13, 14]. Below we only briefly summarize some facts related to a special case which will be needed further on.

Let p be a prime. A regular covering projection $\wp_N : \tilde{X} \to X$ is called *p-elementary abelian* (or just *elmentary abelian* if p is unspecified or clear from the context) whenever the group N is isomorphic to $\mathbb{Z}_p^k$. Such covers were extensively studied in [14]. Let $\mathcal{T}$ be a spanning tree of a graph X, and let $\zeta \colon D(X) \to \mathbb{Z}_p^k$ be such that $-\zeta(x) = \zeta(x^{-1})$ for each $x \in D(X)$, $\zeta(x) = 0$ for each $x \in D(\mathcal{T})$, and such that the image of ζ generates the group $\mathbb{Z}_p^k$. A function ζ satisfying these requirements is called a *voltage assignment* on X, and determines a regular $\mathbb{Z}_p^k$-covering projection $\wp_\zeta \colon \mathrm{Cov}(\zeta) \to X$ as follows. The graph $\mathrm{Cov}(\zeta)$ has $D(X) \times \mathbb{Z}_p^k$ and $V(X) \times \mathbb{Z}_p^k$ as the sets of darts and vertices, respectively, with $\mathrm{beg}(x, \nu) = (\mathrm{beg}\, x, \nu)$ and $\mathrm{inv}(x, \nu) = (\mathrm{inv}\, x, \nu + \zeta(x))$. The corresponding projection $\wp_\zeta$ is defined as the projection onto the first component. A *p-homological* (or just *homological* when p is clear form the context) covering projection is a regular $\mathbb{Z}_p^r$-covering projection, where $r = \dim H_1(X; \mathbb{Z}_p)$ is the dimension of $H_1(X; \mathbb{Z}_p)$, the first homology group of X with $\mathbb{Z}_p$ as the coefficient ring, treated as a vector space over $\mathbb{Z}_p$.

Let α be an automorphism of the graph X. Since α maps a cycle of X to a cycle of X, there is a natural action of α on $H_1(X; \mathbb{Z}_p)$, inducing a linear transformation $\alpha^{\#}$ of $H_1(X; \mathbb{Z}_p)$. The mapping $\# \colon \mathrm{Aut}\, X \to \mathrm{GL}(H_1(X; \mathbb{Z}_p))$ defined by $\alpha \mapsto \alpha^{\#}$ is in fact a group homomorphism. The problem of finding all p-elementary abelian α-admissible covering projections of a graph X can be solved effectively as follows [14, Corollary 6.5]: First choose a spanning tree $\mathcal{T}$ of X. Let $\{e_i \mid i = 1, 2, \ldots r\}$ be the set of edges of X not contained in $\mathcal{T}$, and let x_i be one of the darts of e_i, $i = 1, 2, \ldots, r$. The set of darts $\{x_1, x_2, \ldots, x_r\}$ naturally defines a basis $\mathcal{B}_\mathcal{T}$ of $H_1(X; \mathbb{Z}_p)$. Next, let $A \in \mathrm{GL}(\mathbb{Z}_p^r)$, where $\mathbb{Z}_p^r$ is treated as a column vector space, be the matrix representing $\alpha^{\#}$ relative to the basis $\mathcal{B}_\mathcal{T}$. Then there is a bijective correspondence between all α-admissible p-elementary abelian covering projections (up to equivalence of covering projections) and the invariant subspaces of the transposed matrix A^t. In particular, if U is an A^t-invariant subspace of the column vector space $\mathbb{Z}_p^r$, spanned by a basis $\{u_1, u_2, \ldots, u_k\}$, and Q is a matrix with rows $u_1^t, u_2^t, \ldots, u_k^t$, then the voltage assignment ζ, mapping x_i to the i^{th} coloumn of Q, $i = 1, 2, \ldots r$, and mapping all darts of $\mathcal{T}$ to 0, gives rise to a regular α-admissible covering projection. Note that minimal α-admissible covering projections correspond to minimal invariant subspaces of A^t. Also, two regular α-admissible covering projections are isomorphic if and only if there is a graph automorphism $\beta \in \mathrm{Aut}\, X$ such that the its corresponding matrix B^t maps one of the respective invariant subspaces to the other [14].

Let X be a graph. If $H, H' \leq \mathrm{Aut}\, X$ are two conjugate subgroups, and if a regular covering projection $\tilde{X} \to X$ is H-admissible, then there is an isomorphic covering projection which is H'-admissible. Thus, when faced with the problem of finding all edge-transitive regular covering projections of X (up to isomorphism of covering projections) it suffices to consider all H-admissible covering projections, where H runs through a complete set of representatives of conjugacy classes of minimal edge-transitive subgroups of $\mathrm{Aut}\, X$. This suggests the following simplification when searching for semisymmetric covering projections of X. For each minimal edge-transitive subgroup H, up to conjugacy, it is enough to check, for all pairs (H, G) where G is a minimal arc-transitive subgroup of $\mathrm{Aut}\, X$ containing H, whether G lifts or not.

3 Elementary abelian covers of small cubic graphs

In this section we classify all elementary abelian covers of the 3-*dipole* Dip_3 (that is, the graph with two vertices and three parallel edges), the complete graph on four vertices K_4, and the complete bipartite graph $K_{3,3}$. These three graphs play a central role in the statement of our main result, Theorem 4.4.

Covers of Dip_3

Elementary abelian covers of prime valency dipoles were extensively studied in [14]. We summarize here the results in the special case of the 3-dipole Dip_3.

Proposition 3.1 *Let p be a prime and let $X \to \mathrm{Dip}_3$ be a nontrivial connected edge-transitive $\mathbb{Z}_p^k$-cover of Dip_3. Then one of the following occurs:*

(i) $k = 2$ *and* $X \to \mathrm{Dip}_3$ *is isomorphic to the p-homological covering pojection;*

(ii) $k = 1$, $p = 3$ *and* $X \cong \mathrm{Dip}_3$ *is isomorphic to the covering* $K_{3,3} \to \mathrm{Dip}_3$ *obtained by giving the voltages* 0, 1 *and* 2 *to the three parallel arcs of* Dip_3, *and* 0, 2 *and* 1 *to their respective inverse darts;*

(iii) $k = 1$, $p \equiv 1 \pmod 3$ *and* $X \to \mathrm{Dip}_3$ *is isomorphic to the covering obtained by the voltages* 0, 1 *and* $-\xi$ *as shown in Figure 1 below, where* $\xi \in \mathbb{Z}_p$ *is one of the two elements of order* 3 *in* $\mathbb{Z}_p^*$.

Covers of K_4

Label the vertices of K_4 by the elements of $\mathbb{Z}_4$. The automorphism group $\mathrm{Aut}\, K_4$ is then isomorphic to the symmetric group S_4. It acts regularly on the set of 2-arcs of K_4. The index 2 subgroup G of $\mathrm{Aut}\, K_4$, isomorphic to A_4, acts regularly on the arcs of K_4 and is the unique minimal arc-transitive subgroup of $\mathrm{Aut}\, K_4$. Since K_4 is not bipartite, G is also the unique minimal edge-transitive subgroup of $\mathrm{Aut}\, K_4$. The pair (G, G) is therefore the unique minimal edge-arc-transitive pair of K_4. Let $\rho = (1, 2, 3)$

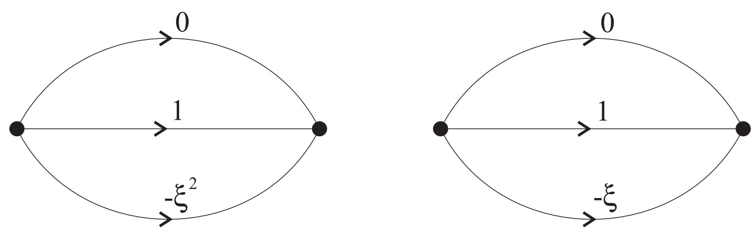

Figure 1: The minimal covers of D_3 in case $p \equiv 1 (\mathrm{mod}\, 3)$.

and $\sigma = (0,1)(2,3)$ be the generators of G. Let the spanning tree $\mathcal{T}$ contain the edges $\{0,1\}$, $\{0,2\}$ and $\{0,3\}$, and let a, b and c denote the elements of $H_1(K_4; \mathbb{Z}_p)$ defined by the tree $\mathcal{T}$ and the darts $1 \to 2$, $2 \to 3$ and $3 \to 1$, respectively. The set $\mathcal{B} = \{a, b, c\}$ is then a basis of the $\mathbb{Z}_p$-vector space $H_1(K_4; \mathbb{Z}_p)$. Let $\#\colon \mathrm{Aut}\, K_4 \to \mathrm{GL}(H_1(K_4; \mathbb{Z}_p))$ be the linear representation of $\mathrm{Aut}\, K_4$ as defined in Section 2, and let $R = [\rho^{\#}; \mathcal{B}, \mathcal{B}]^t$ and $S = [\sigma^{\#}; \mathcal{B}, \mathcal{B}]^t$ be the transposes of matrices representing the linear transformations $\rho^{\#}$ and $\sigma^{\#}$ relative to the basis $\mathcal{B}$, respectively. A straightforward computation shows that:

$$R = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 0 & 0 & 1 \\ -1 & -1 & -1 \\ 1 & 0 & 0 \end{pmatrix}.$$

It transpires that there are no proper non-trivial invariant subspaces of $\langle R, S \rangle$ for odd p. On the other hand, for $p = 2$ there is a proper non-trivial invariant subspace of $\langle R, S \rangle$, namely the 1-dimensional subspace spanned by the vector $(1,1,1)^t$. The corresponding covering graph is isomorphic to the cube Q_3. This implies the following result.

Proposition 3.2 *Let p be a prime, k a positive integer and let $\wp\colon X \to K_4$ be a non-trivial, connected, edge-transitive $\mathbb{Z}_p^k$-covering projection. Then one of the following occurs:*

(i) $k = 3$ and $X \to K_4$ is isomorphic to the p-homological cover of K_4;

(ii) $p = 2$, $k = 1$ and $X \to K_4$ is isomorphic to the canonical double covering $Q_3 \to K_4$.

The above proposition motivates the study of edge-transitive elementary abelian covers of Q_3. All such covers were determined in [12]. It transpires that all of them are arc-transitive.

Covers of $K_{3,3}$

Label the vertices of the complete bipartite graph $K_{3,3}$ by the elements of $\mathbb{Z}_6$ in such a way that the sets $\{0,2,4\}$ and $\{1,3,5\}$ form the bipartition of $K_{3,3}$. Every edge-transitive subgroup of $\mathrm{Aut}\, K_{3,3}$ contains the unique minimal edge-transitive subgroup

H, generated by the permutations $\rho = (0,2,4)$ and $\sigma = (1,3,5)$. It is easy to see that there are precisely three minimal arc-transitive subgroups of $\mathrm{Aut}\, K_{3,3}$ containing H; namely, $G_1 = \langle H, \tau_1\rangle$, $G_2 = \langle H, \tau_2\rangle$ and $G_3 = \langle H, \tau_3\rangle$, where $\tau_1 = (0,1)(2,3)(4,5)$, $\tau_2 = (0,1)(2,5)(4,3)$ and $\tau_3 = (0,1)(2,5,4,3)$. Consequently, the ordered pairs (H, G_i), $i = 1,2,3$, are the only minimal edge-arc-transitive pairs of $K_{3,3}$.

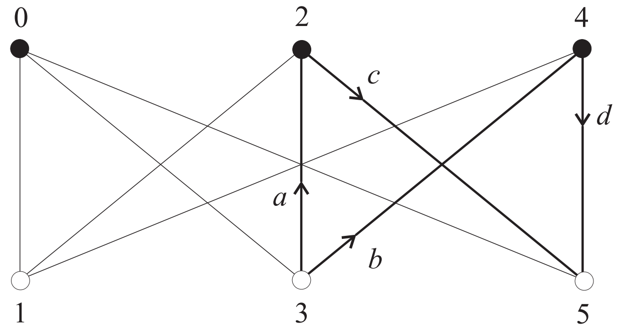

Figure 2: The voltage assignment of $K_{3,3}$.

Let $\mathcal{T}$ be the spanning tree of $K_{3,3}$ containing the edges $\{0,1\}$, $\{0,3\}$, $\{0,5\}$, $\{1,2\}$ and $\{1,4\}$. Let a, b, c and d denote the elements of $H_1(K_{3,3}; \mathbb{Z}_p)$, defined by the tree $\mathcal{T}$ and the darts $3 \to 2$, $3 \to 4$, $2 \to 5$ and $4 \to 5$, respectively. The set $\mathcal{B} = \{a, b, c, d\}$ is then a basis of the $\mathbb{Z}_p$-vector space $H_1(K_{3,3}; \mathbb{Z}_p)$. Let $\#\colon \mathrm{Aut}\, K_{3,3} \to \mathrm{GL}(H_1(K_{3,3}; \mathbb{Z}_p))$ be the linear representation of $\mathrm{Aut}\, K_4$ as defined in Section 2. Further, let $R = [\rho^{\#}; \mathcal{B}, \mathcal{B}]^t$, $S = [(\rho\sigma)^{\#}; \mathcal{B}, \mathcal{B}]^t$, $T_1 = [\tau_1^{\#}, \mathcal{B}, \mathcal{B}]^t$, $T_2 = [\tau_2^{\#}, \mathcal{B}, \mathcal{B}]^t$, and $T_3 = [\tau_3^{\#}, \mathcal{B}, \mathcal{B}]^t$ be the transposes of the matrices representing the linear transformations $\rho^{\#}$, $(\rho\sigma)^{\#}$, $\tau_1^{\#}$, $\tau_2^{\#}$, and $\tau_3^{\#}$ relative to the basis $\mathcal{B}$, respectively. A straightforward computation shows that:

$$R = \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad S = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$T_1 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad T_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad T_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Furthermore, let $\omega_1 = (2,4)$, $\omega_2 = (3,5)$ and $O_i := [\omega_i^{\#}, \mathcal{B}, \mathcal{B}]^t$, for $i = 1,2$. Then

$$O_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad O_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & -0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

The minimal polynomial of S is $m_S(x) = x^3 - 1$ which factors into $(x-1)(x^2+x+1)$ if $p \equiv -1 \pmod 3$, and into $(x-1)(x-\xi)(x-\xi^2)$ if $p \equiv 1 \pmod 3$, where $\xi^2+\xi+1=0$. The set of invariant subspaces of S can be found by computing the kernels of the irreducible factors of $m_S(x)$, valued at S.

Observe first that $K_0 = \mathrm{Ker}\,(S-I) = \langle (1,0,-1,-1)^t, (0,1,1,0)^t \rangle$ is invariant for the matrix R, too. Now if $p \equiv -1 \pmod 3$, then there are no 1-dimensional R-invariant subspaces of K_0. On the other hand, in the case $p \equiv 1 \pmod 3$ there are two 1-dimensional R-invariant subspaces of K_0, that is, $L_1 = \langle (1,-\xi^2,\xi,-1)^t \rangle$ and $L_2 = \langle (1,-\xi,\xi^2,-1)^t \rangle$. As for other S-invariant subspaces we have that, for $p \equiv 1 \pmod 3$, the subspaces $K_1 = \mathrm{Ker}\,(S-\xi I) = \langle (1,-\xi,\xi,\xi^2)^t \rangle$ and $K_2 = \mathrm{Ker}\,(S-\xi^2 I) = \langle (1,-\xi^2,\xi^2,\xi)^t \rangle$ are R-invariant too. If $p \equiv -1 \pmod 3$, then the subspace $J = \mathrm{Ker}\,(S^2+S+I) = \langle (1,0,0,1)^t, (0,1,-1,-1)^t \rangle$ is R-invariant.

For $p \equiv 1 \pmod 3$ the linear transformations T_1, T_2 and T_3 permute the minimal $\langle R, S \rangle$-invariant subspaces K_1, K_2, L_1 and L_2 by the respective rules (K_1, K_2), (L_1, L_2) and (K_1, L_1, K_2, L_2). The above implies that all $\mathbb{Z}_p$-covers associated with K_1, K_2, L_1 and L_2 (as well as all $\mathbb{Z}_p^3$-covers associated with the corresponding complements) are isomorphic. Moreover, there are two non-isomorphic $\mathbb{Z}_p^2$-covers associated with K_1+K_2 (or L_1+L_2) and K_1+L_1 (or K_1+L_2 or K_2+L_1 or K_2+L_2). In the latter case, since none of the transformations T_i, $i=1,2,3$, fixes the subspace K_1+L_1, the associated covering projection is semisymmetric. In fact, the derived covering graph is semisymmetric too, as will be shown in Section 5.

Similarly, for $p \equiv -1 \pmod 3$ the linear transformations T_1, T_2 fix, whereas the linear transformation T_3 interchanges the minimal $\langle R, S \rangle$-invariant subspaces K_0 and J. This implies that all associated covering projections are isomorphic.

Finally, let $p=3$. Define $u_1 = (0,1,1,0)^t$, $v_1 = (1,-1,1,-1)^t$, $v_2 = (1,1,-1,0)^t$, and $v_3 = (-1,0-1,1)^t$, and observe that u_1, v_1, v_2, v_3 form a Jordan basis for the matrix S. Moreover, it can be checked that the notrivial proper invariant subspaces of $(H^{\#})^t = \langle R, S \rangle$ are the following: $V_1 = \langle v_1 \rangle$, $V_2 = \langle v_1, v_2 \rangle$, $K_0 = \langle v_1, u_1 \rangle = \mathrm{Ker}\,(S-I)$, $W_1 = \langle v_1, u_1+v_2 \rangle$, $W_2 = \langle v_1, u_1-v_2 \rangle$, and $V_3 = \langle v_1, v_2, u_1 \rangle = \mathrm{Ker}\,(S-I)^2$. By computation we can check that T_1 and T_2 interchange W_1 with W_2 and fix all others. On the other hand, T_3 interchanges W_1 with W_2 as well as V_2 with K_0, and fixes all others. Hence the covering projections associated with V_2 and K_0 are isomorphic. The same holds for the covering projections associated with W_1 and W_2. Moreover, the only semisymmetric covering projections arise from W_1 (or W_2) since these are the only cases not fixed by any of T_1, T_2 or T_3.

The discussion above is summarized in Table 1 below. Based on the theory developed in [14], the discussion above gives the following proposition.

Proposition 3.3 *Let $X \to K_{3,3}$ be a non-trivial, connected, edge-transitive $\mathbb{Z}_p^k$-covering projection. Then all such pairwise non-isomorphic covering projections arise from voltage assignments given in Table 1.*

Each of the first five rows of this table corresponds to a particular family or a sporadic example, whereas the defining parameters are read from the columns. The *first column gives the corresponding invariant subspace while the next four columns give the voltages (see Figure 2). The last three columns give, respectively, the arithmetic condition for the existence of such a projection, the maximal edge-transitive subgroup of* Aut $K_{3,3}$ *that lifts, and its order.*

Table 1: Edge-transitive elementary abelian covers of $K_{3,3}$

inv. sub.	$\zeta(a)$	$\zeta(b)$	$\zeta(c)$	$\zeta(d)$	condition	group that lifts	its order
K_1	(1)	$(-\xi)$	(ξ)	$(-\xi^2)$	$p \equiv 1 \pmod 3$ $\xi^2+\xi+1=0$	$\langle H, \tau_1 \rangle$	18
K_0 or K_1+K_2	$\begin{pmatrix}1\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\end{pmatrix}$	$\begin{pmatrix}0\\-1\end{pmatrix}$	$\begin{pmatrix}1\\-1\end{pmatrix}$	none	$\langle H, \tau_1, \tau_2 \rangle$	36
K_1+L_1	$\begin{pmatrix}1\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\end{pmatrix}$	$\begin{pmatrix}\xi\\0\end{pmatrix}$	$\begin{pmatrix}0\\\xi\end{pmatrix}$	$p \equiv 1 \pmod 3$ $\xi^2+\xi+1=0$	$\langle H, \omega_1 \rangle$	18
$K_1+K_2+L_1$	$\begin{pmatrix}1\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\1\end{pmatrix}$	$\begin{pmatrix}1\\\xi\\-\xi^2\end{pmatrix}$	$p \equiv 1 \pmod 3$ $\xi^2+\xi+1=0$	$\langle H, \tau_2 \rangle$	18
N	$\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$	none	$\langle H, \tau_1, \tau_2, \tau_3 \rangle$	72
V_1	(1)	(-1)	(1)	(-1)	$p=3$	$\langle H, \tau_1, \tau_2, \tau_3 \rangle$	72
V_2	$\begin{pmatrix}1\\0\end{pmatrix}$	$\begin{pmatrix}-1\\1\end{pmatrix}$	$\begin{pmatrix}1\\-1\end{pmatrix}$	$\begin{pmatrix}-1\\-1\end{pmatrix}$	$p=3$	$\langle H, \tau_1, \tau_2 \rangle$	36
W_1	$\begin{pmatrix}1\\0\end{pmatrix}$	$\begin{pmatrix}-1\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\end{pmatrix}$	$\begin{pmatrix}0\\-1\end{pmatrix}$	$p=3$	$\langle H, \omega_1, \omega_2 \rangle$	36
V_3	$\begin{pmatrix}1\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\1\end{pmatrix}$	$\begin{pmatrix}1\\1\\-1\end{pmatrix}$	$p=3$	$\langle H, \tau_1, \tau_2, \tau_3 \rangle$	72

A word of comment on the data regarding some of the graphs obtained from Table 1 is in order. First, all graphs are arc-transitive except for those obtained from rows 3 and 8 which are semisymmetric. In fact, the graph in row 8 is the Gray graph, the smallest cubic semisymmetric graph, as was shown in [16]. Regarding the semisymmetry of the family of graphs associated with row 3, see Theorem 5.1. Finally, note that the graphs associated with rows 6, 7 and 9, respectively, are the cubic arc-transitive graphs whose respective codes in the Foster Census [4] are **18** (the Pappus graph), **54**, and **162C**.

4 Main results

In this section we analyse the structure of cubic graphs admitting an edge-transitive action of a solvable group of automorphisms. We start by giving three lemmas. The first one is a mere observation, whereas the second one is a bit more complex. It deals with the case when the quotient graph is the tripod $K_{1,3}$, and is central to the proof of Theorem 4.4. The third lemma gives an extension of a particular case covered by Table 1 in Proposition 3.3.

An *n-semistar* is a graph with one vertex and n semiedges.

Lemma 4.1 *Let X be a connected cubic simple graph admitting an edge-transitive group G of automorphisms. Let N be a normal subgroup of G, and let X_N be the quotient graph with respect to the action of N on X. Then X_N admits an edge-transitive action of the solvable group G/N, and is one of the following graphs:*

(i) a connected cubic simple graph;

(ii) the 3-dipole Dip_3;

(iii) the tripod $K_{1,3}$;

(iv) the complete graph K_2;

(v) the 3-semistar s_3.

Proof. The normality of N implies that the quotient graph is a $\{1,3\}$-graph, that is, a graph whose vertices have valencies 1 or 3. Moreover, X_N is obviously edge-transitive. It is easy to see that every edge-transitive $\{1,3\}$-graph is isomorphic to one of the above graphs. ■

Lemma 4.2 *Let X be a connected cubic simple graph admitting an edge-transitive group G of automorphisms. If G contains a normal subgroup $N \cong \mathbb{Z}_p^k$ such that X_N is isomorphic to the tripod $K_{1,3}$, then X is G-semisymmetric and one of the following occurs:*

(i) $k = 1$ and $X \cong K_{3,3}$;

(ii) $k = 2$ and X is isomorphic to the Pappus graph;

(iii) $k = 3$ and X is isomorphic to the Gray graph.

Proof. Since G has a normal subgroup with respect to which the quotient graph is isomorphic to $K_{1,3}$, a graph which is not vertex-tansitive, the action of G on X cannot be vertex-trasitive, and hence is semisymmetric.

Let U_1, U_2, and U_3 be the N-orbits corresponding to three vertices of degree 1 of the tripod $K_{1,3}$, and let W be the N-orbit corresponding to the vertex of valency 3 of the tripod. Clearly, N acts faithfully and hence regularly on W. Thus $|W| = p^k$. Since all the sets U_i are of the same cardinality and $|W| = 3|U_1|$, we have $p = 3$, and hence $|W| = 3^k$ and $|U_i| = 3^{k-1}$, $i = 1, 2, 3$. Moreover, $|E(X)| = 3^{k+1}$.

Let M be a Sylow 3-subgroup of G. By [20, Theorem 3.4] the group M acts transitively on the edge set $E(X)$. By [15, Proposition 2.4], the order of a vertex stabilizer G_v, $v \in V(X)$, is divisible by 3 but not by 9. It follows that $|M| = 3^{k+1}$, implying that M acts regularly on the edge set $E(X)$. Clearly, N is normal in M of index 3. Let μ be an element of order 3 in G_v, $v \in W$, mapping U_i to U_{i+1}. Then $\mu \in M \setminus N$. For each $i \in \{1, 2, 3\}$, let K_i be the kernel of the action of N on U_i. As $|N| = 3|U_i|$, we have that $K_i \cong \mathbb{Z}_3$. Clearly, μ permutes K_1, K_2 and K_3 by conjugation. Let $L = \langle K_1, K_2, K_3 \rangle$. Note that L is normal in N, $L^\mu = L$ and that $M = \langle N, \mu \rangle$. Therefore L is normal in M.

We now count the number of orbits of the action of L on W and $U = U_1 \cup U_2 \cup U_3$. Let $|L| = 3^l$. Observe that $l \leq 3$. Then the number of orbits of the action of L on W is 3^{k-l}. Moreover, the number of orbits of the action of L on U is 3^{k-l+1}. Namely, for each $i \in \{1, 2, 3\}$, the kernel of the action of L on U_i is K_i. Hence $|L/K_i| = 3^{l-1}$. But L/K_i acts semiregularly on U_i. It follows that the number of orbits of L on U_i equals $3^{-l+1}|U_i| = 3^{k-l}$. Therefore the number of orbits of L on U is 3^{k-l+1}. Recall that L is normal in M, and that the latter acts regularly on $E(X)$. The quotient graph X_L is a bipartite $\{1, 3\}$-graph with bipartition sets of respective sizes 3^{k-l} and 3^{k-l+1}. But as it is connected, it can be easily seen that $3^{k-l} = 1$, and so $k = l$. Therefore $L = N$. Recalling that $k = l \leq 3$, we consider three different cases.

Suppose first that $k = 1$. Then $N \cong \mathbb{Z}_3$ and so $X \cong K_{3,3}$. Moreover, N acts trivially on one part of the bipartition and cyclically permutes the other part.

Next, let $k = 2$. Then $N = \mathbb{Z}_3^2$. Let $\{e_1, e_2\}$ be the standard bases of N. We may assume that $K_i = \langle e_i \rangle$ for $i = 1, 2$. Since μ takes by conjugation K_i to K_{i+1}, $i \in \mathbb{Z}_3$, it follows that $K_3 = \langle -e_1 - e_2 \rangle$. Further, by computation we see that μ normalizes the subgroup $T = \langle e_1 - e_2 \rangle \cong \mathbb{Z}_3$. Therefore T is normal in $M = \langle N, \mu \rangle$. Observe that T, being a subgroup of N, acts semiregularly on W, and moreover, acts semiregularly on U since it intersects each K_i trivially. Also, it acts semiregularly on $E(X)$ since it is contained in M. It follows that $X \to X_T$ is a regular $\mathbb{Z}_3$-covering projection, where X_T is a connected cubic graph with 9 edges. Hence $X_T \cong K_{3,3}$. From Table 1 in Proposition 3.3 it may be seen that X is isomorphic to the Pappus graph. Moreover, N acts regularly on one part of the bipartition and has three orbits of length 3 on the other part.

Finally, suppose that $k = 3$. Then $N \cong \mathbb{Z}_3^3$. Let $\{e_1, e_2, e_3\}$ be the standard basis of N. We may assume that $K_i = \langle e_i \rangle$, $i = 1, 2, 3$. Set $T = \langle e_1 - e_2, e_2 - e_3 \rangle \cong \mathbb{Z}_3^2$. Observe that μ normalizes T, implying that T is normal in M. As in the preceeding paragraph T acts semiregulary on X and so the graph X is a regular $\mathbb{Z}_3^2$-cover of $X_T \cong K_{3,3}$. Comparing Table 1 in Proposition 3.3 with the Foster Census [4], the graph X is isomorphic either to the graph with Foster code **54**, or to the Gray graph. In the first case, It can be checked that the unique minimal edge-transitive group has exactly one normal subgroup isomorphic to $\mathbb{Z}_3^2$. This normal subgroup has six orbits of length 9, and so the corresponding quotient is not the tripod, a contradiction. Consequently, the only remaining possibility is that X be isomorphic to the Gray graph. Moreover, N acts regularly on one part of the bipartition and has three orbits of length 3 on the other part. ■

Lemma 4.3 *Let G be an edge-transitive subgroup of* $\mathrm{Aut}\,\mathrm{Gray}$. *Then there exists a regular $\mathbb{Z}_3^2$-covering projection* $\mathrm{Gray} \to K_{3,3}$ *along which G projects, if and only if $|G| \in \{81, 162\}$ or $|G| = 324$ and G has a normal subgroup isomorphic to $\mathbb{Z}_3^2$.*

Proof. From row 8 of Table 1 in Proposition 3.3 we deduce that the maximal edge-transitive subgroup of Aut Gray which projects along Gray $\to K_{3,3}$ has order 324. More precisely, there are two conjugacy classes of subgroups of order 324 in Aut Gray, the first one consisting of a normal subgrop and the second one consisting of four subgroups. For each of these four subgroups there exists a corresponding covering projection Gray $\to K_{3,3}$. Now, as it was checked with MAGMA [1], each of the groups from the statement of the lemma is contained in one of these four subgroups of order 324. ■

Theorem 4.4 *Let X be a connected cubic simple graph admitting an edge-transitive solvable subgroup G of automorphisms. Then G contains a normal subgroup K, possibly trivial, such that one of the following occurs:*

(i) X is a K-regular cover of K_4, where G/K is isomorphic to one of the two edge-transitive subgroups of $\mathrm{Aut}\,K_4$;

(ii) X is a K-regular cover of the dipole Dip_3, *where G/K is isomorphic to one of the four edge-transitive subgroups of* $\mathrm{Aut}\,\mathrm{Dip}_3$;

(iii) X is a K-regular cover of $K_{3,3}$, where G/K is isomorphic to one of the five edge-transitive subgroups of $\mathrm{Aut}\,K_{3,3}$ *which do not project along the regular covering projection* $K_{3,3} \to \mathrm{Dip}_3$;

(iv) X is a K-regular cover of the Gray graph Gray, *and G/K is isomorphic to one of the five edge-transitive subgroups of* Aut Gray *which do not project along the regular covering projection* Gray $\to K_{3,3}$.

Moreover, the covering projection $X \to X_K$ can be decomposed into a sequence of (minimal) elementary abelian covering projections.

Remark. Let us mention that the subgroups isomorphic to G/K, appearing in (i)-(iv) above, are respectively: A_4 and S_4 in case (i); A_3, S_3 and their direct products with $\mathbb{Z}_2$ in case (ii); the groups $\langle \rho, \sigma, \omega_1 \rangle$, $\langle \rho, \sigma, \omega_2 \rangle$, $\langle \rho, \sigma, \omega_1, \omega_2 \rangle$, $\langle \rho, \sigma, \tau_3 \rangle$, and $\mathrm{Aut}\,K_{3,3} = \langle \tau_1, \tau_2, \tau_3 \rangle$, with the notation of Section 3, in case (iii); and the only normal subgroup of order 324 in Aut Gray, all three subgroups of order 648 in Aut Gray, and Aut Gray, a group of order 1296, itself. The relative computations regarding the subgroups of Aut Gray were done with the help of MAGMA [1].

PROOF. Let X be a minimal counterexample to the statement of the theorem, and let $N \triangleleft G$ be the minimal normal subgroup of G. By [18], N is elementary abelian, say, $N \cong \mathbb{Z}_p^k$. Applying Lemma 4.1 we now consider the five possibilities for the quotient graph X_N.

Suppose first that X_N is a connected cubic simple graph. Then the quotient projection is valency preserving and hence $X \to X_N$ is a regular covering projection. By minimality of X there exists a regular covering projection $X_N \to Y$ such that G/N projects, where Y is isomorphic to one of K_4, Dip_3, $K_{3,3}$, or the Gray graph. But the composition of these two coverings is a regular covering projection $X \to Y$ along which G projects, a contradiction.

Suppose next that $X_N \cong \mathrm{Dip}_3$. Since this quotient projection is a covering projection such that G projects, we have an immediate contradiction.

Suppose now that $X_N \cong K_{1,3}$. In view of Lemma 4.2, X is isomorphic to one of the exceptional graphs, that is, $K_{3,3}$, the Pappus graph, or the Gray graph. Clearly, if $X \cong K_{3,3}$ then X falls in (ii) or (iii). Next, let X be isomorphic to the Pappus graph. From row 6 of Table 1 in Proposition 3.3 we have that G projects along $X \to K_{3,3}$, and so the pair (X, G) falls in (ii) or (iii). Finally, suppose that X is isomorphic to the Gray graph. If G projects along $X \to K_{3,3}$ then the pair (X, G) falls in (ii) or (iii), and if G does not project along $X \to K_{3,3}$, then the pair (X, G) falls in (iv). All these contradictions show that this case cannot occur.

Next, let $X_N \cong K_2$. Then N acts transitively and hence regularly on $E(X)$. Thus $2p^k = 3|V(X)|$, and so $p = 3$. If $k = 1$, then $X \cong \mathrm{Dip}_3$ and X falls in (ii). Otherwise, consider the line graph $L(X)$ which is a 4-valent Cayley graph of $\mathbb{Z}_3^k$. But then $k = 2$. Therefore $X \cong K_{3,3}$ and X falls in (ii) or (iii). These contradictions show that this case cannot occur.

Finally, suppose that $X_N \cong s_3$. Then the quotient projection, being valency-preserving, is a regular covering projection onto a monopole. Hence X is a Cayley graph of the group N. So $3p^k = 2|E(X)|$ and consequently $p = 2$. By connectivity of X we have $k \leq 3$. Thus, X is isomorphic either to K_4 or to Q_3. But Q_3 is the canonical double cover of K_4 and so the full automorphism group of Q_3 projects. Hence X falls in (i). This contradiction completes the proof of Theorem 4.4. ∎

Theorem 4.4 has the following immediate consequences.

Corollary 4.5 *Let X be a connected simple cubic graph admitting an edge-transitive solvable subgroup of automorphisms. Then X is a regular cover either of the 3-dipole Dip_3 or of the complete graph K_4. Moreover, the correspondig covering projection decomposes into a sequence of (minimal) elementary-abelian covering projection.*

Corollary 4.6 *Let X be a connected simple cubic graph admitting an arc-transitive solvable subgroup G of automorphisms. If X is a regular cover of $K_{3,3}$, then it is at most $(G, 3)$-arc-transitive. In all other cases X is at most $(G, 2)$-arc-transitive.*

Corollary 4.7 *Let X be a connected simple cubic graph admitting a semisymmetric solvable subgroup G of automorphisms. Then one of the following occurs.*

(i) If G projects along a regular covering projection $X \to \mathrm{Gray}$, then there exists an integer $k \in \{0,1,2\}$ such that X is at most $(G, 2k+1)$-path-transitive. Moreover, G has two orbits on $2i$-paths, $i \leq k$, of X.

(ii) If G projects along a regular covering projection $X \to K_{3,3}$, then there exists an integer $k \in \{0,1\}$ such that X is at most $(G, 2k+1)$-path-transitive. Moreover, G has two orbits on $2i$-paths, $i \leq k$, of X.

(iii) In all other cases X is at most 1-path-transitive, and G has at least two orbits on 2-paths of X.

In view of Corollary 4.5, a connected simple cubic graph admitting a solvable group of automorphisms can be constructed from Dip_3 or K_4 via a sequence of minimal elementary abelian covers. These graphs can be thought of as being arranged into a lattice, with Dip_3 and K_4 as minimal elements. The distance of a graph in this lattice from the set of minimal elements $\{\mathrm{Dip}_3, K_4\}$ defines its level. (Note that the lattice changes if the objects, rather than just graphs, are ordered pairs (X, G), where G is solvable and acts edge-transitively on a cubic graph X; with an arrow between the two objects whenever G projects along an elementary abelian cover. In that sense, the set of minimal elements includes also ordered pairs $(K_{3,3}, G)$, where G is one of the exceptional groups from (iii) of Theorem 4.4, and (Gray, G), where G is one of the exceptional groups from (iv) of Theorem 4.4.) This point of view is useful when one is faced with the problem of constructing graphs with specific symmetry properties. For example, in the next section we shall give constructions of cubic semisymmetric graphs as elementary abelian covers of $K_{3,3}$, the Heawood graph $\mathcal{H}$ and the Moebius-Kantor graph $GP(8,3)$. These graphs are of respective levels 2, 2 and 3.

5 New families of semisymmetric cubic graphs

The object of this section is twofold. First, we show that the graphs, denoted here by $K_{3,3}^{p,p}$ where $p \equiv 1 \pmod 3$, belonging to the infinite family of semisymmetric $\mathbb{Z}_p^2$-covering projections of $K_{3,3}$ from row 3 of Table 1 in Proposition 3.3, are semisymmetric. Second, we give constructions of further three infinite families of semisymmetric elementary abelian covering projections. Two of them are associated with the Heawood graph and one with the Moebius-Kantor graph, that is, the generalized Petersen graph $GP(8,3)$.

Semisymmetric covers of $K_{3,3}$

Theorem 5.1 *Let $p \equiv 1 \pmod 3$ be a prime. Then $K_{3,3}^{p,p}$ is a semisymmetric graph with edge stabilizers isomorphic to $\mathbb{Z}_2$.*

PROOF. Let $A = \mathrm{Aut}\, K_{3,3}^{p,p}$ and recall, from row 3 of Table 1, that $\langle H, \omega_1 \rangle$ is the largest subgroup of $\mathrm{Aut}\, K_{3,3}$ that lifts. Let Γ denote the lift of this group, and note that $|\Gamma| = 18p^2$. Clearly, Γ is semisymmetric with edge stabilizers isomorphic to $\mathbb{Z}_2$. It therefore suffices to see that $A = \Gamma$. This is what we do now basing our arguments on a thorough analysis of 12-cycles.

Let us first analyze possible closed walks in $K_{3,3}$ which lift to 12-cycles in $K_{3,3}^{p,p}$. We call a 12-cycle in $K_{3,3}^{p,p}$ *homological* if its projection in $K_{3,3}$ is a closed walk traversing every edge the same number of times in both directions. Observe that such a walk is made of three 4-cycles and misses out precisely one of the vertices of $K_{3,3}$ (see Figure 3 below). In fact, for every vertex of $K_{3,3}$ there exists a single such walk (modulo taking a translation or the inverse of a walk). It follows that there is a total of $6p^2$ homological 12-cycles in $K_{3,3}^{p,p}$. It may be seen that every 12-cycle of $K_{3,3}^{p,p}$ arises in this way provided

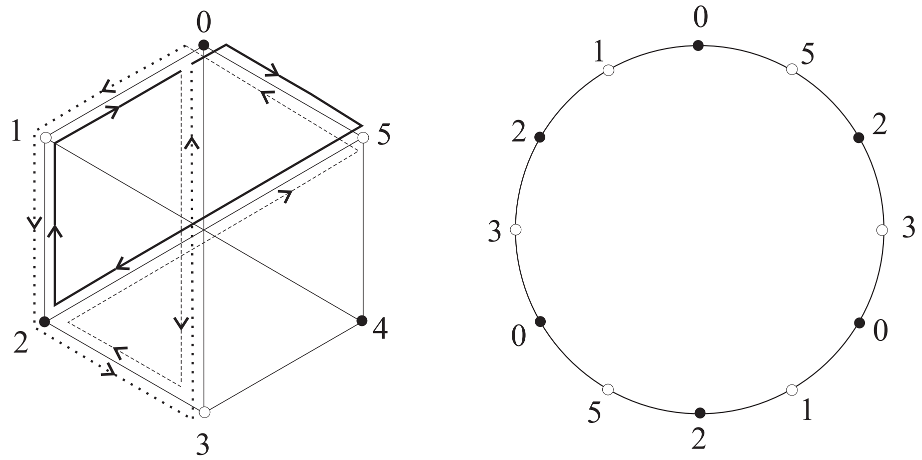

Figure 3: The configuration (walk) in $K_{3,3}$ missing out fibre $\tilde{4}$ and the corresponding 12-cycle in $K_{3,3}^{p,p}$.

$p > 7$.(We leave out the rather technical details.) For $p = 7$, we used MAGMA [1] to check that $K_{3,3}^{7,7}$ is indeed semisymmetric, with edge stabilizers isomorphic to $\mathbb{Z}_2$. We now assume that $p > 7$ and hence that all 12-cycles in $K_{3,3}^{p,p}$ are homological.

Let us call a vertex of $K_{3,3}^{p,p}$ *even* if it belongs to an even fibre $\tilde{0}$, $\tilde{2}$ or $\tilde{4}$, and *odd* if it belongs to an odd fibre $\tilde{1}$, $\tilde{3}$ or $\tilde{5}$. Similarly, a 12-cycle of $K_{3,3}^{p,p}$ is said to be *even* if it misses out an even fibre , and *odd* if it misses out an odd fibre. We let the symbols $\mathcal{C}^+$ and $\mathcal{C}^-$ denote the two respective sets of even and odd 12-cycles in $K_{3,3}^{p,p}$.. As the number of 12-cycles coincides with the order $6p^2$ of the graph $K_{3,3}^{p,p}$, we have that every vertex lies on precisely twelve 12-cycles. Consider an even 12-cycle and an odd vertex on it (see Figure 3). Observe that its antipodal vertex on that 12-cycle belongs to the same fibre. Analogously, on an odd 12-cycle any two antipodal even vertices belong to the same fibre.

For vertices in odd fibres consider the equivalence relation obtained by taking the transitive (and reflexive) hull of the relation of 'being antipodal on even 12-cycles'. It may be easily seen that the equivalence classes coincide with the three odd fibres. Denote by G the largest subgroup of A that fixes the two parts of the bipartition of $K_{3,3}^{p,p}$ as well as the sets $\mathcal{C}^+$ and $\mathcal{C}^-$. Let $v \in K_{3,3}^{p,p}$ be an odd vertex and $g \in G_v$. In view of the above remarks, it follows that g fixes the whole fibre containing v. Hence all odd fibres are blocks of imprimitivity for G. But then the same holds also for all even fibres. In other words G coincides with the lifted group Γ.

The following consequences are now at hand. If every automorphism of $K_{3,3}^{p,p}$ fixes $\mathcal{C}^+$ and $\mathcal{C}^-$ then $[A : \Gamma] \leq 2$. Similarly, if there are automorphisms in A interchanging $\mathcal{C}^+$ and $\mathcal{C}^-$, then $[A : \Gamma] \leq 4$. This puts the upper bound for the order of A to $2^3 \cdot 3^2 \cdot p^2$. Let P be a Sylow p-subgroup of A isomorphic to $\mathbb{Z}_p^2$. Then $\Gamma = N_A(P)$ coincides with the normalizer of P in A. By the above comments we have $[A : \Gamma] \in \{1, 2, 4\}$. Therefore by the Sylow theorem, P is normal in A and so the whole of A projects, forcing $A = \Gamma$. It follows that X is semisymmetric with edge stabilizers isomorphic to $\mathbb{Z}_2$, as required. ■

Note that $K_{3,3}^{7,7}$, the smallest graph in the above family, has 294 vertices.

There is another infinite family of semisymmetric graphs associated with $K_{3,3}$ (see [16]). They are obtained as regular $\mathbb{Z}_n$-covers of $K_{3,3}$, where $n = p_1 p_2 \dots p_k$, with $p_i \equiv 1 \pmod 3$, $i = 1, 2, \dots k$, being distinct primes. The corresponding voltage assignments are given by $\zeta(a) = 1$, $\zeta(b) = -r$, $\zeta(c) = s$ and $\zeta(d) = -rs$ (see Figure 2), where r and s generate two distinct subgroups of order 3 in $\mathbb{Z}_n^*$. As opposed to the graphs $K_{3,3}^{p,p}$, the graphs in this family have trivial edge stabilizers, for the maximum group of $\operatorname{Aut} K_{3,3}$ that lifts is H. The smallest graph in this family, a $\mathbb{Z}_{91}$-cover of $K_{3,3}$, has 546 vertices.

Let us remark again that the maximal possible semisymmetric subgroup of $\operatorname{Aut} K_{3,3}$, that is, $\langle H, \omega_1, \omega_2 \rangle$, lifts in the case of the Gray graph. But since the Gray graph has additional non-projectable automorphisms, the order of edge stabilizers is 2^4.

Semisymmetric covers of the Heawood graph

In [5] a list of all cubic semisymmetric graphs of order up to 768 was given. Apart from the Gray graph on 54 vertices and the graph on 110 vertices, associated with the primitive action of $\mathrm{PSL}(2, 11)$, the smallest graph in the list has 112 vertices. It transpires that this graph is the smallest member of an infinite family of semisymmetric minimal regular $\mathbb{Z}_p^3$-covers of the Heawood graph $\mathcal{H}$, where $p \equiv 1, 2$ or $4 \pmod 7$. Besides this family and three sporadic examples of regular $\mathbb{Z}_7^6$-covers of $\mathcal{H}$, there is another infinite family of semisymmetric minimal elementary abelian covering projections of $\mathcal{H}$ with the group of covering transformation isomorphic to $\mathbb{Z}_p^5$, where $p \equiv 1, 2$ or $4 \pmod 7$. We now present these families together with the non-minimal ones (see Theorem 5.2 below).

The Heawood graph $\mathcal{H}$ is obtained via an arc-transitive regular $\mathbb{Z}_7$-covering projection of the 3-dipole Dip_3 given in part (iii) of Proposition 3.1. The vertex-set of $\mathcal{H}$ can be identified with $\mathbb{Z}_7 \times \mathbb{Z}_2$ in such a way that, for all $x \in \mathbb{Z}_7$, a vertex $(x,0) \in V(\mathcal{H})$ is adjacent to vertices $(x+1,1)$, $(x+2,1)$, and $(x+4,1)$.

The covering projections onto $\mathcal{H}$ are now given in terms of the voltage assignment $\zeta\colon D(\mathcal{H}) \to \mathbb{Z}_p^k$, $k \leq 8$, assumed trivial on the spanning tree $\mathcal{T}$, induced by the edges

$$\{(x,0),(x+1,1)\},\ x \in \{0,\ldots,4,6\}, \quad \text{and} \quad \{(x,0),(x+2,1)\},\ x \in \mathbb{Z}_7.$$

Further, let the dart $(5,0) \to (6,1)$ be denoted by a_7, and for any $i \in \mathbb{Z}_7$, let a_i denote the dart $(i,0) \to (i+4,1)$.

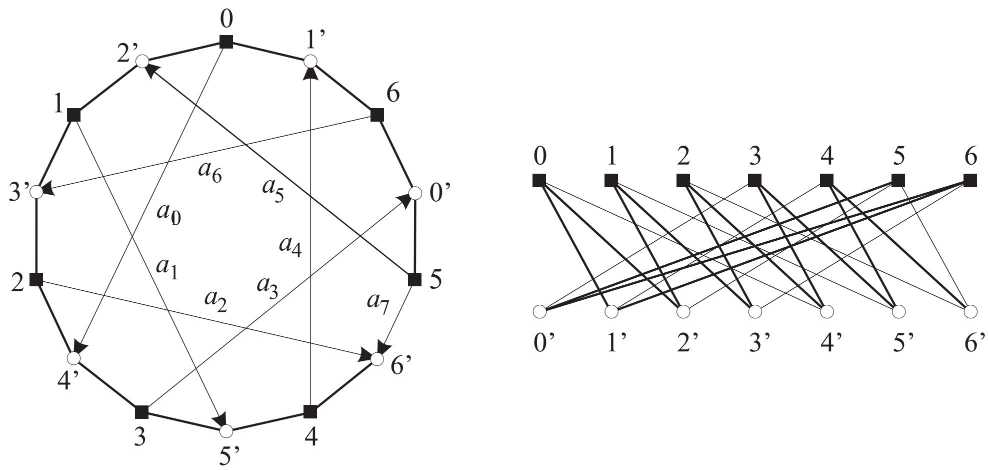

Figure 4: Two drawings of the Heawood graph.

Theorem 5.2 *(see [5].) Let p be a prime, let $\mathcal{T}$ be the spanning tree of the Heawood graph $\mathcal{H}$, and let $a_7, a_0, \ldots, a_6$ be the cotree darts of $\mathcal{H}$.*

If $p \equiv 3, 5,$ or $6 \pmod 7$, then there are no connected semisymmetric regular p-elementary abelian covering projections onto $\mathcal{H}$.

If $p \equiv 0, 1, 2,$ or $4 \pmod 7$, then the isomorphism classes of connected semisymmetric regular p-elementary abelian covering projections onto $\mathcal{H}$ are given in Table 1.

Table 2: Edge-transitive elementary abelian covers of the Heawood graph

row	$\zeta(a_7)$	$\zeta(a_0)$	$\zeta(a_1)$	$\zeta(a_2)$	$\zeta(a_3)$	$\zeta(a_4)$	$\zeta(a_5)$	$\zeta(a_6)$	condition
1	$\begin{pmatrix}0\\0\\0\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\1\end{pmatrix}$	$\begin{pmatrix}1\\\alpha+1\\\alpha\end{pmatrix}$	$\begin{pmatrix}\alpha\\-1\\-1\end{pmatrix}$	$\begin{pmatrix}-1\\-1\\-\alpha-1\end{pmatrix}$	$\begin{pmatrix}-\alpha-1\\-\alpha\\-1\end{pmatrix}$	$p\equiv 1,2,$ or $4 \pmod 7$; $\alpha^2+\alpha+2=0$
2	$\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix}$	$\begin{pmatrix}\alpha\\-1\\-\alpha\\1\\\alpha+1\end{pmatrix}$	$\begin{pmatrix}\alpha\\-\alpha-1\\1\\1\\\alpha\end{pmatrix}$	$\begin{pmatrix}-1\\-\alpha\\1\\\alpha+1\\-1\end{pmatrix}$	$p\equiv 1,2,$ or $4 \pmod 7$; $\alpha^2+\alpha+2=0$
3	$\begin{pmatrix}\lambda+3\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}1\\1\\0\\0\end{pmatrix}$	$\begin{pmatrix}1\\0\\1\\0\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$	$\begin{pmatrix}-\lambda-2\\1\\\alpha+1\\\alpha\end{pmatrix}$	$\begin{pmatrix}-\lambda-2\\\alpha\\-1\\-1\end{pmatrix}$	$\begin{pmatrix}1\\-1\\-1\\-\alpha-1\end{pmatrix}$	$\begin{pmatrix}1\\-\alpha-1\\-\alpha\\-1\end{pmatrix}$	$p\equiv 1,2,$ or $4 \pmod 7$, and $p\equiv 1 \pmod 3$; $\alpha^2+\alpha+2=0$, $\lambda^2+\lambda+1=0$
4	$\begin{pmatrix}\lambda^2+3\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}1\\1\\0\\0\end{pmatrix}$	$\begin{pmatrix}1\\0\\1\\0\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$	$\begin{pmatrix}-\lambda^2-2\\1\\\alpha+1\\\alpha\end{pmatrix}$	$\begin{pmatrix}-\lambda^2-2\\\alpha\\-1\\-1\end{pmatrix}$	$\begin{pmatrix}1\\-1\\-1\\-\alpha-1\end{pmatrix}$	$\begin{pmatrix}1\\-\alpha-1\\-\alpha\\-1\end{pmatrix}$	$p\equiv 1,2,$ or $4 \pmod 7$, and $p\equiv 1 \pmod 3$; $\alpha^2+\alpha+2=0$, $\lambda^2+\lambda+1=0$
5	$\begin{pmatrix}1\\0\\0\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\1\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\0\\1\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\0\\0\\1\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\0\\0\\0\\1\end{pmatrix}$	$\begin{pmatrix}6\\1\\2\\3\\4\\5\end{pmatrix}$	$\begin{pmatrix}6\\5\\4\\3\\2\\1\end{pmatrix}$	$p=7$
6	$\begin{pmatrix}1\\0\\0\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\1\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\0\\1\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\0\\0\\1\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\0\\0\\0\\1\end{pmatrix}$	$\begin{pmatrix}3\\1\\2\\3\\4\\5\end{pmatrix}$	$\begin{pmatrix}2\\5\\4\\3\\2\\1\end{pmatrix}$	$p=7$
7	$\begin{pmatrix}1\\0\\0\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\1\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\0\\1\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\0\\0\\1\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\0\\0\\0\\1\end{pmatrix}$	$\begin{pmatrix}0\\1\\2\\3\\4\\5\end{pmatrix}$	$\begin{pmatrix}5\\5\\4\\3\\2\\1\end{pmatrix}$	$p=7$

Semisymmetric covers of the Moebius-Kantor graph

Finally, let us present an infinite family of semisymmetric regular $\mathbb{Z}_p^2$-covers of the Moebius-Kantor graph. These covers are extensively studied in [12]. The vertex set of the Moebius-Kantor graph $GP(8,3)$ is identified with $\mathbb{Z}_8 \times \mathbb{Z}_2$ and its edges have the form

$$E = \{\{(i,0),(i+1,0\} \mid i \in \mathbb{Z}_8\} \cup \{\{(i,1),(i+3,1)\} \mid i \in \mathbb{Z}_8\} \cup \{\{(i,0),(i,1)\} \mid i \in \mathbb{Z}_8\}.$$

Define the voltages on the arcs of the inner cycle and on the spokes to be trivial, and the arcs on the oriented outer cycles to be, respectively, $(1,0)^t$, $(0,1)^t$, $(i,i-1)^t$, $(0,-i)^t$, $(-1,0)^t$, $(0,-1)^t$, $(-i,-i-1)^t$, $(0,i)^t$, where $i^2 = -1$ and $p \equiv 1 \pmod 4$. This gives rise to a semisymmetric $\mathbb{Z}_p^2$-covering projection for any $p \equiv 1 \pmod 4$.

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