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1. Introduction

Write $\Delta(a,\rho)=\{\zeta\in\mathbb{C}: |\zeta-a|<\rho\}$, $\Delta=\Delta(0,1)$. We say that a continuous function on $b\Delta(a,\rho)$ extends holomorphically from $b\Delta(a,\rho)$ if it has a continuous extension to $\overline{\Delta}(a,\rho)$ which is holomorphic on $b\Delta(a,\rho)$.

The initial motivation for the work whose results are presented here was the following still open

Question 1.1 [G2] *Let $D=\{\zeta\in\mathbb{C}: |Im\ \zeta|<1\}$ and let f be a continuous function on $\overline{D}$. Suppose that f extends holomorphically from every circle $b\Delta(t,1)$, $t\in\mathbb{R}$. Is f holomorphic on D?*

It is known that the answer is positive under the additional assumption that f is real-analytic in a neighbourhood of $\overline{D}$. This follows from recent work of M. Agranovsky and the author [AG] or from recent work of L. Ehrenpreis [E].

The family of cirles in Question 1.1 is a one parameter family. Even for two-parameter families the analogous question is open:

Question 1.2. *Let D be as in Question 1.1 and let f be a continuous function on D. Let $0<\lambda<1$ and assume that f extends holomorphically from every circle $\Gamma\subset D$ of radius λ. Is f holomorphic on D?*

We begin by an example. Let $\mathcal{C}$ be the family of all circles in $\mathbb{C}$ surrounding the origin. There are continuous functions f on $U=\mathbb{C}\setminus\{0\}$ which extend holomorphically from each $\Gamma\in\mathcal{C}$ and are not holomorphic on U. An example is $f(z)=1/\overline{z}$.

Let $a\in\mathbb{C}$ and $\rho>0$. As in [1], let $\Lambda_{a,\rho}=\{(z,w)\in\mathbb{C}^2: (z-a)(w-\overline{a})=\rho^2,\ 0<|z-a|<\rho\}$. This is a closed complex submanifold of $\mathbb{C}^2\setminus\Sigma$ attached to the real two plane $\Sigma=\{(z,\overline{z}): z\in\mathbb{C}\}$ along the circle $b\Lambda_{a,\rho}=\{(z,\overline{z}): z\in b\Delta(a,\rho)\}$. We call $\Lambda_{a,\rho}$ the *variety associated with* $b\Delta(a,\rho)$.

It is easy to see that the function f extends holomorphically from $b\Delta(a,\rho)$ if and only if the function $(z,\overline{z})\mapsto F(z,\overline{z})=f(z)$ defined on $b\Lambda_{a,\rho}$ has a continuous extension to $\Lambda_{a,\rho}\cup b\Lambda_{a,\rho}$ which is holomorphic and bounded on $\Lambda_{a,\rho}$. Suppose that $\mathcal{C}$ is an open connected set of circles and let f be a continuous function on $U=\bigcup\{\Gamma: \Gamma\in\mathcal{C}\}$ which extends holomorphically from each $\Gamma\in\mathcal{C}$. In the paper we look at all varieties associated with circles in $\mathcal{C}$, at the function F on $\tilde{U}=\{(z,\overline{z}): z\in U\}$ and simultaneously at holomorphic extensions of F into all these varieties. We show that these extensions match at the intersections of these varieties and give rise to a holomorphic function of two variables. We show that the union Ω of all varieties associated with circles in $\mathcal{C}$ is an open connected set attached to Σ along $\tilde{U}$ and that the function f extends holomorphically from each circle in $\mathcal{C}$ if and only if the function F has a continuous extension $\tilde{F}$ to $\Omega\cup\tilde{U}$ which is holomorphic on Ω and bounded on each compact subset of varieties $\Lambda_{a,\rho}$ associated with circles in $\mathcal{C}$ (Theorem 5.1). We show that the domain Ω contains a wedge with the edge $\tilde{U}$. The function f will be holomorphic on U if and only if $\tilde{F}$ will depend only on the first variable. All this alows us to use some known facts from several complex variables when looking at various families $\mathcal{C}$. In particular, we use the edge of the wedge theorem to show that holomorphic extendibility from certain families of circles implies the real analyticity of f (Theorem 6.1) which allows us to obtain a partial answer to Question 2.

Let $\mathcal{P}$ be a subset of $\mathbb{C}\times(0,\infty)$. Define

$$\begin{aligned}\mathcal{C}(\mathcal{P}) &= \{b\Delta(a,\rho)\colon\ (a,\rho)\in\mathcal{P}\}\\ U(\mathcal{P}) &= \bigcup\{\Gamma:\ \Gamma\in\mathcal{C}(\mathcal{P})\} = \bigcup\{b\Delta(a,\rho)\colon\ (a,\rho)\in\mathcal{P}\}\\ \Omega(\mathcal{P}) &= \bigcup\{\Lambda_{a,\rho}\colon\ (a,\rho)\in\mathcal{P}\}.\end{aligned}$$

2. Varieties $\Lambda_{a,\rho}$

Proposition 2.1 *Let $(z,w)\in\mathbb{C}^2\setminus\Sigma$. Given $R>0$ there is a unique a such that $(z,w)\in\Lambda_{a,R}$:*

$$a = z+\left[\sqrt{1+4R^2/|z-\overline{w}|^2}-1\right](z-\overline{w})/2 \tag{2.1}$$

Proof. It is easy to see [1] that $(z,w)\in\Lambda_{a,R}$ if and only if there is a $t>0$ such that $a=z+t(z-\overline{w})$ and $R=\sqrt{t(t+1)}|z-\overline{w}|$. Finding t from $R^2=t(t+1)|z-\overline{w}|^2$ gives (2.1) and completes the proof.

Proposition 2.2 *Let $(a_1,\rho_1)\neq(a_2,\rho_2)$. Then Λ_{a_1,ρ_1} meets Λ_{a_2,ρ_2} if and only if $a_1\neq a_2$ and one of the circles $b\Delta(a_1,\rho_1)$, $b\Delta(a_2,\rho_2)$ surrounds the other. If this is the case then Λ_{a_1,ρ_1} and Λ_{a_2,ρ_2} meet transversely and the intersection consists of one point.*

Proof. Suppose that $(z,w)\in\Lambda_{a_1,\rho_1}\bigcap\Lambda_{a_2,\rho_2}$. If $a_1=a_2$ then, by our assumption, $\rho_1\neq\rho_2$ so $\Lambda_{a_1,\rho_1}\bigcap\Lambda_{a_2,\rho_2}=\emptyset$. Consequently $a_1\neq a_2$. There are $t_i>0$ such that $a_i==z+t_i(z-\overline{w})$ and $\rho_i=\sqrt{t_i(t_i+1)}|z-\overline{w}|$, $i=1,2$. Since $a_1\neq a_2$ it follows that $t_1\neq t_2$ which implies that one of the circles $b\Delta(a_1,\rho_1)$, $b\Delta(a_2,\rho_2)$ surrounds the other. By Proposition 2.1, (z,w) is the unique point of $\Lambda_{a_1,\rho_1}\cap\Lambda_{a_2,\rho_2}$.

Conversely, assume that one of the circles surrounds the other and that the centers are different. With no loss of generality assume that the surrounding circle is centered at the origin. So assume that $b\Delta(0,R)$ surrounds $b\Delta(b,\rho)$, $b\neq 0$. Computing the z-coordinates of the intersection of the corresponding varieties

$$w=R_2/z\ \ (0<|z|<R),\quad w=\overline{b}+\rho^2/(z-b)\ \ (0<|z-b|<\rho) \tag{2.2}$$

we get

$$(z-b)^2+\frac{b\overline{b}+\rho^2-R^2}{\overline{b}}(z-b)+\rho^2\frac{b}{\overline{b}}=0,$$

so the roots z_1,z_2 satisfy $|z_1-b||z_2-b|=\rho^2$. Assume for a moment that $|z_1-b|=\rho$, so $z_1=b+\rho\zeta$ where $|\zeta|=1$. By (2.2) it follows that $w_1=\overline{b}+\rho/\zeta$ and $(b+\rho\zeta)(\overline{b}+\rho/\zeta)=R^2$ which is impossible since $|b|+\rho<R$. Thus, one of the roots, say z_1, satisfies $|z_1-b|<\rho$ and since $|b|+\rho<R$ it follows that $|z_1|<R$. Obviously $z_1\neq b, z_1\neq 0$ so $\Lambda_{0,R}$ meets $\Lambda_{a,\rho}$.

Suppose that the varieties $\Lambda_{0,R}$ and $\Lambda_{a,\rho}$ intersect at (z,w) nontransversely. This means that

$$R^2/z = \overline{a} + \rho^2/(z-a) \quad \text{and} \quad R^2/z^2 = -\rho^2/(z-a)^2 \tag{2.3}$$

Suppose first that $\rho/(z-a) = +R/z$. The first equality in (2.3) gives $(R-\rho)^2 = a\overline{a}$ so $R = \rho + |a|$ which is not possible since $b\Delta(R,0)$ surrounds $b\Delta(a,\rho)$. If $\rho/(z-a) = -R/z$ then the first equality in (2.3) gives $(R+\rho)^2 = a\overline{a}$ so $R+\rho = \pm|a|$. Neither is possible since $b\Delta(R,0)$ surrounds $b\Delta(a,\rho)$. Thus, $\Lambda_{0,R}$ intersects $\Lambda_{a,\rho}$ transversely. This completes the proof.

We will intersect $\Lambda_{a,\rho}$ with two-planes perpendicular to the Lagrangian two-plane Σ. Note that the two dimensional subspace perpendicular to Σ is $i\Sigma = \{(z,-\overline{z})\colon\ z \in \mathbb{C}\}$.

Proposition 2.3 *Let $z \in \mathbb{C},\ t > 0$ and $\varphi \in \mathbb{R}$. Then $(z,\overline{z}) + (te^{i\varphi}, -te^{-i\varphi}) \in \Lambda_{a,R}$ if and only if $a = z + \sqrt{t^2+R^2}e^{i\varphi}$.*

Proof. By Proposition 2.1, $(z+te^{i\varphi}, \overline{z} - te^{-i\varphi}) \in \Lambda_{a,R}$ if and only if

$$a = te^{i\varphi} + (\sqrt{1+R^2/t^2} - 1)(2te^{i\varphi})/2 = \sqrt{t^2+R^2}e^{i\varphi}.$$

We will also intersect $\Lambda_{a,R}$ with complex lines $z = const.$

Proposition 2.4 *Let $z \in \mathbb{C},\ t > 0$ and $\varphi \in \mathbb{R}$. Then $(z,\overline{z}) + (0,te^{i\varphi}) \in \Lambda_{a,R}$ if and only if*

$$a = z - \frac{2R^2}{\sqrt{4R^2+t^2}+t}e^{-i\varphi}.$$

Proof. By Proposition 2.1, $(z, \overline{z} + te^{i\varphi}) \in \Lambda_{a,R}$ if and only if

$$a = z + \big(\sqrt{1+4R^2/t^2} - 1\big)(-te^{-i\varphi})/2 = z - \frac{2R^2}{\sqrt{4R^2+t^2}+t}e^{-i\varphi}.$$

If $R > 0$ and $a_1, a_2 \in \mathbb{C},\ a_1 \neq a_2$, then $\Lambda_{a_1,R} \bigcap \Lambda_{a_2,R} = \emptyset$. In fact, $\Lambda_{a_2,R} = T(\Lambda_{a_1,R})$ where T is the translation parallel to Σ given by $T(z,w) = (z-(a_2-a_1), w - \overline{(a_2-a_1)})$.

3. Domains $\Omega(\mathcal{P})$

Let $\mathcal{P}$ be an open subset of $\mathbb{C} \times (0,\infty)$. Proposition 2.1 implies that $\Omega(\mathcal{P})$ is open. In fact, by Proposition 2.1 the set $\bigcup\{\Lambda_{b,R}\colon\ |b-a| < \delta\}$ is an open neighbourhood of $\Lambda_{a,R}$ for each $\delta > 0$. Assume, in addition, that $\mathcal{P}$ is connected. We show that $\Omega(\mathcal{P})$ is connected. Since each $\Lambda_{a,\rho}$ is homeomorphic to the punctured disc it is enough to show that for each $(a_0,\rho_0),\ (a_1,\rho_1) \in \mathcal{P}$ there is a path in $\Omega(\mathcal{P})$ joining a point in Λ_{a_0,ρ_0} with a point in Λ_{a_1,ρ_1}. Let $(a_0,\rho_0),(a_1,\rho_1) \in \mathcal{P}$. Since $\mathcal{P}$ is connected there is a path $(a,\rho)\colon\ [0,1] \mapsto \mathcal{P}$ such that $(a(0),\rho(0)) = (a_0,\rho_0),\ (a(1),\rho(1)) = (a_1,\rho_1)$. Since a and ρ are bounded on $[0,1]$ it follows that for w large enough, each $\Lambda_{a(t),\rho(t)}$ is a uniformly small perturbation of $\{(z,w)\colon z = a(t)\},\ 0 \le t \le 1$. Thus we may choose w_0 so large that the line $\{(z,w_0)\colon\ z \in \mathbb{C}\}$ intersects $\Lambda_{a(t),\rho(t)}$ at one point and transversely. Call the intersection point $(z(t),w(t))$.

By transversality, $t \mapsto (z(t), w(t))$ is continuous on $[0,1]$ and provides a path joining a point of Λ_{a_0,ρ_0} with a point of Λ_{a_1,ρ_1}.

Let $a_0 \in \mathbb{C}$ and $R > 0$. We will need domains of the form $\Omega = \bigcup\{\Lambda_{a,R}\colon\ |a - a_0| < \delta\}$ for small $\delta > 0$. In this case $\Omega = \Omega(Q)$ where $Q = \{(a, R)\colon\ |a - a_0| < \delta\}$ is not an open subset of $\mathbb{C} \times (0,\infty)$. We will show that in this case there is a domain $\mathcal{P}$ in $\mathbb{C} \times (0,\infty)$ such that $\Omega = \Omega(\mathcal{P})$. In our case $\bigcup\{b\Delta(a,R)\colon\ |a - a_0| < \delta\}$ is the annulus $A = \{z \in \mathbb{C}\colon\ R - \delta < |z - a_0| < R + \delta\}$ and so Ω is the union of varieties associated with circles of radius R contained in A. We shall show that whenever a circle $\Gamma \subset A$ surrounds the hole the associated variety is contained in Ω. Thus, $\Omega = \Omega(\mathcal{P})$ where $\mathcal{P} = \{(a,\rho) \in \mathbb{C} \times (0,\infty)\colon\ |a| + \rho < R + \delta,\ |a| - \rho > -(R - \delta)\}$.

Proposition 3.1 *Let $0 < \rho < R$. The variety $\Lambda_{0,\rho}$ is contained in the union of varieties $\{\Lambda_{a,R}\colon\ |a| < R - \rho\}$ and this is the smallest family of varieties $\Lambda_{a,R}$ whose union contains $\Lambda_{0,\rho}$.*

Remark In other words, to cover $\Lambda_{a,\rho}$ we have to take the family of varieties associated with all circles of radius R which surround the circle $b\Delta(0,\rho)$.

Proof. Let $0 < t < \rho$ and $\alpha \in \mathbb{R}$. Proposition 2.1 implies that the point $(te^{i\alpha}, \rho^2 e^{-i\alpha}/t) \in \Lambda_{0,\rho}$ belongs to $\Lambda_{a,R}$ if and only if

$$\begin{aligned} a &= e^{i\alpha}\big[t + \big[\sqrt{1 + 4R^2/(\rho^2/t - t)^2} - 1\big]\big(t - \rho^2/t\big)/2\big] \\ &= e^{i\alpha}\frac{-2(R^2 - \rho^2)}{(t + \rho^2/t) + \sqrt{(t + \rho^2/t)^2 + 4(R^2 - \rho^2)}} \end{aligned}$$

It is easy to see that when t increases from 0 to ρ the expression

$$\frac{R^2 - \rho^2}{(t + \rho^2/t) + \sqrt{(t + \rho^2/t)^2 + 4(R^2 - \rho^2)}}$$

increases from 0 to $R - \rho$ and so a moves along the segment with endpoints $0, -e^{i\alpha}(R - \rho)$ from 0 to $-e^{i\alpha}(R - \rho)$. This completes the proof.

Proposition 3.2 *Let $0 < R < \rho$. The variety $\Lambda_{0,\rho}$ is contained in the union of varieties $\{\Lambda_{a,R}\colon\ |a| < \rho - R\}$ and this is the smallest family of varieties $\Lambda_{a,R}$ whose union contains $\Lambda_{0,\rho}$.*

Remark In other words, to cover $\Lambda_{0,\rho}$ we have to take the family of varieties $\Lambda_{a,R}$ associated with all circles of radius R which are surrounded by the circle $b\Delta(0,\rho)$.

Proof. As before, $(te^{i\alpha}, \rho^2 e^{-i\alpha}/t) \in \Lambda_{0,\rho}$ belongs to $\Lambda_{a,R}$ if and only if

$$a = 2e^{i\alpha}\frac{\rho^2 - R^2}{(t + \rho^2/t) + \sqrt{(t + \rho^2/t)^2 + 4(R^2 - \rho^2)}}.$$

Again, as t increases from 0 to ρ the expression

$$\frac{2\rho^2 - R^2}{(t + \rho^2/t) + \sqrt{(t + \rho^2/t)^2 + 4(R^2 - \rho^2)}}$$

increases from 0 to $\rho - R$ and consequently a moves along the segment with endpoints $0,\ e^{i\alpha}(\rho - R)$ from 0 to $e^{i\alpha}(\rho - R)$. This completes the proof.

Proposition 3.3 *Let $z_0 \in \mathbb{C},\ 0 < r < R$ and write $\gamma = (R+r)/2$. Let $A = \{z \in \mathbb{C}\colon\ r < |z - z_0| < R\}$. Let Ω be the union of all $\Lambda_{a,\gamma}$ such that $b\Delta(a,\gamma) \subset A$ and let Ω_1 be the union of all $\Lambda_{b,\rho}$ such that $b\Delta(b,\rho) \subset A$ surrounds the hole. Then $\Omega = \Omega_1$.*

Proof. Obviously $\Omega \subset \Omega_1$. To show that $\Omega_1 \subset \Omega$ let $b\Delta(a,\rho) \subset A$ surround the hole. We have to show that $\Lambda_{a,\rho}$ is contained in the union of $\Lambda_{b,\gamma}$ such that $b\Delta(b,\gamma) \subset A$. This is obvious if $\rho = \gamma$. Suppose that $\rho < \gamma$. By Proposition 3.1, $\Lambda_{b,\rho}$ is contained in the union of all $\Lambda_{a,\gamma}$ such that $b\Delta(a,\gamma)$ surrounds $b\Delta(b,\rho)$. Since $b\Delta(b,\rho)$ surrounds the hole each such $b\Delta(a,\gamma)$ must surround the hole and so it must be contained in A since $\gamma = (R+r)/2$. Suppose now that $\rho > \gamma$. By Proposition 3.2, $\Lambda_{b,\rho}$ is contained in the union of $\Lambda_{a,\gamma}$ such that $b\Delta(a,\gamma)$ is surrounded by $b\Delta(b,\rho)$. In particular, each such $b\Delta(a,\gamma)$ is contained in $\Delta(z_0,R)$ and thus surrounds the hole since $\gamma = (R+r)/2$. So each such $b\Delta(a,\gamma)$ is contained in A. This completes the proof.

Proposition 3.4 *Let $z_0,\ r,\ R,\ \gamma$ and Ω be as in Proposition 3.3. Then the boundary of Ω is the union of $\{(z,\overline{z})\colon\ z \in \overline{A}\}$ and all $\Lambda_{a,\gamma}$ such that $b\Delta(a,\gamma)$ is tangent to both $b\Delta(z_0,r)$ and $b\Delta(z_0,R)$.*

Proof. This is an easy consequence of Proposition 2.1.

Proposition 3.5 *Suppose that $b\Delta(a_1,\rho_1) \subset \overline{\Delta}(a_2,\rho_2)$ and that $b\Delta(a,\rho_1)$ meets $b\Delta(a_2,\rho_2)$ at one point z_0. There is a $\delta > 0$ such that if $\mathcal{P}_i = \{(a,\rho)\colon\ |a - a_i| < \delta,\ |\rho - \rho_i| < \delta\},\ i = 1,2,$ then $\Omega(\mathcal{P}_1) \bigcap \Omega(\mathcal{P}_2)$ is connected. Moreover, there is a neighbourhood $\mathcal{S}$ of z_0 in $\mathbb{C}$ such that $\Omega(\mathcal{P}_1) \bigcap \Omega(\mathcal{P}_2)$ contains a wedge with the edge $\tilde{\mathcal{S}} = \{(z,\overline{z})\colon\ z \in \mathcal{S}\}$.*

Remark Under wedge here we understand a domain of the form $E + V$ where E is an open subset of Σ and V is an open cone in $i\Sigma$ with vertex at the origin. We call E the edge of the wedge $E + V$.

Proof. Choose $\delta > 0$ so small that no circle in $\mathcal{C}(\mathcal{P}_1)$ surrounds a circle from $\mathcal{P}_2$. Since $\mathcal{P}_1, \mathcal{P}_2$ are open it follows that $\Omega(\mathcal{P}_1) \cap \Omega(\mathcal{P}_2)$ is open. Suppose that $(z_1,w_1),\ (z_2,w_2) \in \Omega(\mathcal{P}_1) \cap \Omega(\mathcal{P}_2)$. We show that there is a path $(z,w)\colon\ [0,1] \to \Omega(\mathcal{P}_1) \cap \Omega(\mathcal{P}_2)$ joining (z_1,w_1) and (z_2,w_2). There are $(b_1,r_1),\ (b_2,r_2) \in \mathcal{P}_1$ and $(c_1,R_1),(c_2,r_2) \in \mathcal{P}_2$ such that $(z_i,w_i) = \Lambda_{b_i,r_i} \cap \Lambda_{c_i,R_i},\ i = 1,2$. Obviously $b\Delta(c_1,R_1)$ surrounds $b\Delta(b_1,r_1)$, $b\Delta(c_2,R_2)$ surrounds $b\Delta(b_2,r_2)$ and $c_i \neq b_i,\ i = 1,2$. It is easy to choose continuous functions $(b,r)\colon\ [0,1] \to \mathcal{P}_1,\ (c,R)\colon\ [0,1] \to \mathcal{P}_2,\ b(0) = b_1,\ b(1) = b_2,\ r(0) = r_1,\ r(1) = r_2,$ $c(0) = c_1, c(1) = c_2, R(0) = R_1, R(1) = R_2$, such that for each $t,\ 0 \le t \le 1,\ b\Delta(c(t),R(t))$ surrounds $b\Delta(b(t),r(t))$ and $b(t) \neq c(t)$. It follows by Proposition 2.1 that for each such t, $\Lambda_{b(t),r(t)}$ meets $\Lambda_{c(t),R(t)}$ in one point $(z(t),w(t))$ and transversely. The transversality implies that the map $t \mapsto (z(t),w(t))$ is continuous on $[0,1]$. It has all the required properties.

Recall that by Proposition 2.3, $(\zeta,\overline{\zeta})+(te^{i\varphi},-te^{-i\varphi}) \in \Lambda_{a,R}$ for $t > 0$ and φ real, if and only if $a = \zeta + \sqrt{t^2+R^2}e^{i\varphi}$. There is a $\varphi_0 \in \mathbb{R}$ such that $z_0 + \rho_j e^{i\varphi_0} = a_j,\ j = 1,2$. Clearly $\rho_2 > \rho_1$. By the preceding discussion, given $j = 1,2$, we have $(\zeta,\overline{\zeta})+(te^{i\varphi},-te^{-i\varphi}) \in \Lambda_{a,\rho_j}$ with $t > 0$ and $\varphi \in \mathbb{R}$ if and only if $a = \zeta + \sqrt{t^2+\rho_j^2}e^{i\varphi}$.

It is easy to see that there is an $\eta > 0$ such that $\zeta + \sqrt{t^2+\rho_1^2}e^{i\varphi} \in a_1 + \delta\Delta$, $\zeta + \sqrt{t^2+\rho_2^2}e^{i\varphi} \in a_2 + \delta\Delta$ provided that

$$|\zeta - z_0| < \eta, \quad 0 < t < \eta, \quad \text{and} \quad |\varphi - \varphi_0| < \eta \tag{3.1}$$

which implies that $(\zeta,\overline{\zeta}) + (te^{i\varphi}, -te^{-i\varphi}) \in \Omega(\mathcal{P}_1)\cap\Omega(\mathcal{P}_2)$ whenever (3.1) holds. So $\{(\zeta,\overline{\zeta}) + (te^{i\varphi}, -te^{-i\varphi})\colon\ |\zeta - z_0| < \eta,\ \ 0 < t < \eta,\ \ |\varphi-\varphi_0| < \eta\}$ is a wedge domain contained in $\Omega(\mathcal{P}_1)\cap\Omega(\mathcal{P}_2)$. This completes the proof.

Remark Proposition 3.5 implies that if F_i are continuous functions on $\Omega(\mathcal{P}_i)\cup\tilde{\mathcal{S}}$ which are holomorphic on $\Omega(\mathcal{P}_i)$, $i = 1, 2$, and if $F_1 \equiv F_2$ on $\tilde{\mathcal{S}}$ then $F_1 \equiv F_2$ on $\Omega(\mathcal{P}_1)\cap\Omega(\mathcal{P}_2)$.

4. Holomorphic extensions, local case

Proposition 4.1 *A continuous function f on $b\Delta(a,\rho)$ extends holomorphically from $b\Delta(a,\rho)$ if and only if the function F, defined on $b\Lambda_{a,\rho}$ by $F(z,\overline{z}) = f(z)$ $(z\in b\Delta(a,\rho))$ has a continuous extension to $\Lambda_{a,\rho}\cup b\Lambda_{a,\rho}$ which is bounded and holomorphic on $\Lambda_{a,\rho}$.*

Proof. Suppose that f extends holomorphically from $b\Delta(a,\rho)$. This means that there is a continuous function $\tilde{f}$ on $\overline{\Delta}(a,\rho)$ which is holomorphic on $\Delta(a,\rho)$ and such that $\tilde{f}\equiv f$ on $b\Delta(a,\rho)$. Then $\tilde{F}(z,\overline{a}+\rho^2/(z-a)) = \tilde{f}(z)$ $(0 < |z-a| \le \rho)$ is a continous extension of F from $b\Lambda_{a,\rho}$ to $\Lambda_{a,\rho}\cup b\Lambda_{a,\rho}$ which is holomorphic and bounded on $\Lambda_{a,\rho}$. Conversely, let $\tilde{F}$ be continuous on $\Lambda_{a,\rho}\cup b\Lambda_{a,\rho}$, holomorphic and bounded on $\Lambda_{a,\rho}$ and such that $\tilde{F}\equiv F$ on $b\Lambda_{a,\rho}$. Define $\tilde{f}(z) = \tilde{F}(z,\overline{a}+\rho^2/(z-a))$ to get a continous function on $\overline{\Delta}(a,\rho)\setminus\{a\}$, holomorphic and bounded on $\Delta(a,\rho)\setminus\{a\}$ and such that $\tilde{f}\equiv f$ on $b\Delta(a,\rho)$. Since $\tilde{f}$ is bounded it has a removable singularity at a and so it is holomorphic on $\Delta(a,\rho)$. This completes the proof.

Remark The same reasoning implies that whenever G is a continuous function on $\Lambda_{a,\rho}\cup b\Lambda_{a,\rho}$ which is holomorphic and bounded on $\Lambda_{a,\rho}$ then for each $(z,w)\in\Lambda_{a,\rho}$

$$|G(z,w)| \le \max\{|G(\xi,\eta)|\colon\ (\xi,\eta)\in b\Lambda_{a,\rho}\}.$$

We shall need the following result proved in [G3].

Lemma 4.2 *Let $0 < r < R$, let $\gamma = (R+r)/2$ and let $A = \{z\in\mathbb{C}\colon\ r < |z| < R\}$. Let f be a continuous function on $\overline{A}$ which extends holomorphically from every circle $b\Delta(a,\gamma)$ which is tangent to both $b\Delta(0,r)$ and $b\Delta(0,R)$. Then, on $\overline{A}$, f is a uniform limit of polynomials in z and $1/\overline{z}$. In particular, f extends holomorphically from every circle contained in $\overline{A}$ which surrounds the hole.*

Notice that the assumptions in Lemma 4.2 are that $\overline{A}$ is the union of the circles $e^{i\omega}\Gamma$, $\omega\in\mathbb{R}$, where Γ is a circle which surrounds the origin and is not centered at the origin, and f extends holomorphically from $e^{i\omega}\Gamma$ for each $\omega\in\mathbb{R}$.

Let $0 < r < R$ and let $A = \{z\in\mathbb{C}\colon\ r < |z| < R\}$. In the rest of this section we shall denote by $\Omega(A)$ the union of all $\Lambda_{a,\rho}$ associated with circles $b\Delta(a,\rho)\subset A$ that surround the origin. By Proposition 3.3, $\Omega(A)$ is equal to the union of all $\Lambda_{a,\gamma}$, $\gamma = (R+r)/2$, associated with circles $b\Delta(a,\gamma)\subset A$ that surround the origin.

Here is our local theorem.

Theorem 4.3 *Let* $0 < r < R$ *and let* $A = \{z \in \mathbb{C}\colon\ r < |z| < R\}$. *Let* f *be a continuous function on* $\overline{A}$. *The following are equivalent*

(i) f *extends holomorphically from every circle of radius* $(R+r)/2$ *tangent to both* $b\Delta(0,r)$ *and* $b\Delta(0,R)$

(ii) f *extends holomorphically from every circle in* $\overline{A}$ *surrounding the hole*

(iii) the function $F(z,\overline{z}) = f(z)$ *defined on* $\{(z,\overline{z})\colon\ z \in \overline{A}\}$ *extends to a bounded continuous function on* $\overline{\Omega(A)}$ *which is holomorphic on* $\Omega(A)$.

Proof. The equivalence of (i) and (ii) follows from Lemma 4.2. Suppose that (iii) holds. Then for each a, ρ such that $b\Delta(a,\rho)$ surrounds the origin the function $F(z,\overline{z}) = f(z)$ has a bounded continuous extension from $b\Lambda_{a,\rho}$ to $\Lambda_{a,\rho} \cup b\Lambda_{a,\rho}$ which is holomorphic on $\Lambda_{a,\rho}$ which, by Proposition 4.1 implies that f extends holomorphically from $b\Delta(a,\rho)$. By continuity it follows that this holds for every circle in $\overline{A}$ which surrounds the origin. This proves that (iii) implies (ii). Conversely, suppose that (ii) holds. By Lemma 4.2 it follows that on $\overline{A}$, the function f is a uniform limit of polynomials in z and $1/\overline{z}$ which implies that on $\{(z,\overline{z})\colon\ z \in \overline{A}\}$ the function $F(z,\overline{z}) = f(z)$ is a uniform limit of polynomials in z and $1/w$. Each such polynomial $P(z,1/w)$ is a bounded continuous function on $\overline{\Omega(A)}$ which is holomorphic on $\Omega(A)$. For each $\Lambda_{a,\rho} \subset \overline{\Omega(A)}$ such a polynomial is bounded and continuous on $\Lambda_{a,\rho} \cup b\Lambda_{a,\rho}$ and holomorphic on $\Lambda_{a,\rho}$ hence by the remark after Proposition 4.1, for each $(z,w) \in \Lambda_{a,\rho}$ we have $|P(z,1/w)| \leq \max\{|P(\xi,1/\eta)|\colon\ (\xi,\eta) \in b\Lambda_{a,\rho}\}$. It follows that $\max\{|P(z,1/w)|\colon\ (z,w) \in \overline{\Omega(A)}\} = \max\{|P(z,1/w)|\colon\ (z,w) \in \{(\zeta,\overline{\zeta})\colon\ \zeta \in \overline{A}\}\}$ which implies that the uniform convergence of polynomials in z and $1/w$ on $\{(z,\overline{z})\colon\ z \in \overline{A}\}$ implies the uniform convergence to a bounded continuous function on $\overline{\Omega(A)}$. In particular, if on $\overline{A}$ the function f is the uniform limit of a sequence $P_n(z,1/\overline{z})$, where P_n are polynomials then $P_n(z,1/w)$ converges uniformly on $\overline{\Omega(A)}$ to a bounded continuous function $\tilde{F}$ that extends F and is holomorphic on $\Omega(A)$. Thus (iii) holds. This completes the proof.

The function algebra on the annulus $\overline{A}$ generated by z and $1/\overline{z}$ arose naturally in [G3]. By our Theorem 4.3 this is the algebra of all functions of the form $f(z) = F(z,\overline{z})$ $(z \in \overline{A})$ where F is a bounded continuous function on the closure of the wedge domain $\Omega(A)$ with the edge $\{(z,\overline{z})\colon\ z \in A\}$ which is holomorphic on $\Omega(A)$.

Proposition 4.4 *Suppose that* f *satisfies the conditions of Theorem 4.3. Then* f *is holomorphic on* A *(and in fact, extends holomorphically into the hole) if and only if the continuous extension* $\tilde{F}(z,w)$ *of* $F(z,\overline{z}) = f(z)$ *from* $\tilde{A} = \{(z,\overline{z})\colon\ z \in A\}$ *to* $\tilde{A} \cup \Omega(A)$ *which is holomorphic on* $\Omega(A)$, *depends only on the variable* z.

Proof. Suppose that f is holomorphic on A. By (i) the function f extends holomorphically to $\Delta(0,R)$; denote the extension by $\tilde{f}$. The function $(z,w) \mapsto \tilde{f}(z)$ is holomorphic on $\Delta(0,R) \times \mathbb{C}$ and equal to $\tilde{F}(z,w)$ on $\Omega(A)$ so F depends only on the first variable. Conversely, if $\tilde{F}(z,w)$ depends only on the first variable, $\tilde{F}(z,w) = g(z)$ then obviously g is holomorphic on $\Delta(0,R)$ and $g = f$ on A. This completes the proof.

In Theorem 4.3 $\Omega(A)$ is an unbounded wedge domain contained in $\mathbb{C}^2 \setminus \Sigma$ with the edge $\tilde{A} = \{(z,\overline{z})\colon\ z \in A\}$ contained in Σ. The closure of $\Omega(A)$ misses the complex line $\{(z,w) :$ $w = 0\}$ so we can use the map $\Phi(z,w) = (z,1/w)$ to get another version of Theorem 4.3

and thus another description of continuous functions on $\overline{A}$ which extend holomorphically from each circle in A surrounding the origin. The map Φ maps $\mathbb{C}^2 \setminus \{(z,w): w=0\}$ biholomorphically onto itself and the image of $\Sigma \setminus \{(0,0)\}$ under Φ is the totally real manifold $\mathcal{L} = \{(z, 1/\overline{z}): z \in \mathbb{C} \setminus \{0\}\}$. If $b\Delta(a,\rho)$ surrounds the origin then the analytic disc

$$\begin{aligned} \Theta_{a,\rho} &= \Phi(\Lambda_{a,\rho}) \cup \{(a,0)\} \\ &= \{(z,w): (z-a)(\frac{1}{w} - \overline{a}) = \rho^2,\ 0 < |z-a| < \rho\} \cup \{(a,0)\} \\ &= \{(z, \frac{z-a}{\overline{a}z - \overline{a}a + \rho^2}): |z-a| < \rho\} \end{aligned}$$

is contained in $\mathbb{C}^2 \setminus \mathcal{L}$ and is attached to $\mathcal{L}$ along $b\Theta_{a,\rho} = \{(z, 1/\overline{z}): z \in b\Delta_{a,\rho}\}$. Clearly a continuous function f on a circle $b\Delta(a,\rho) \subset A$ which surrounds the origin extends holomorphically from $b\Delta(a,\rho)$ if and only if the function $G(z, 1/\overline{z}) = f(z)$ defined on $b\Theta_{a,\rho}$ has a continuous extension to $\Theta_{a,\rho} \cup b\Theta_{a,\rho}$ which is holomorphic on $\Theta_{a,\rho}$.

A function f is a uniform limit of polynomials in z and $1/\overline{z}$ on $\overline{A}$ if and only if on $\{(z,1/\overline{z}): z \in \overline{A}\}$ the function $G(z,1/\overline{z}) = f(z)$ $(z \in \overline{A})$ is a uniform limit of polynomials. Let $\Xi(A)$ be the union of all $\Theta_{a,\rho}$ such that $b\Delta(a,\rho) \subset A$ surrounds the origin. $\Xi(A)$ is a domain in $\mathbb{C}^2 \setminus \mathcal{L}$ attached to $\mathcal{L}$ along $\{(z,1/\overline{z}): z \in A\}$. It is foliated by the discs $\Theta_{a,\gamma}$ where $\gamma = (R+r)/2$ and $b\Delta(a,\gamma) \subset A$.

Here is an equivalent form of (iii) in Theorem 4.3

Theorem 4.5 *(i), (ii) or (iii) in Theorem 4.3 are equivalent to*

(iv) the function $G(z,1/\overline{z}) = f(z)$ extends from $\{(z,1/\overline{z}): z \in \overline{A}\} \subset \mathcal{L}$ to a continuous function on $\overline{\Xi(A)}$ which is holomorphic on $\Xi(a)$

Remark In the next section we will work with large families of circles and in general there will be no line $(z,w): z \in \mathbb{C}$ missing all $\Lambda_{a,\rho}$. So the above reasoning will not be possible in general.

5. Holomorphic extensions, general case

In this section we look at large families of circles.

Theorem 5.1 *Let $\mathcal{P}$ be an open connected subset of $\mathbb{C} \times (0,\infty)$. Then $\Omega(\mathcal{P})$ is a domain in $\mathbb{C}^2 \setminus \Sigma$ attached to Σ along $\tilde{U}(\mathcal{P}) = \{(z,\overline{z}): z \in U(\mathcal{P})\}$. If f is a continuous function on $U(\mathcal{P})$ then the following are equivalent:*

(i) f extends holomorphically from each $\Gamma \in \mathcal{C}(\mathcal{P})$

(ii) the function F, defined on $\tilde{U}(\mathcal{P})$ by $F(z,\overline{z}) = f(z)$, has a continuous extension to $\tilde{U}(\mathcal{P}) \cup \Omega(\mathcal{P})$ which is holomorphic on $\Omega(\mathcal{P})$ and bounded on each $\Lambda_{a,\rho}$, $(a,\rho) \in \mathcal{P}$.

Remark By the remark after Proposition 4.1 the boundedness of F on each $\Lambda_{a,\rho}$, $(a,\rho) \in \mathcal{P}$ implies that F is bounded on $\bigcup\{\Lambda_{a,\rho}: (a,\rho) \in K\}$ for each compact subset K of $\mathcal{P}$.

Proof. It is obvious that (ii) implies (i). Suppose that (i) holds. Then for each $(a,\rho) \in \mathcal{P}$ the function F extends from $b\Lambda_{a,\rho} = \{(z,\overline{z}): z \in b\Delta(a,\rho)\}$ to a continuous function $F_{a,\rho}$ on $\Lambda_{a,\rho} \cup b\Lambda_{a,\rho}$, holomorphic on $\Lambda_{a,\rho}$. This extension is unique and is bounded on $\Lambda_{a,\rho}$ by $\max\{|F(z,w): (z,w) \in b\Lambda_{a,\rho}\}$. Theorem 4.3 implies that for each $(a_0,\rho_0) \in \mathcal{P}$ there is an open neighbourhood $\mathcal{W} \subset \mathcal{P}$ of (a_0,ρ_0) such that the function F extends from $\tilde{U}(\mathcal{W}) = \{(z,\overline{z})\colon\ z \in U(\mathcal{W})\}$ to a continuous function on $\tilde{U}(\mathcal{W}) \cup \Omega(\mathcal{W})$ which is holomorphic and bounded on $\Omega(\mathcal{W})$. On each $\Lambda_{a,\rho}$, $(a,\rho) \in \mathcal{W}$, the extension coincides with $F_{a,\rho}$. In particular, if (a,ρ), $(a',\rho') \in \mathcal{W}$ are such that $\Lambda_{a,\rho}$ and $\Lambda_{a',\rho'}$ meet, then $F_{a,\rho} = F_{a',\rho'}$ on $\Lambda_{a,\rho} \cap \Lambda_{a',\rho'}$. To show that the extensions $F_{a,\rho}$ give rise to a well defined function F on $\Omega(\mathcal{P})$ we must show that whenever (a,ρ), $(a',\rho') \in \mathcal{P}$ are such that $\Lambda_{a,\rho}$ and $\Lambda_{a',\rho'}$ meet, then $F_{a,\rho} = F_{a',\rho'}$ on $\Lambda_{a,\rho} \cap \Lambda_{a',\rho'}$. This is obvious if $\Lambda_{a,\rho} = \Lambda_{a',\rho'}$. If $\Lambda_{a,\rho} \neq \Lambda_{a',\rho'}$ and $\Lambda_{a,\rho}$ and $\Lambda_{a',\rho'}$ meet then, by Proposition 2.2 they meet at one point and transversely and this happens if and only if the circles $b\Delta(a,\rho)$, $b\Delta(a',\rho')$ have different centers and one surrounds the other. With no loss of generality assume that $b\Delta(a,\rho)$ surrounds $b\Delta(a',\rho')$. Since $\mathcal{P}$ is open and connected there are an $(a_0,\rho_0) \in \mathcal{P}$ and a real analytic map $t \mapsto (A(t),R(t))$ from $[0,1]$ to $\mathcal{P}$ such that

(a) $b\Delta(a_0,\rho_0) \subset \overline{\Delta}(a,\rho)$ and $b\Delta(a_0,\rho_0)$ meets $b\Delta(a,\rho)$ at one point z_0

(b) $A(0) = a_0,\ R(0) = \rho_0,\ A(1) = a',\ R(1) = \rho'$

(c) for each t, $0 < t \leq 1$, $b\Delta(a,\rho)$ surrounds $b\Delta(A(t),R(t))$

(d) $A(t) \neq a\ (0 \leq t \leq 1)$.

Define $\{(z(t),w(t))\} = \Lambda_{A(t),R(t)} \cap \Lambda_{a,\rho}\ \ (0 < t \leq 1)$, $\ (z(0),w(0)) = (z_0,\overline{z_0})$. Since $\Lambda_{A(t),R(t)}$ are complex manifolds depending in a real analytic way on t transversality implies that $t \mapsto (z(t),w(t))$ is real analytic on $(0,1]$ and it is easy to see that it is continuous at $t = 0$.

For each $(b,r) \in \mathcal{P}$ there is an open neighbourhood $\mathcal{V}$ of (b,r) in $\mathcal{P}$ such that the function F, defined on $\Omega(\mathcal{V})$ by $F|\Lambda_{a,\rho} = F_{a,\rho}$ $((a,\rho) \in \mathcal{V})$ is well defined and holomorphic on $\Omega(\mathcal{V})$. It follows that the functions $t \mapsto F_{A(t),R(t)}(z(t),w(t))$ and $t \mapsto F_{a,\rho}(z(t),w(t))$ are real analytic on $(0,1]$. Proposition 3.5 implies that there is an $\varepsilon > 0$ such that

$$F_{A(t),R(t)}(z(t),w(t)) = F_{a,\rho}(z(t),w(t))\ \ (0 < t < \varepsilon), \tag{5.1}$$

so the real analyticity implies that (5.1) holds for all t, $0 \leq t \leq 1$. In particular, for $t = 1$ we obtain $F_{a',\rho'}|\Lambda_{a',\rho'} \cap \Lambda_{a,\rho} = F_{a,\rho}|\Lambda_{a',\rho'} \cap \Lambda_{a,\rho}$ which we wanted to show. This completes the proof.

6. Holomorphic extensions from circles and real analyticity

Let $0 < r < R$ and let $A = \{z \in \mathbb{C}\colon\ r < |z| < R\}$ and let f be a continuous function on A which extends holomorphically from each circle $\Gamma \subset A$ surrounding the origin. Then f is not necessarily smooth on A. We show this by an example. For $z = re^{i\theta} \neq 0$ define

$$f(z) = \sqrt{1 + e^{2i\theta}}. \tag{6.1}$$

After choosing the branch of the square root the function f is well defined and continuous on $\mathbb{C} \setminus \{0\}$. It is constant on each ray emanating from the origin and is not smooth along $\{\pm it\colon\ t > 0\}$. Since $e^{i\theta} \mapsto \sqrt{1+e^{i\theta}}$ is the boundary function of the function $\zeta \mapsto \sqrt{1+\zeta}$ that belongs to the disc algebra it follows that on $b\Delta$, $\sqrt{1+\zeta}$ is a uniform limit of polynomials in ζ. In particular, on $\mathbb{C} \setminus \{0\}$, $f(z) = \sqrt{1+z/\overline{z}}$ is a uniform limit of polynomials in $z/\overline{z}$ and consequently f extends holomorphically from every circle surrounding the origin.

The example also shows that such a function can be real analytic on a part of A without being real analytic everywhere on A.

Theorem 6.1 *Let $\Gamma_1, \Gamma_2, \Gamma_3$ be three circles passing through the origin such that the triangle with vertices at their centers contains the origin. Let f be a continuous function on a neighbourhood of $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ which extends holomorphically from each circle sufficiently close to any of the circles $\Gamma_1, \Gamma_2, \Gamma_3$. Then f is real analytic in a neighbourhood of the origin.*

Remark By Lemma 4.2 it is enough to assume the holomorphic extendibility from circles Γ such that those near Γ_i have the same radius as Γ_i, $1 \leq i \leq 3$.

Corollary 6.2 *Let Γ_1, Γ_2 be two circles passing through the origin such that the open discs they surround are disjoint. Suppose that f is a continuous function in a neighbourhood of $\Gamma_1 \cup \Gamma_2$ which extends holomorphically from each circle sufficiently close to either Γ_1 or Γ_2. Then f is real analytic in a neighbourhood of the origin.*

Proof of Theorem 6.1. Moving the circles slightly if necessary we may assume with no loss of generality that the triangle with vertices at the centers contains the origin in its interior. Let $\delta > 0$ and let $\Gamma_i = b\Delta(a_i, \rho_i)$, $\mathcal{P}_i = \{(a, \rho)\colon |a - a_i| < \delta, |\rho - \rho_i| < \delta\}$, $1 \leq i \leq 3$, and suppose that f is a continuous function on $U(\mathcal{P}_1) \cup U(\mathcal{P}_2) \cup U(\mathcal{P}_3)$ which extends holomorphically from each $b\Delta(a, \rho)$, $(a, \rho) \in \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3$. For each i, $1 \leq i \leq 3$, apply Theorem 5.1 to show that the function F, defined by $F(z, \overline{z}) = f(z)$, has a continuouus extension from $\tilde{U}(\mathcal{P}_i) = \{(z, \overline{z})\colon z \in U(\mathcal{P}_i)\}$ to $\tilde{U}(\mathcal{P}_i) \cup \Omega(\mathcal{P}_i)$ which is holomorphic on $\Omega(\mathcal{P}_i)$.

Fix j, $1 \leq j \leq 3$. By Proposition 2.3 there is a $\gamma_j > 0$ such that if $D_j = \{(\zeta, \overline{\zeta})\colon |\zeta| < \gamma_j\}$ and $V_j = \{t(e^{i\varphi}, -e^{i\varphi})\colon 0 < t < \gamma_j, |\varphi - \arg a_j| < \gamma_j\}$ then the wedge domain $D_j + V_j$ is contained in $\Omega(\mathcal{P}_j)$. So there is a $\delta > 0$ such that if $D = \{(\overline{\zeta})\colon |\zeta| < \delta\}$ and $W_j = \{t(e^{i\varphi}, -e^{i\varphi})\colon 0 < t < \delta, |\varphi - \arg a_j| < \gamma\}$ then W_i are pairwise disjoint and F has a continuous extension to $D \cup (D + W_1) \cup (D + W_2) \cup (D + W_3)$, which is holomorphic on $(D + W_1) \cup (D + W_2) \cup (D + W_3)$.

Recall that D is a subset of Σ, a Lagrangian subspace of $\mathbb{C}^2$, and $W_j \subset i\Sigma$ are, by our assumptions about the centers a_j, such that the convex hull of $W_1 \cup W_2 \cup W_3$ is all of $i\Sigma$. Hence by Epstein's generalization of the edge of the wedge theorem [R], $F(z, \overline{z}) = f(z)$ extends holomorphically into a neighbourhood of the origin in $\mathbb{C}^2$. This completes the proof.

Remark Note that the conditions about position of the centers cannot be dropped. Indeed, some condition of this sort is necessary as seen from example (6.1). Let f be as in (6.1), let $t > 0$ be very small and let $c > 0$ be very large. Let Γ_+ and Γ_- be the circles passing through the point it with centers at c, $-c$, respectively. Then f extends holomorphically from each circle sufficiently close to either Γ_+ or Γ_-, yet f is not smooth in a neighbourhood of the point it.

Known results imply that the answer to Question 1.2 is positive when $\lambda < 1/2$. In this case the family of circles of radius λ contained in $\{\zeta \in \mathbb{C}\colon -1 < \text{Im } \zeta < 1\}$ contains a rotation invariant family of circles such that the center of the rotation is an exterior point of the circles and thus the positive answer follows from [G1, Corollary 1] or [G4]. Using

Theorem 6.1 one can do better:

Theorem 6.2 *Let* $0 < \lambda < 2/3$ *and let* $D = \bigcup\{\Delta(t,1)\colon\ -\lambda < t < \lambda\}$. *Let* f *be a continuous function on* D *which extends holomorphically from each circle of radius* λ *contained in* D. *Then* f *is holomorphic on* D.

Proof. We already know how to prove this for $0 < \lambda < 1/2$. Suppose that $1/2 \leq \lambda < 2/3$. Let $2\lambda - 1 < \rho < r < 1 - \lambda$. Since $\rho > 2\lambda - 1$ it follows that each circle of radius λ that surrounds $\overline{\Delta}(0,\rho)$ is contained in D and f extends holomorphically from such a circle. It follows by Theorem 4.3 that f extends holomorphically from each circle of radius not exceeding λ that surrounds $\overline{\Delta}(0,\rho)$.

Since $\lambda \geq 1/2$ it follows that $r < 1 - \lambda \leq 1/2 \leq \lambda$ so f extends holomorphically from each circle sufficiently close to $b\Delta(0,r)$. Further, since $r < 1-\lambda$ it follows that each point z of $\overline{\Delta}(0,r)$ is the common point of two circles of radius λ contained in $\{\zeta\colon\ -1 < \text{Im}\ \zeta < 1\}$, meeting only at z, such that the open discs they surround are disjoint and have centers on the real line through z which is parallel to $\mathbb{R}$. Since $2\lambda + 2r + 2\lambda < 2\lambda + 2(1-\lambda) + 2\lambda = 2(1+\lambda)$ it follows that each such pair of circles is contained in D. By our assumption f extends holomorphically from each circle of radius λ sufficiently close to either of these two circles. Theorem 6.1 implies that f is real analytic in a neighbourhood of z. Thus, f is real analytic on $\Delta(0,R)$ for some $R > r$ and extends holomorphically from each circle which is sufficiently close to $b\Delta(0,r)$. To prove that f is holomorphic one invokes [1, Th. 11.1]. Alternatively, one can prove directly that if $\mathcal{P} = \{(a,\rho)\colon\ b\Delta(a,\rho) \subset \Delta(0,R)$ then for each sufficiently large $w \in \mathbb{C}$ the integral

$$\frac{1}{2\pi i}\int_{b\Delta(a,\rho)} \frac{f(\zeta)d\zeta}{\zeta - w}$$

is a real analytic function of (a,ρ) on $\mathcal{P}$ which vanishes identically on an open neighbourhood of $(0,r)$ in $\mathcal{P}$. Consequently

$$\frac{1}{2\pi i}\int_{b(a,\rho)} \frac{f(\zeta)d\zeta}{\zeta - w} = 0 \quad ((a,\rho) \in \mathcal{P})$$

for all large w which implies that f extends holomorphically from each circle in $\Delta(0,R)$ which implies that f is holomorphic on $\Delta(0,R)$ and thus f is holomorphic on D. This completes the proof.

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REFERENCES

[AG] M. Agranovsky and J. Globevnik: Analyticity on circles for rational and real-analytic functions of two real variables.
Preprint.

[E] L. Ehrenpreis: Three problems at Mount Holyoke.
Radon Transforms and Tomography. Contemp. Math. 278 (2001) 123-130

[G1] J. Globevnik: Analyticity on rotation invariant families of curves.
Trans. Amer. Math. Soc. 280 (1983) 247-254

[G2] J. Globevnik: Analyticity on families of curves. Talk at Bar Ilan University, November 1987

[G3] J. Globevnik: Testing analyticity on rotation invariant families of curves.
Trans. Amer. Math. Soc. 306 (1988) 401-410

[G4] J. Globevnik: Holomorphic extensions and rotation invariance.
Compl. Variables 24 (1993) 49-51

[R] W. Rudin: *Lectures on the edge of the wedge theorem.*
Regional Conf. Ser. in Math. No. 6. American Mathematical Society, Providence, 1971

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