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Abstract

In this paper, we study the distance choosability — the list counterpart of the distance constrained labelings. In particular, we show that the Alon-Tarsi theorem for choosability in graphs has an analogous version for the choosability of distance constrained labelings, a notion stemming from channel assignment. We apply this result to paths and cycles for labeling with a condition at distance two.

1 Introduction

One of the main issues concerning the efficient use of radio spectra in telecommunication is the design and analysis of efficient algorithms for frequency

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assignment. The main task is to assign transmitters frequencies from the shortest possible range while maintaining an adequate quality of the signal. The signal clarity depends on the possible interference of simultaneous transmitting from close sources. The level of interference depends mainly on the distance of transmitters, but also on other factors, e.g. on the hilliness of the terrain (see [5]).

The graph theoretic model of the frequency assignment transforms the geometric instance into a discrete structure. In the most simplest case we draw a graph whose vertices represent transmitters and adjacent transmitters should use different frequencies. Clearly, it is a graph coloring but in this simple version the coloring approach for channel assignment provides only rough level on channel separation. In practice several levels of channel separation are used, and so we adopt a notion of distance constrained labeling [11, 12], where close vertices should have labels separated by a specified parameter that depends on the graph distance. These labelings were investigated in [10, 14, 16, 19] also for their interesting graph-theoretic properties.

Moreover we consider a situation, where transmitters may support only a limited list of frequencies (due to physical or organizational reasons). Then our aim is to select labels for vertices from the given lists such that all distance constraints are met. We call this the *distance constrained list labeling* problem (we omit the terms distance constrained if no confusion may appear).

We note here that the list version of graph coloring was introduced by Vizing [18] and independently by Erdős, Rubin, and Taylor [6] and extensively studied by many authors, for several surveys see [1, 13, 17, 20].

Graph labeling generalizes graph coloring, so we expect that its computational complexity should not be simpler in general. The graph coloring problem belongs among the hardest combinatorial optimization problems, it is strongly *NP*-hard and also inapproximable within the factor $n^{1/7-\varepsilon}$, where n denotes the number of vertices of the graph [4].

On the other hand, the First-fit coloring heuristic might bring satisfactory results for restricted classes of graphs that might appear in practice [8], and is also easily extendible to the list coloring problem. The First-fit method selects colors for vertices of a graph G as follows: Order the vertices arbitrarily, process them one by one and assign to a vertex the smallest available color, that means the least number from the assigned list that has not been yet used on an adjacent vertex.

It is well known, that if the vertices of the graph can be ordered such that each vertex is adjacent to at most d predecessors, then the lists of size $d+1$

assure that the First-fit algorithm finds a suitable coloring. This ordering can be transformed to an acyclic orientation with maximum indegree at most d, where edges are oriented towards successors. Observe that for d-regular graphs, this approach gives the relatively weak bound of list size $d+1$ at each vertex, which is very close to the maximum degree.

Alon and Tarsi [2] extended this result to special orientations (see Theorem 1). In specific cases a suitable orientation might bring a twice better upper bound than the First-fit method, e.g., in the case of bipartite k-regular graphs. However, the proof is not constructive. It means, we only know that such list coloring with bounded number of colors exists but it is not known how to find a feasible coloring in polynomial time.

We reproduce the statement of Theorem 1 in Section 3. The hard part of its practical application is the proof of the inequality between the number of even and odd Eulerian subgraphs for a particular orientation of the underlying graph. Even from graphs derived from cycles it might not be an easy task. On the other hand, this method is up to now the only way to prove a statement that every 4-regular graph composed of a Hamiltonian cycle and a set of disjoint triangles is 3-choosable [9].

The main contribution of this paper is the proof of an analog of the Alon-Tarsi theorem for distance constrained list labelings. A similar approach for so called T-colorings was shown in [3]. Furthermore, as an example for the application of the extended theorem we provide sharp upper bounds on the sizes of lists for the list labelings of cycles and paths with constraints $(2,1)$, as an extension of the labeling results of [10], also with a possible application in linear and circular transmitter networks.

2 Preliminaries

We assume that the set of natural numbers does not contain zero, i.e. $\mathbb{N} = \{1, 2, \dots\}$.

Let $G = (V, E)$ be a finite undirected loopless multigraph with vertices $V = \{v_1, \dots, v_n\}$ and a multiset of edges E over $\binom{V}{2}$. We say that a multigraph H is a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

The symbols C_n and P_n denote the cycle and the path on n vertices. The graph distance between a pair of vertices is the number of edges of the shortest path connecting these two vertices.

An orientation of a multigraph G provides a directed multigraph $\vec{D}$, on the same vertex set V. For each edge $(u, v) \in E(G)$, the graph $\vec{D}$ contains one of ordered pairs $[u, v]$ or $[v, u]$. We define the *indegree* and the *outdegree* of a vertex $u \in V$ as follows:

$$\operatorname{indeg}(u) = |\{[v, u] \in E(\vec{D})\}| \qquad \operatorname{outdeg}(u) = |\{[u, v] \in E(\vec{D})\}|.$$

A directed multigraph is called *Eulerian* if for every vertex u : $\operatorname{indeg}(u) = \operatorname{outdeg}(u)$. The set of all Eulerian subgraphs of a directed multigraph $\vec{D}$ is denoted by $\mathcal{E}(\vec{D})$. We also define sets $\mathcal{E}_o(\vec{D})$ and $\mathcal{E}_e(\vec{D})$ as the sets of Eulerian subgraphs with odd and with even numbers of edges, respectively. Thus, $\mathcal{E}(\vec{D}) = \mathcal{E}_o(\vec{D}) \cup \mathcal{E}_e(\vec{D})$.

A *list assignment* of G is a function L which assigns to each vertex $u \in V(G)$ a *list* $L(u) \subseteq \mathbb{N}$ of *admissible* colors for u. An *L-coloring* is a function $c : V(G) \to \mathbb{N}$ such that $c(u) \in L(u)$ for each $u \in V(G)$ and such that $c(u) \neq c(v)$ whenever u and v are adjacent. If G admits an L-coloring, it is *L-colorable*. We say that G is *k-choosable*, if it admits an L-coloring for every list assignment L such that $|L(u)| \geq k$ for each $u \in V(G)$. The smallest k for which G is k-choosable is called the *choice number* of G and it is denoted by $\chi^\ell(G)$.

In this paper we address a generalization of graph coloring called distance constrained labelings. For a sequence $P = (p_1, \ldots, p_k)$ of positive integers (called *distance constraints*) and a list-assignment L of a graph G we define an *L_P-labeling* as a mapping $c : V(G) \to \mathbb{N}$ such that

(1) $\forall u \in V(G) : c(u) \in L(u)$, and

(2) $|c(u) - c(v)| \geq p_i$ for every $u, v \in V(G)$ at distance at most $i \leq k$.

It follows from the definition that without loss of generality we may assume that numbers in P appear in the nonincreasing order. To avoid a possible misunderstanding we note here that P stands for distance constraints only in Section 3, while in Section 4 the symbol P_n means the path on n vertices.

The non-list version of the above concept can be obtained if we select all lists of available labels as the integer interval $[1, n]$ for some fixed n. Then the smallest n for which G has a labeling satisfying distance constraints P is denoted by $\chi_P(G)$. The graph invariant $\lambda_P(G) = \chi_P(G) - 1$ is also used in the literature, but since we consider list labelings we find $\chi_P(G)$ more convenient than $\lambda_P(G)$.

Similarly, as above, the minimum number k for which G has an L_P-labeling, for every list assignment L with lists of size k, is denoted by $\chi^{\ell}_P(G)$. Observe that in general $\chi_P(G) \leq \chi^{\ell}_P(G)$.

For only one distance constraint $P = (1)$ the $L_{(1)}$-labeling becomes equal to the list coloring, i.e., $\chi^{\ell}_{(1)}(G) = \chi^{\ell}(G)$. Similarly for $p_1 = p_2 = \cdots = p_k = 1$, we get $\chi^{\ell}_{(1,\ldots,1)}(G) = \chi^{\ell}(G^{(k)})$, where $G^{(k)}$ is the k-th power of G, i.e. the graph that arises from G by adding edges connecting vertices at distance at most k. Therefore, the notion of distance constrained labeling generalizes the concept of coloring of powers of graphs as well.

3 Alon-Tarsi theorem for distance constrained labeling

One of the most interesting results in the theory of list-colorings is the following result [2].

Theorem 1 (Alon and Tarsi) *Let $\vec{D}$ be an orientation of a graph G satisfying $|\mathcal{E}_e(\vec{D})| \neq |\mathcal{E}_o(\vec{D})|$. If L is a list assignment of G such that $|L(u)| \geq \mathrm{indeg}(u) + 1$ for all vertices $u \in V(G)$, then the graph G is L-colorable.*

Through this section we assume that $P = (p_1, \ldots, p_k)$ is a fixed k-tuple of distance constraints. Denote by G^P the multigraph with the same vertex set as G such that every two vertices of distance $i \in \{1, \ldots, k\}$ are connected by a cluster of $2p_i - 1$ multiple edges.

We extend Theorem 1 to the concept of distance constrained list labelings as follows:

Theorem 2 *Let L be a list assignment of a graph G and P be a tuple of distance constraints. Suppose that for an orientation $\vec{D}$ of G^P it holds that $|\mathcal{E}_e(\vec{D})| \neq |\mathcal{E}_o(\vec{D})|$ and $\mathrm{indeg}(u) + 1 \leq |L(u)|$ for every vertex $u \in V(\vec{D})$. Then G admits an L_P-labeling.*

Proof: We follow the ideas from [2, 3]. Assign to every vertex u a variable x_u and consider a polynomial in $\mathbb{N}^{|V(G)|}$:

$$f^P_G = \prod_{i=1}^{k} \prod_{j=1-p_i}^{p_i-1} \prod_{\substack{u,v \in V(G)\\ \mathrm{dist}(u,v)=i}} (x_u - x_v - j).$$

In the definition of polynomial f_G^P as well as in the forthcoming definition of the polynomial $g_{\vec{D}}$, we assume that vertices of G are linearly ordered, and that u is a predecessor of v in the innermost product. If c is an L-labeling of G, then c is an L_P-labeling if and only if the polynomial f_G^P evaluates to a nonzero value at $x_u = c(u)$, $u \in V$ (since every term of the product has a nonzero value in such a case). Let

$$r_u(x_u) = x_u^{|L(u)|} - \prod_{s \in L(u)} (x_u - s).$$

Then the degree of r_u is at most $|L(u)| - 1$ and $r_u(c(u)) = c(u)^{|L(u)|}$ if $c(u) \in L(u)$, since $\prod_{s \in L(u)}(x_u - s) = 0$ in such a case.

We expand f_G^P into a linear combination of monomials and recursively replace every occurrence of $x_u^{|L(u)|}$ by $r_u(x_u)$, until the degree of every variable x_u in every monomial of the modified polynomial h_G^P is at most $|L(u)| - 1$. Observe that the new polynomial h_G^P retains the value of f_G^P for all selections $x_u \in L(u)$.

Now consider the multigraph G^P, its orientation $\vec{D}$ and a polynomial

$$g_{\vec{D}} = \prod_{[u,v] \in E(\vec{D})} (x_u - x_v),$$

where the term $(x_u - x_v)$ appears in $g_{\vec{D}}$ as many times as the edge (u, v) in $E(G^P)$. Observe that for different orientations $\vec{D}$ the polynomial $g_{\vec{D}}$ is the same or multiplied by the (-1) factor. It follows from the construction of G^P that every monomial with a nonzero coefficient in $g_{\vec{D}}$ appears also with the same coefficient in f_G^P (upto (-1) factor).

Corollary 2.3. of [2] shows that in the polynomial $g_{\vec{D}}$, the coefficient by the monomial $M = \prod_{u \in V} x_u^{\text{indeg}(u)}$ is equal to $|\mathcal{E}_o(\vec{D})| - |\mathcal{E}_e(\vec{D})|$, perhaps with the multiplicative (-1) factor. By our assumptions, M has nonzero coefficients in all polynomials $g_{\vec{D}}$, f_G^P, and h_G^P, since:

(1) M was not reduced, because it contains no $x_u^{|L(u)|}$, and

(2) monomial M cannot be obtained by such reduction from another monomial $\prod x_u^{d_u}$, because the reduction decreases the value $\sum d_u \leq |E(G^P)|$ and the sum of degrees in M attains the upper bound $|E(G^P)|$.

We may summarize that in this moment we know that h_G^P contains a nonzero coefficient by monomial M, and the degree of every variable in every monomial is at most $\text{indeg}(u) = |L(u)| - 1$.

If we assume that the value of h_G^P is zero for all possible selections $x_u \in L(u)$, we get a contradiction with Lemma 2.1. in [2], since such polynomial $h_G^P \equiv 0$.

We conclude the proof by the argument that any selection of variables $x_u \in L(u)$ having a non-zero value of h_G^P is in one-to-one correspondence with a feasible L_P-labeling of the graph G. □

4 Distance choosability for paths and cycles

It is not difficult to show that for k distance constraints $(1, \ldots, 1)$ and the path P_n on n vertices $\chi_{(1,\ldots,1)}(P_n) = \chi^{\ell}_{(1,\ldots,1)}(P_n) = \min(n, k)$. Recently, Prowse and Woodall [15] proved that the equality $\chi_{(1,\ldots,1)}(C_n) = \chi^{\ell}_{(1,\ldots,1)}(C_n)$ holds also for cycles.

In this section we apply Theorem 2 to $L_{(2,1)}$-labelings and prove that the equality $\chi_{(2,1)}(G) = \chi^{l}_{(2,1)}(G)$ holds for both paths and cycles.

Proposition 3 *Let P_n be the path on $n \geq 2$ vertices. Then,*

$$\chi^{\ell}_{(2,1)}(P_n) = \begin{cases} 3, & n = 2 \\ 4, & n = 3, 4 \\ 5, & n \geq 5. \end{cases}$$

Proof: Since $\chi^{\ell}_{(2,1)}(P_n) \geq \chi_{(2,1)}(P_n)$ and the above equality is satisfied for $\chi^{\ell}_{(2,1)}(P_n)$ replaced by $\chi_{(2,1)}(P_n)$ (see [10]) it will be enough if we prove that $\chi^{\ell}_{(2,1)}(P_n)$ is smaller than or equal to the right hand side of the above expression. Assume that the vertices of the path P_n appear in order $v_1, v_2, \ldots, v_n$. When $n = 2$ the statement is straightforward. In the case $n \geq 5$, the First-fit algorithm works assuming that every vertex has 5 available colors and vertices are ordered as they appear on the path.

For $n = 3$ or 4, let $\vec{D}_3, \vec{D}_4$ be the oriented graphs obtained from $P_3^{(2,1)}$ and $P_3^{(2,1)}$ as depicted in Fig. 1.

Both graphs $\vec{D}_3$ and $\vec{D}_4$ have maximum indegree 3. It is easy to calculate that $|\mathcal{E}_o(\vec{D}_3)| = 9$ and $|\mathcal{E}_e(\vec{D}_3)| = 1$. Graphs from $\mathcal{E}_o(\vec{D}_4)$ may have only 3 or 5 edges. Their number is 15 and 21, respectively. Similarly, graphs

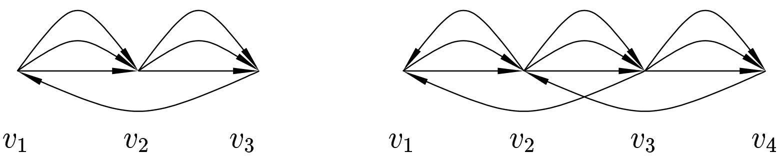

Figure 1: Orientations of graphs $P_3^{(2,1)}$ and $P_4^{(2,1)}$.

from $\mathcal{E}_e(\vec{D}_4)$ may have 0,2,6, or 8 edges. Their number is 1,2,18, and 9, respectively. Thus, $|\mathcal{E}_o(\vec{D}_4)| = 36$ and $|\mathcal{E}_e(\vec{D}_4)| = 30$. Now, the statement follows by Theorem 2. $\square$

It is proved in [10] that $\chi_{(2,1)}(C_n) = 5$ for every $n \geq 3$. We will extend this result to distance constrained list labelings.

Theorem 4 *If C_n is a cycle on $n \geq 3$ vertices, then $\chi^{\ell}_{(2,1)}(C_n) = 5$.*

Proof: Since $\chi^{\ell}_{(2,1)}(C_n) \geq \chi_{(2,1)}(C_n) = 5$ it will be enough if we prove $\chi^{\ell}_{(2,1)}(C_n) \leq 5$. Assume that the vertices in the graph C_n appear in the order $v_1, v_2, \ldots, v_n = v_0, v_1$.

Suppose first that $n = 3$. Let L be a list assignment of C_3 with lists of size 5. Then assume that a_1 is the smallest color in $L(v_1) \cup L(v_2) \cup L(v_3)$ and $a_1 \in L(v_1)$. Now let a_2 be the smallest color from $L(v_2) \cup L(v_3)$ greater than $a_1 + 1$. We may assume that $a_2 \in L(v_2)$. Finally, let a_3 be the greatest color from $L(v_3)$. Now it is easy to see that the labeling c with $c(v_i) = a_i$ for each $i = 1, 2, 3$ is an $L_{(2,1)}$-labeling.

Consider the case $n \geq 4$. Let $\vec{C}$ be the digraph constructed from $C_n^{(2,1)}$ so that every edge is oriented towards the vertex with index greater by one or by two. All computations on indices i of v_i are done modulo n with the only exception that for diagonals of C_4 we use the classical addition. These orientations are depicted in Fig. 2. We call the edges of form $[v_i, v_{i+1}]$ *short*, while those of type $[v_i, v_{i+2}]$ are *long*. We use the following simpler notation without the argument $\vec{C}$, i.e. $\mathcal{E}_o = \mathcal{E}_o(\vec{C})$ and $\mathcal{E}_e = \mathcal{E}_e(\vec{C})$.

Suppose now that $n = 4$. Then $\text{indeg}(v_1) = \text{indeg}(v_2) = 3$ and $\text{indeg}(v_3) = \text{indeg}(v_4) = 4$. Observe that each Eulerian subgraph with odd number of edges contains exactly one of the arcs $[v_1, v_3]$ and $[v_2, v_4]$. Now it is easy to evaluate that the Eulerian subgraphs from $\mathcal{E}_o$ may have 3, 7, or 11 edges. The numbers of such subgraphs are 18, 162, and 18, respectively. Similarly, $\vec{C}$ has 1, 81, 27, 81, 27, and 1 Eulerian subgraphs with 0, 4, 6, 8, 10, or

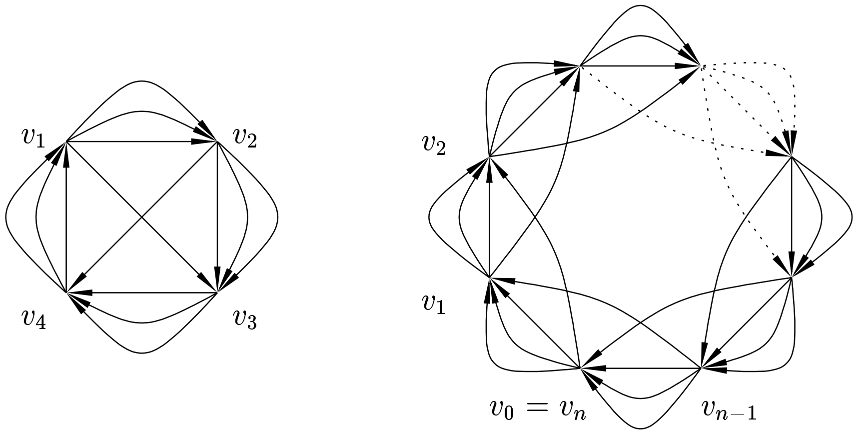

Figure 2: Orientations of graphs $C_4^{(2,1)}$ and $C_n^{(2,1)}$.

12 edges, respectively. Thus, $|\mathcal{E}_o| = 198$ and $|\mathcal{E}_e| = 218$, and the equality $\chi^{\ell}_{(2,1)}(C_4) = 5$ follows directly from Theorem 2.

We now prove the case of cycles of length at least 5. The basic idea is as follows: We derive recursive formulas for the numbers of Eulerian subgraphs with specified degrees. Then we provide an explicit formula for the difference of the numbers of even and odd Eulerian subgraphs and show that it never attains zero. Because the indegree of every vertex in the proposed orientation $\vec{C}$ is 4, we conclude that for lists of size 5 a feasible $L_{(2,1)}$-labeling of C_n always exists.

Suppose that $n \geq 5$. Let $S \subseteq \{0, 1, 2, 3, 4\}$. We denote by $\mathcal{E}_e^S$ and $\mathcal{E}_o^S$ the Eulerian subgraphs of $\vec{C}$ with indegrees in S and with even and odd number of edges, respectively. Moreover, we assume that for every number from S there exists at least one vertex in $\vec{C}$ with indegree equal to this number. For simplicity, in the expression $\mathcal{E}^S$ we write the set S without braces, thus $\mathcal{E}^{0,1}$ means the same as $\mathcal{E}^{\{0,1\}}$.

Assume an Eulerian tour on $\vec{C}$ that winds i-times around the cycle. Then all its vertices are of indegree either i or $i-1$. Thus, whenever S contains i, j with $|i - j| \geq 2$, then no such Eulerian subgraph exists, hence $\mathcal{E}_o^S = \mathcal{E}_e^S = \emptyset$. The total number of edges in $\vec{C}$ is even. The complement operation $\phi : \mathcal{E} \to \mathcal{E}$ with $\phi(H) = (V, E(\vec{C}) \setminus E(H))$ is one-to one mapping between sets $\mathcal{E}_o^i$ and $\mathcal{E}_o^{4-i}$, or $\mathcal{E}_o^{i,i+1}$ and $\mathcal{E}_o^{4-i,3-i}$. Due to this complementarity we get that $|\mathcal{E}_o^0| = |\mathcal{E}_o^4|$, $|\mathcal{E}_o^{0,1}| = |\mathcal{E}_o^{3,4}|$, etc. The same equalities hold for even subgraphs as well. Thus, it is sufficient to consider only graph classes $\mathcal{E}_o^0, \mathcal{E}_o^{0,1}, \mathcal{E}_o^1, \mathcal{E}_o^{1,2}, \mathcal{E}_o^2$, and their counterparts for even Eulerian subgraphs.

The subgraph with no edges is even Eulerian, hence $|\mathcal{E}_o^0| = 0$ and $|\mathcal{E}_e^0| = 1$. The number of edges of a 1-regular digraph is equal to the number of its vertices, hence $|\mathcal{E}_o^1| = 0$ for n even and $|\mathcal{E}_e^1| = 0$ for n odd. In the other case there are 3^n cycles composed of short edges and one Eulerian subgraph composed by all long edges. We have $|\mathcal{E}_o^1| = 3^n+1$ for odd n and $|\mathcal{E}_e^1| = 3^n+1$ for n even.

Now, we calculate the size of $\mathcal{E}_o^{0,1}$ and $\mathcal{E}_e^{0,1}$. In $\vec{C}$, we remove the edge $[v_{n-1}, v_1]$, and cut the vertex $v_n = v_0$ into two new vertices v_0, v_n such that the new v_0 is incident with all outgoing edges and the new v_n with all incoming. The graph $\vec{P}$ obtained in this way is isomorphic to the monotone orientation of the graph $P_{n+1}^{(2,1)}$. Denote by x_n^o (and by x_n^e) the number of directed paths of odd (and of even) length from the vertex v_0 to v_n in $\vec{P}$. The numbers x_n^o and x_n^e are defined by the recursive relations:

$$x_1^o = 3, \quad x_2^o = 1, \quad x_n^o = 3x_{n-1}^e + x_{n-2}^e$$
$$x_1^e = 0, \quad x_2^e = 9, \quad x_n^e = 3x_{n-1}^o + x_{n-2}^o.$$

Observe that the number of graphs from $\mathcal{E}_o^{0,1} \cup \mathcal{E}_o^1$ (resp. $\mathcal{E}_e^{0,1} \cup \mathcal{E}_e^1$) which contain an edge incident with the vertex v_0 and do not contain $[v_{n-1}, v_1]$ is x_n^o (resp. x_n^e). Similarly, there are x_{n-2}^e (and x_{n-2}^o) graphs from $\mathcal{E}_o^{0,1} \cup \mathcal{E}_o^1$ (and $\mathcal{E}_e^{0,1} \cup \mathcal{E}_e^1$) which contain the long edge $[v_{n-1}, v_1]$ and no edge incident with v_0. Finally, only the subgraph induced by all n long edges has the property that it contains both the edge $[v_{n-1}, v_1]$ and an edge incident with the vertex v_0. In total we got that

$$\begin{aligned} |\mathcal{E}_o^{0,1} \cup \mathcal{E}_o^1| &= x_n^o + x_{n-2}^e + (n \bmod 2) \\ |\mathcal{E}_e^{0,1} \cup \mathcal{E}_e^1| &= x_n^e + x_{n-2}^o + 1 - (n \bmod 2), \end{aligned}$$

where $(n \bmod 2)$ is 1 for n odd and 0 for n even.

Since every 2-regular digraph has an even number of edges, it follows that $\mathcal{E}_o^2 = \emptyset$. Now we show that $\mathcal{E}_e^2$ contains precisely $2 \cdot 3^n$ graphs. We can connect v_i and v_{i+1} with two edges in three different ways, and it follows that the number of graphs from $\mathcal{E}_e^2$ which contain only short edges is 3^n. If a graph from $\mathcal{E}_e^2$ contains a long edge then it contains all long edges of $\vec{C}$. Similarly as above, we obtain that $\mathcal{E}_e^2$ contains exactly 3^n such graphs. Thus, $|\mathcal{E}_e^2| = 2 \cdot 3^n$.

We proceed by calculating $|\mathcal{E}_o^{1,2} \cup \mathcal{E}_o^2|$. Denote by $\mathcal{P}_n$ the set of subgraphs in $\vec{P}$ that define a 2-flow from v_0 to v_n with an odd number of edges. More formally such graphs satisfy $\mathrm{outdeg}(v_i) = \mathrm{indeg}(v_i)$ for all $i = 1, \ldots, n-1$ and $\mathrm{indeg}(v_n) = \mathrm{outdeg}(v_0) = 2$. We call such graphs *odd 2-paths.*

Note that every odd 2-path starts/ends with either two short edges or one short and one long edge. In what follows $p_n(a, b)$ (where $a, b \in \{l, s, *\}$) denotes the numbers of graphs from $\mathcal{P}_n$ with the following property: if $a = l$ then this 2-path starts with one long and one short edge, if $a = s$ then it starts with two short edges and if $a = *$ then the type of the edges by which this 2-path starts is not important. Similarly b specifies the way this graph ends. For example, $p_n(*, *) = |\mathcal{P}_n|$ and $p_n(s, l)$ is the number of graphs from $\mathcal{P}_n$ which start with two short edges and end with one long and one short edge. For simplicity, we write $p_n = p_n(*, *)$. Observe that for every $x \in \{l, s, *\}$ and $n \geq 2$:

$$\begin{aligned} p_n(x, l) &= p_n(l, x) = p_n(*, x) - 3p_{n-1}(*, x) \\ p_n(x, s) &= p_n(s, x) = 3p_{n-1}(*, x). \end{aligned}$$

From these relations we derive that for $n \geq 3$:

$$\begin{aligned} p_n(l, l) &= p_n - 6p_{n-1} + 9p_{n-2} \\ p_n(s, l) &= 3p_{n-1} - 9p_{n-2} \\ p_n(s, s) &= 9p_{n-2}. \end{aligned}$$

In a similar way as above define an *even 2-path* in $\vec{P}$. Denote by $\mathcal{Q}_n$ the set of even 2-paths in $\vec{P}$. We define the parameter $q_n(a, b)$ for graphs from $\mathcal{Q}_n$ and obtain analogous equations as above for $p_n(a, b)$. Similarly we write $q_n = q_n(*, *)$.

Now consider an odd 2-path in $\vec{P}$. The three possible ways of building of such path as an extension of shorter one are depicted in Fig. 3. This directly implies two recursive relations between p_n and q_n:

$$p_n = 3p_{n-1} + q_{n-1} + 6q_{n-2} \quad \text{and} \quad q_n = 3q_{n-1} + p_{n-1} + 6p_{n-2}.$$

Note that $p_1 = 0$, $p_2 = 9$, $q_1 = 3$, and $q_2 = 9$. Thus, the above two relations provide the recursive definition of sequences (p_n) and (q_n). We now focus our attention back to graphs from $\mathcal{E}_o^{1,2} \cup \mathcal{E}_o^2$ and $\mathcal{E}_e^{1,2} \cup \mathcal{E}_e^2$. In exactly p_n graphs from $\mathcal{E}_o^{1,2} \cup \mathcal{E}_o^2$ which do not contain the edge $v_{n-1}v_1$, the vertex v_0 has outdegree two. Now we count the number of graphs from $\mathcal{E}_o^{1,2} \cup \mathcal{E}_o^2$, in which v_0 has outdegree one. Such graphs necessarily contain the long edge

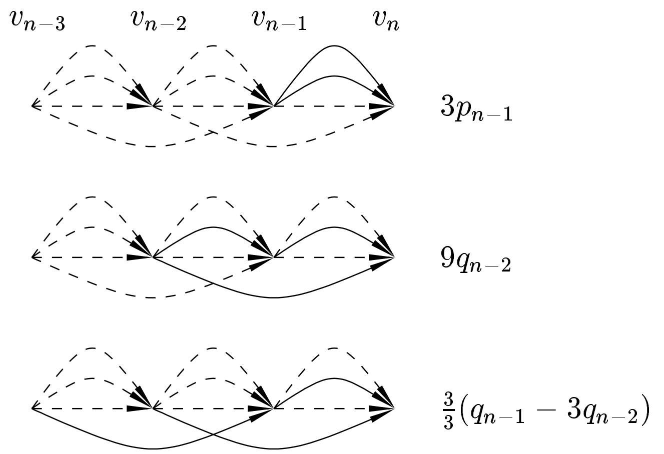

Figure 3: Composition of a 2-path from shorter 2-paths.

$[v_{n-1}, v_1]$. Denote by e^+ (resp. e^-) the edge which goes out (resp. in) from v_0. The number of graphs in which both e^+ and e^- are short is exactly $q_n(s,s)$. Similarly the number of graphs for which e^+ and e^- are long edges is $q_n(l,l)/9$. Finally the number of graphs in which e^+ is long and e^- is short or vice-versa is $q_n(l,s)/3 + q_n(s,l)/3$. Thus, we obtain that

$$\begin{aligned} |\mathcal{E}_o^{1,2} \cup \mathcal{E}_o^2| &= p_n + q_n(s,s) + \frac{q_n(l,s) + q_n(s,l)}{3} + \frac{q_n(l,l)}{9} \\ &= \frac{5}{3}p_n + \frac{1}{9}q_n - 2p_{n-1} + \frac{2}{3}q_{n-1}. \end{aligned}$$

In a similar way we calculate $|\mathcal{E}_e^{1,2} \cup \mathcal{E}_e^2|$, where the only difference is that we must encounter also graphs which contain the edge $v_{n-1}v_1$ and in the same time where v_0 has outdegree two. Such graphs necessarily contain all long edges and their number is exactly 3^n. Observe also that such graphs are specific only for $\mathcal{E}_o^2$ — they cannot have an odd number of edges.

$$|\mathcal{E}_e^{1,2} \cup \mathcal{E}_e^2| = \frac{5}{3}q_n + \frac{1}{9}p_n - 2q_{n-1} + \frac{2}{3}p_{n-1} + 3^n.$$

Now, the term $|\mathcal{E}_o| - |\mathcal{E}_e|$ can be expressed as

$$
\begin{aligned}
|\mathcal{E}_o| - |\mathcal{E}_e| &= 2\Big[|\mathcal{E}_o^0| + |\mathcal{E}_o^{0,1} \cup \mathcal{E}_o^1| + |\mathcal{E}_o^{1,2} \cup \mathcal{E}_o^2|\Big] - \mathcal{E}_o^2 \\
&- 2\Big[|\mathcal{E}_e^0| + |\mathcal{E}_e^{0,1} \cup \mathcal{E}_e^1| + |\mathcal{E}_e^{1,2} \cup \mathcal{E}_e^2|\Big] + \mathcal{E}_e^2 \\
&= 2\Big[x_n^o + x_{n-2}^e + (n \bmod 2) + \frac{5}{3}p_n + \frac{1}{9}q_n - 2p_{n-1} + \frac{2}{3}q_{n-1}\Big] \\
&- 2\Big[1 + x_n^e + x_{n-2}^o + 1 - (n \bmod 2) \\
&+ \frac{5}{3}q_n + \frac{1}{9}p_n - 2q_{n-1} + \frac{2}{3}p_{n-1} + 3^n\Big] + 2 \cdot 3^n \\
&= 2\Big[2(n \bmod 2) - 2 + x_n^o - x_n^e - x_{n-2}^o + x_{n-2}^e \\
&+ \frac{14}{9}(p_n - q_n) - \frac{8}{3}(p_{n-1} - q_{n-1})\Big]
\end{aligned}
$$

To simplify the above expression define $x_n = x_n^o - x_n^e$ for every n. Then, we infer that $x_1 = 3$, $x_2 = -8$, and $x_n = -3x_{n-1} - x_{n-2}$. Similarly, let $r_n = p_n - q_n$ for every n. Then, we obtain that $r_1 = -3$, $r_2 = 0$, and $r_n = 2r_{n-1} - 6r_{n-2}$. The parameters x_n and r_n could be expressed explicitly for $n \geq 1$ as:

$$
\begin{aligned}
x_n &= \Big(-\frac{1}{2} - \frac{3\sqrt{5}}{10}\Big)\Big(\frac{-3-\sqrt{5}}{2}\Big)^n + \Big(-\frac{1}{2} + \frac{3\sqrt{5}}{10}\Big)\Big(\frac{-3+\sqrt{5}}{2}\Big)^n \\
r_n &= \Big(-\frac{1}{2} - \frac{i}{\sqrt{5}}\Big)(1 - i\sqrt{5})^n + \Big(-\frac{1}{2} + \frac{i}{\sqrt{5}}\Big)(1 + i\sqrt{5})^n.
\end{aligned}
$$

We continue our calculations as follows[2]:

$$
\begin{aligned}
|\mathcal{E}_o| - |\mathcal{E}_e| &= 4(n \bmod 2) - 4 + 2x_n - 2x_{n-2} + \frac{28}{9}r_n - \frac{16}{3}r_{n-1} \\
&= 4(n \bmod 2) - 4 - 2\Big(\frac{-3-\sqrt{5}}{2}\Big)^n - 2\Big(\frac{-3+\sqrt{5}}{2}\Big)^n \\
&- 2(1 - i\sqrt{5})^n - 2(1 + i\sqrt{5})^n
\end{aligned}
$$

[2]E.g. we get the coefficient by $(\frac{-3-\sqrt{5}}{2})^n$ as: $2(-\frac{1}{2} - \frac{3\sqrt{5}}{10}) - 2(-\frac{1}{2} - \frac{3\sqrt{5}}{10})/(\frac{-3-\sqrt{5}}{2})^2 = 2(\frac{1}{2} + \frac{3\sqrt{5}}{10})(\frac{4}{14+6\sqrt{5}} - 1) = \frac{5+3\sqrt{5}}{5}(\frac{2(7-3\sqrt{5})}{4} - 1) = \frac{(5+3\sqrt{5})(5-3\sqrt{5})}{10} = -2$

n	5	6	7	8	9	10
$\|\mathcal{E}_o\| - \|\mathcal{E}_e\|$	-58	-1352	2102	638	19172	-45362

Table 1: The difference $|\mathcal{E}_o| - |\mathcal{E}_e|$ for small n

For $n = 5, \ldots, 10$ the values of the difference $|\mathcal{E}_o| - |\mathcal{E}_e|$ are summarized in Table 1. For $n \geq 11$, the absolute value of the term $(\frac{-3-\sqrt{5}}{2})^n$ outweighs the other terms, so after the numerical evaluation we get that:

$$\begin{aligned}
\Big||\mathcal{E}_o| - |\mathcal{E}_e|\Big| &\geq 2\left|\frac{-3-\sqrt{5}}{2}\right|^n - 2\left|\frac{-3+\sqrt{5}}{2}\right|^n \\
&- 2\left|1 - i\sqrt{5}\right|^n - 2\left|1 + i\sqrt{5}\right|^n - 4(n \bmod 2) - 4 \\
&\geq 2 * 2.618^n - 2 * 0.382^n - 4 * 2.450^n - 8 \\
&\geq 2 * 2.618^{11} - 2 * 0.382^{11} - 4 * 2.450^{11} - 8 > 0
\end{aligned}$$

This argument encloses the proof of Theorem 4. □

5 Conclusion and open problems

In this study we have proved and analog of the Alon-Tarsi theorem for distance constrained labelings and have applied this result on paths and cycles and distance constraints $(2, 1)$. Since this method brings a non constructive improvement up to the factor at most two the list sizes against the classical First-fit algorithm, we believe that the methods and calculations involved for computing odd and even Eulerian subgraphs are interesting in the theoretical sense rather than applicable in practice.

On the other hand, by use of this method we proved that the equality $\chi_{(2,1)}(G) = \chi^{\ell}_{(2,1)}(G)$ holds for all cycles and paths. This is a special case of the general inequality

$$\chi_P(G) \leq \chi^{\ell}_P(G).$$

It would be interesting to classify all graphs for which the equality holds with respect to given distance constraints P. Our intuition says that the class of paths and cycles would be a good candidate for such result, so we would like to express this fact as the following conjecture:

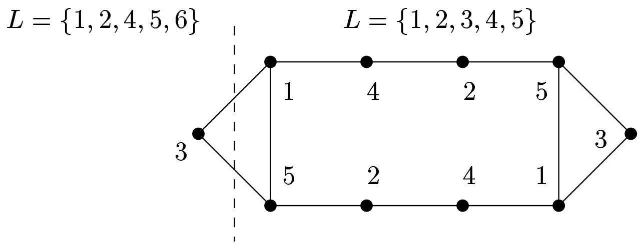

Figure 4: A graph satisfying the strict inequality $\chi_{(2,1)}(G) < \chi^{\ell}_{(2,1)}(G)$

Conjecture 5 *For all distance constraints P and for all $n > 0$, it holds that*

$$\chi_P(P_n) = \chi^{\ell}_P(P_n) \qquad \textit{and} \qquad \chi_P(C_n) = \chi^{\ell}_P(C_n).$$

In [6] are shown constructions of graphs satisfying the strict inequality for the standard graph coloring and choosability, i.e. $P = (1)$. The easiest example of such graph is the complete bipartite graph K_{k,k^k} satisfying $\chi(K_{k,k^k}) = 2 < k < \chi^{\ell}(K_{k,k^k})$.

We expect that in the case of distance constrained labeling it would be easier to construct such examples of graphs as there are more constraints on the possible labeling.

As an example we would like to present a graph depicted in Fig. 4. A feasible labeling showing $\chi_{(2,1)}(G) = 5$ is indicated by the numbers by the vertices, while for the indicated lists it is impossible to find a feasible $L_{(2,1)}$-labeling. Therefore, $\chi^{\ell}_{(2,1)}(G) > 5 = \chi_{(2,1)}(G)$.

Another possible direction of the further research would consider the relation of distance constrained labeling with similar distance constraints. Observe that in the non-list version we may assume that distance constraints have no common divisor [7, 10]:

Let $P = (p_1, p_2, \dots, p_s)$ and we denote by kP the tuple $(kp_1, kp_2, \dots, kp_s)$. Then for every graph G it holds that

$$k(\chi_P(G) - 1) = \chi_{kP}(G) - 1.$$

It could be an interesting adventure to search for some necessary as well as sufficient conditions under which an analogous equality holds for list labelings:

$$k(\chi^{\ell}_P(G) - 1) = \chi^{\ell}_{kP}(G) - 1.$$

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References

[1] N. Alon, *Restricted colorings of graphs*, In: *Surveys in combinatorics*, Lond. Math. Soc. Lect. Note Ser. 187, (1993) 1–33.

[2] N. Alon and M. Tarsi, *Colorings and orientations of graphs*, Combinatorica **12** (1992) 125–134.

[3] N. Alon and A. Zaks, *T-choosability in graphs*, Discrete Applied Math. **82** (1998) 1–13.

[4] M. Bellare, O. Goldreich and M. Sudan. *Free bits, PCPs and non-approximability - towards tight results*, SIAM J. Comp. **27** (1998) 804–915.

[5] A. Eisenblätter, M. Grötschel, A. M. Koster, *Frequency Assignment and Ramifications of Coloring*, ZIB Report 00-47, December 2000, (to appear in Disc. Math. Graph Theory).

[6] P. Erdős, A. L. Rubin, and H. Taylor, *Choosability in graphs*, Congr. Numer. **26** (1980) 125–157.

[7] J. Fiala, *Locally injective Homomorphisms*, Doctoral Thesis, Charles University, 2000.

[8] J. Fiala, A. V. Fishkin, and F. V. Fomin, *Off-line and on-line distance constrained labeling of graphs*, In *Proceedings of the 9th European Symposim on Algorithms ESA'01*, LNCS 2161 (2001) 464–475.

[9] H. Fleischner and M. Stiebitz, *A solution to a colouring problem of P. Erdös*, Discrete Math. **101** (1992) 39–48.

[10] J. Griggs and R. Yeh, *Labeling graphs with a condition at distance 2*, SIAM J. of Discrete Math. **5** (1992) 586–595.

[11] W. K. Hale, *Frequency assignment: Theory and applications*, Proc. of the IEEE **68** (12) (1980) 1497–1514.

[12] J. van den Heuvel, R. A. Leese, and M. A. Shepherd, *Graph labeling and radio channel assignment*, Journal of Graph Theory **29** (1998), 263–283.

[13] J. Kratochvíl, Zs. Tuza and M. Voigt, *New trends in the theory of graph colorings: Choosability and list coloring*, In: *Contemporary trends in discrete mathematics*, Discrete Math. Theor. Comput. Sci. 49, (1999) 183–197.

[14] D. D.-F. Liu and R. K. Yeh, *On Distance Two Labellings*, Ars Combinatoria **47** (1997) 13–22.

[15] A. Prowse and D. R. Woodall, *Choosability of powers of circuits*, manuscript, 2000.

[16] D. Sakai, *Labeling chordal graphs: distance two condition*, SIAM J. of Discrete Math. **7** (1994) 133–140.

[17] Zs. Tuza, *Graph colorings with local constraints—a survey*, Discuss. Math., Graph Theory **17** (2) (1997) 161–228.

[18] V. G. Vizing, *Coloring the vertices of a graph in prescribed colors* (in Russian), Metody Diskret. Analiz. **29** (1976) 3–10.

[19] M. A. Whittlesey, J. P. Georges, and D. W. Mauro, *On the λ-number of Q_n and related graphs*, SIAM J. of Discrete Math. **8** (1995) 499–506.

[20] D. R. Woodall, *List colourings of graphs*, In *Surveys in Combinatorics* London Math. Soc. Lecture Note Series 288, (2001) 269-301.