University of Ljubljana
Institute of Mathematics, Physics and Mechanics
Department of Mathematics
Jadranska 19, 1000 Ljubljana, Slovenia

Preprint series, Vol. 40 (2002), 825

CHARACTERIZING SUBGRAPHS OF HAMMING GRAPHS

Sandi Klavžar Iztok Peterin

ISSN 1318-4865

May 22, 2002

Ljubljana, May 22, 2002

Characterizing subgraphs of Hamming graphs

Sandi Klavžar *
Department of Mathematics, PeF
University of Maribor
Koroška cesta 160, 2000 Maribor, Slovenia
email: sandi.klavzar@uni-lj.si

Iztok Peterin
Faculty of Electrical Engineering and Computer Science
University of Maribor
Smetanova ulica 17, 2000 Maribor, Slovenia
email: iztok.peterin@uni-mb.si

May 21, 2002

Abstract

Cartesian products of complete graphs are known as Hamming graphs. Using embeddings into Cartesian products of quotient graphs we characterize subgraphs, induced subgraphs, and isometric subgraphs of Hamming graphs. For instance, a graph G is an induced subgraph of a Hamming graph if and only if there exists a labeling of $E(G)$ fulfilling the following two conditions: (i) incident edges receive the same label if and only if they lie on a common triangle; (ii) for any vertices u and v at distance at least two, there exist two labels such that they appear on any induced u, v-path.

Key words: Hamming graphs; Induced subgraphs; Isometric subgraphs; Edge-labelings; Cartesian products; Quotient graphs

1 Introduction

Hamming graphs are, by definition, Cartesian products of complete graphs. For different characterizations of these graphs see [2, 3, 19, 20, 21]. Hamming graphs can be recognized in linear time and space [13, 14]. Isometric subgraphs of Hamming graphs, called *partial Hamming graphs*, have been intensively studied by now, cf. [1, 4, 6, 23, 24]. One of the most important subclasses of partial Hamming graphs is formed by weak

*Supported by the Ministry of Education, Science and Sport of Slovenia under the grant 0101-P-504.

retracts of Hamming graphs; these graphs are known as quasi-median graphs. Quasi-median structures have been independently discovered several times, cf. [3] and references therein.

Graham and Winkler [10] proved that every graph has a best representation as an isometric subgraph of a Cartesian product, cf. also [11, 15]. The key construction is based on embeddings into Cartesian products of the so-called quotient graphs with respect to a certain relation defined on the edge set of a graph. Feder [8, 9] followed with a similar approach in order to obtain such representations for stronger embeddability conditions: 2-isometric representation, weak retract representation and Cartesian prime factorization.

In this paper we take the opposite direction, treating isometry as the strongest property. We namely consider subgraphs (see also [22]), induced subgraphs, and isometric subgraphs of Hamming graphs. We show, roughly speaking, that embeddings into Cartesian product of quotient graphs can be applied also to subgraphs and induced subgraphs of Hamming graphs. Of course, as the embeddability conditions are rather weak in these two cases, we cannot expect to obtain some "best" (say unique) representation for subgraphs and induced subgraphs.

In the rest of this section we fix the notation and introduce the scheme for embeddings into Cartesian products of quotient graphs. In Section 2 we first observe that embeddings of graphs as subgraphs into general Cartesian product graphs are equivalent to embeddings into Hamming graphs (of two factors). Then we characterize subgraphs of Hamming graphs via certain edge labelings of graphs. In Section 3 we give a similar characterization for induced subgraphs of Hamming graphs. As far as we know, no characterization of induced subgraphs of Hamming graphs was previously known. In the last section we briefly discuss partial Hamming graphs, adding one more characterization of these graphs to the literature.

We consider finite, undirected graphs, without loops or multiple edges.

The *Cartesian product* $G \Box H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ in which the vertex (a, x) is adjacent to the vertex (b, y) whenever $ab \in E(G)$ and $x = y$, or $a = b$ and $xy \in E(H)$. For a fixed vertex a of G, the vertices $\{(a, x) \mid x \in V(H)\}$ induce a subgraph of $G \Box H$ isomorphic to H, called an H-*layer* of $G \Box H$. Analogously we define G-layers. A subgraph of $G \Box H$ is called *nontrivial* if it intersects at least two G-layers and at least two H-layers. The map $p_G : G \Box H \to G$ defined by $p_G(u, v) = u$, is called a *projection*. Clearly, the image of an edge under a projection is either an edge or a vertex.

As we already mentioned, Cartesian products of complete graphs are known as *Hamming graphs*. They can be alternatively described as follows. For $i = 1, 2, \ldots, n$ let $r_i \geq 2$ be given integers. Let G be the graph whose vertices are the n-tuples $b_1 b_2 \ldots b_n$ with $b_i \in \{0, 1, \ldots, r_i - 1\}$. Two vertices are adjacent if the corresponding tuples differ in precisely one place—one coordinate. Then it is easy to see that G is isomorphic to $K_{r_1} \Box K_{r_2} \Box \cdots \Box K_{r_n}$. For an edge uv of $H = K_{r_1} \Box K_{r_2} \Box \cdots \Box K_{r_n}$ we define the *color map* $c : E(H) \to \{1, 2, \ldots, n\}$ with $c(uv) = i$, where u and v differ in coordinate i.

A subgraph H of G is called *isometric* if $d_H(u, v) = d_G(u, v)$ for all $u, v \in V(H)$, where $d_G(u, v)$ denotes the length of a shortest u, v-path. Note that an isometric subgraph is induced.

We now introduce the central concept of this paper. Let G be a connected graph and let $\mathcal{F} = \{F_1, F_2, \ldots, F_k\}$ be a partition of $E(G)$. The *quotient graph* G/F_i has connected components of $G \setminus F_i$ as vertices, two components C and C' being adjacent whenever there exists an edge of F_i connecting a vertex of C with a vertex of C'. For each i, define a map $f_i : V(G) \to V(G/F_i)$ by $f_i(v) = C$, where C is the component of $G \setminus F_i$ containing v. Then let

$$f : G \to G/F_1 \Box G/F_2 \Box \cdots \Box G/F_k$$

be the natural coordinate-wise mapping, that is,

$$f(v) = (f_1(v), f_2(v), \ldots, f_k(v)).$$

We call f the *quotient map* of G with respect to $\mathcal{F}$. Note that f need not be one-to-one in general and that it is possible that some quotient graphs are the one vertex graph. However, all the partitions $\mathcal{F}$ introduced later will lead to one-to-one mappings with nontrivial quotient graphs.

A partition $\{F_1, F_2, \ldots, F_k\}$ of $E(G)$ naturally leads to an edge-labeling $\ell : E(G) \to \{1, 2, \ldots, k\}$ by setting $\ell(e) = i$, where $e \in F_i$. Unless stated otherwise, a *labeling* (or more precisely a k*-labeling*) of G will mean an edge-labeling (with k labels).

Finally, two edges $e = xy$ and $f = uv$ of a graph G are in the Djoković-Winkler [7, 24] relation Θ if $d_G(x, u) + d_G(y, v) \neq d_G(x, v) + d_G(y, u)$. Relation Θ is reflexive and symmetric, Θ^* stands for the transitive closure of Θ. Let $\mathcal{F}$ be the partition of $E(G)$ induced by Θ^*. Graham and Winkler [10] proved that the corresponding quotient map is an isometry and called it the *canonical isometric embedding* of G.

2 Subgraphs of Hamming graphs

Subgraphs of Cartesian product graphs were first investigated by Lamprey and Barnes [17, 18] and later characterized in [16] using certain vertex-labelings. Additional results are given in [5]. In this section we give another characterization of them, this time using edge-labelings. Before giving the result, we note that the general problem is equivalent to that of subgraphs of Hamming graphs, as the next lemma asserts. Recall that a subgraph G of $G_1 \Box G_2$ is nontrivial if the projections $p_{G_1}(G)$ and $p_{G_2}(G)$ both contain at least two vertices. More generally, G is a nontrivial subgraph of $G_1 \Box G_2 \Box \cdots \Box G_k$, if $p_{G_i}(G)$ contains at least two vertices for $i = 1, 2, \ldots, k$.

Lemma 2.1 *A graph G is a nontrivial subgraph of the Cartesian product of graphs if and only if G is a nontrivial subgraph of the Cartesian product of two complete graphs.*

Proof. Clearly, we only need to prove that a nontrivial subgraph of the Cartesian product of graphs can be embedded as a nontrivial subgraph into the Cartesian product of two complete graphs. So let G be a nontrivial subgraph of $G_1 \Box G_2 \Box \cdots \Box G_k = G_1 \Box (G_2 \Box \cdots \Box G_k)$. Setting $H_1 = G_2 \Box \cdots \Box G_k$, we get that G is a nontrivial subgraph of $G_1 \Box H_1$. But now we can connect any nonadjacent vertices of G_1 and of H_1 without violating that G is a nontrivial subgraph. We conclude that G is a nontrivial subgraph of $K_{|V(G_1)|} \Box K_{|V(H_1)|}$. $\Box$

By Lemma 2.1 we only need to consider 2-labelings of graphs with respect to their embeddability into Cartesian products. Thus, for a given 2-labeling of G we pose the following condition.

Condition A. *Let G be a 2-labeled graph. Let C be an induced cycle of G that possesses both labels. Then the labels change at least three times while passing the cycle.*

Note that if G has a 2-labeling obeying Condition A, then the edges of a triangle have the same label.

Theorem 2.2 *Let G be a connected graph. Then G is a nontrivial subgraph of the Cartesian product of graphs if and only if there exists a 2-labeling of G that fulfills Condition A.*

Proof. Let G be a nontrivial subgraph of the Cartesian product of graphs. By Lemma 2.1 we can restrict to products of two complete graphs.

Let G be a nontrivial subgraph of $K_n \Box K_m$, $n, m \geq 2$ and let $e \in E(G)$. Then we set

$$\ell(e) = \begin{cases} 1; & p_{K_n}(e) \text{ is an edge,} \\ 2; & p_{K_n}(e) \text{ is a vertex.} \end{cases}$$

Let $C = v_1 v_2 \ldots v_k$ be an induced cycle of G that possesses both labels. Suppose that labels change only twice on C, that is, the labels along C are $1, \ldots, 1, 2, \ldots, 2$, where $v_1 v_2$ is the first edge with label 1. By the definition of the Cartesian product and ℓ, vertex v_1 is of degree at least 4 on C. As this is not possible, the labeling ℓ fulfills Condition A.

Conversely, let ℓ be a 2-labeling of a graph G that satisfies Condition A. Let $\mathcal{F} = \{F_1, F_2\}$ be the partition of $E(G)$ induced by ℓ and let f be the quotient map of G with respect to $\mathcal{F}$. We are going to show that G is a nontrivial subgraph of $G/F_1 \,\square\, G/F_2$, where G/F_1 and G/F_2 both contain at least two vertices.

We claim that f is one-to-one. Let $u \neq v$ be vertices of G. Suppose first that $e = uv \in E(G)$ and assume without loss of generality that $\ell(uv) = 1$, that is, $uv \in F_1$. We claim that u and v are in different components of $G \setminus F_1$. Let P be an arbitrary path connecting the endvertices of e not containing e. Suppose that every edge of P has label 2 and select P to be shortest possible among all such paths. The edge e together with the path P forms a cycle C. By Condition A, C is not induced. Let w and w' be two non-consecutive but adjacent vertices of C. Select w and w' such that the distance between w and w' along the cycle is as short as possible. Then by the minimality of P we have $\ell(ww') = 1$. But then we find an induced cycle C' containing the edge ww' and the w, w'-path on C containing only labels 2. Hence we are in conflict with Condition A and so $f(u) \neq f(v)$.

Suppose next $d_G(u,v) \geq 2$. If every u,v-path contains at least one edge with label 1, or if every u,v-path contains at least one edge with label 2, we are done. Indeed, then u and v are mapped into different vertices in at least one of G/F_1 and G/F_2. Thus assume that there are u,v-paths P and Q such that all edges of P receive label 1 and all edges of Q label 2. Let P and Q be shortest among all such paths. Let w be the first common vertex of P and Q traversing these two paths from u to v. (Note that we may have $w = v$.) Then the u,w-subpath P' of P together with the u,w-subpath Q' of Q form a cycle C. If C is induced we violate Condition A. Hence assume C is not induced. Then we have three possibilities. The first is that there is an edge between nonconsecutive vertices of P'. By the minimality of P, the label of this edge is 2. The second case is that there is an edge between nonconsecutive vertices of Q'. By the minimality of Q, the label of this edge must be 1. The last possibility is that there is an edge between a vertex of P' and a vertex of Q'. Such an edge can be labeled 1 or 2. In any of the three cases, we can select the corresponding edge in such a way that we obtain an induced cycle that violates Condition A. We conclude that $f(u) \neq f(v)$ holds also in this case, which proves the claim.

To conclude the proof we just need to observe that an edge of G is mapped by the quotient map to an edge of $G/F_1 \,\square\, G/F_2$. Hence G is a subgraph of $G/F_1 \,\square\, G/F_2$. Moreover, since each of G/F_1 and G/F_2 contains at least two vertices, it also follows that G is a nontrivial subgraph. □

Corollary 2.3 *A graph G is a nontrivial subgraph of a Hamming graph if and only if there exists a 2-labeling of G that fulfills Condition A.*

Proof. Combine Theorem 2.2 with Lemma 2.1. □

Theorem 2.2 and its proof are illustrated in Fig. 1. A graph G is given together with a 2-labeling fulfilling Condition A. On the figures of $G \setminus F_1$ and $G \setminus F_2$, the connected components are assigned numbers 1, 2, and 3, that represent the vertices of $G/F_1 = K_3$ and $G/F_2 = K_3$. The images of vertices under the quotient map are also given and finally the embedding of G into $K_3 \Box K_3$ is shown.

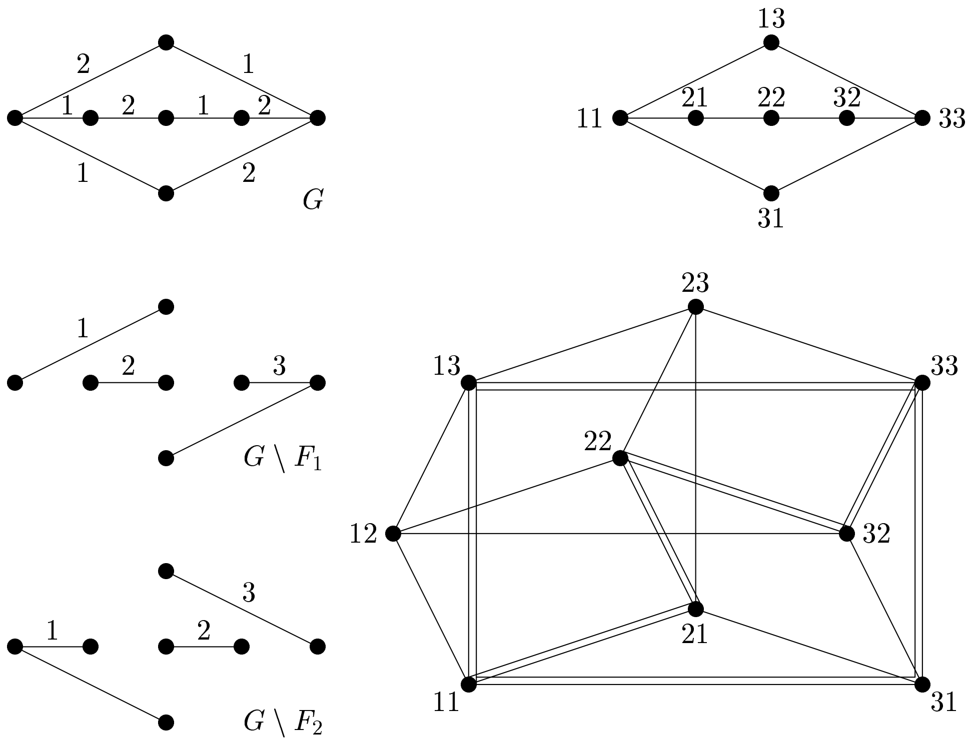

Figure 1: Subgraph of a Hamming graph and its quotient embedding

The characterization of subgraphs of Cartesian product graphs G from [16] involves vertex-labelings, where labels can use integers between 2 and $|V(G)| - 1$. Hence the present approach seems to be more convenient which we demonstrate on the following example ([16, Corollary 3]).

Let G be a bipartite graph with radius 2 and suppose that G contains no subgraph isomorphic to $K_{2,3}$. We claim that G is a nontrivial subgraph of the Cartesian product of two complete graphs. Let u be a vertex of G such that all vertices of G are at distance at most two from it and let v be a neighbor of u. Assign label 1 to uv and to the edges between vertices in $N(u) \setminus v$ and $N(v) \setminus u$. Assign label 2 to all the remaining edges. Note that all induced cycles of G are 4-cycles. Moreover, because G is $K_{2,3}$-free, it follows immediately that every induced 4-cycle is labeled 1,2,1,2.

3 Induced subgraphs of Hamming graphs

In this section we characterize induced subgraph of Hamming graphs. We first state two labeling conditions needed for the result.

Condition B. *Let G be a labeled graph. Then edges of a triangle have the same label.*

Condition C. *Let G be a labeled graph and let u and v be arbitrary vertices of G with $d_G(u,v) \geq 2$. Then there exist labels i and j, $i \neq j$, such that i and j appear on any induced u,v-path.*

Let G be a labeled graph fulfilling Condition C. Let C_k, $k \geq 4$, be an induced cycle of G and let u and v be vertices of C_k with $d_{C_k}(u,v) = 2$. Then the labels of a u,v-path of length 2 on C_k are different. Hence, by Condition C, the other u,v-path along C_k contains these two labels. Therefore:

Corollary 3.1 *Let G be a labeled graph fulfilling Condition C and let C_k, $k \geq 4$, be an induced cycle of G. Then every label of C_k is presented more than once on C_k.*

For the main result of this section we also need:

Lemma 3.2 *If labels i and j appear on every induced u,v-path, then they appear on every u,v-path.*

Proof. Suppose that labels i and j appear on every induced u,v-path. Let $P = x_1x_2\ldots x_r$, $x_1 = u$, $x_r = v$, be a u,v-path of minimal length that does not contain both labels i and j. Then P is not induced, hence we have an edge $e = x_kx_\ell$ with $\ell - k > 1$. We may assume that e is selected such that $\ell - k$ is as small as possible. By the minimality of P, the path $x_1x_2\ldots x_kx_\ell x_{\ell+1}\ldots x_r$ contains both labels i and j. Hence the label of the edge x_kx_l is either i or j. Assume without loss of generality it is i. Then, using minimality again, label j appears on the path $x_1x_2\ldots x_k$ or on $x_\ell x_{\ell+1}\ldots x_r$. It follows that i does not appear on the path $x_kx_{k+1}\ldots x_\ell$. But then the label i appears only once on the cycle $C = x_kx_{k+1}\ldots x_\ell x_k$. If C is a triangle, we have a contradiction with Condition B, otherwise with Corollary 3.1. □

Theorem 3.3 *Let G be a connected graph. Then G is an induced subgraph of a Hamming graph if and only if there exists a labeling of G that fulfills Conditions B and C.*

Proof. Let G be an induced subgraph of $H = K_{n_1}\Box K_{n_2}\Box\cdots\Box K_{n_k}$. Denote $p_i = p_{K_{n_i}}$ and consider the labeling of $E(G)$ induced by the color map c of H.

Condition B is clear. Indeed, if u, v, and w induce a triangle, then they all lie in the same layer of H and so the edges uv, uw, and vw receive the same label. We next show Condition C. Let u and v be two vertices of G with $d_G(u,v) \geq 2$. Suppose that there is no label that appears on all induced u,v-paths. Then $p_i(u) = p_i(v)$ for all i, contrary to $d_G(u,v) \geq 2$. Suppose now that all induced u,v-paths have exactly one label in common, say i. We have $p_j(u) = p_j(v)$ for all $j \neq i$ and $p_i(u) \neq p_i(v)$. Vertices $p_i(u)$ and $p_i(v)$ are adjacent in K_{n_i}, since K_{n_i} is a complete graph. Hence u and v are adjacent in H and therefore also in G which is impossible.

Conversely, let ℓ be a labeling of G that fulfills Conditions B and C. Let $\mathcal{F} = \{F_1, F_2, \ldots, F_k\}$ be the partition of $E(G)$ induced by ℓ and let f be the quotient map of G with respect to $\mathcal{F}$.

We claim that f embeds G as an induced subgraph into $G/F_1 \,\square\, G/F_2 \,\square\, \cdots \,\square\, G/F_k$. We show first that f is one-to-one. Let x and y be adjacent vertices of G and let $\ell(xy) = i$. Suppose that there exists an x,y-path $P = x_1x_2 \ldots x_r$ in $G \setminus F_i$, where $x_1 = x$ and $x_r = y$. We can assume that P is shortest among all x,y-paths in $G \setminus F_i$. If P is induced in $G - xy$ we have a contradiction with Condition B when $r = 3$ and a contradiction with Corollary 3.1 when $r > 3$. Thus P is not induced in $G - xy$, $r > 3$, and there are adjacent vertices x_j and x_k with $k > j + 1$. By the minimality of P we have $\ell(x_jx_k) = i$. We can select j and k such that $k - j$ is minimal among all such vertices x_j and x_k. Then the cycle $C = x_jx_{j+1} \ldots x_{k-1}x_kx_j$ is induced. If C is a triangle we have a contradiction with Condition B, otherwise we have a contradiction with Corollary 3.1. Suppose next that vertices x and y are not adjacent in G. Then by Condition C and Lemma 3.2, there exist labels i and j such that on every x,y-path we find labels i and j. So x and y are in different components in both $G \setminus F_i$ and $G \setminus F_j$. Already the first fact assures that $f(x) \neq f(y)$ and we conclude that f is one-to-one. Let xy be an edge with $\ell(xy) = i$. Then, by the above, x and y are in different components of $G \setminus F_i$. Moreover, they belong to the same component on any of the $G \setminus F_j$, $j \neq i$. It follows that f maps edges to edges and the claim is proved.

Hence $G = f(G)$ is an induced subgraph of $G/F_1 \,\square\, G/F_2 \,\square\, \cdots \,\square\, G/F_k$. To complete the proof we show that G is also an induced subgraph of the Hamming graph

$$K_{|G/F_1|} \square K_{|G/F_2|} \square \cdots \square K_{|G/F_k|}\,.$$

Let x and y be nonadjacent vertices of G. Then, by the same reasoning as above, x and y are in different components of at least two graphs $G \setminus F_i$. It follows that $f(x)$ and $f(y)$ differ in at least two coordinates which remains valid after adding edges to the factor graphs. $\square$

Note that the quotient graphs obtained in the proof of Theorem 3.3 need not be complete. For instance, consider the path P_4 together with the labeling 1, 2, 1.

Theorem 3.3 (and its proof) are illustrated in Fig. 2, where an admissible labeling is assigned to C_7, that is in turn embedded into $K_2 \square K_3 \square K_2$.

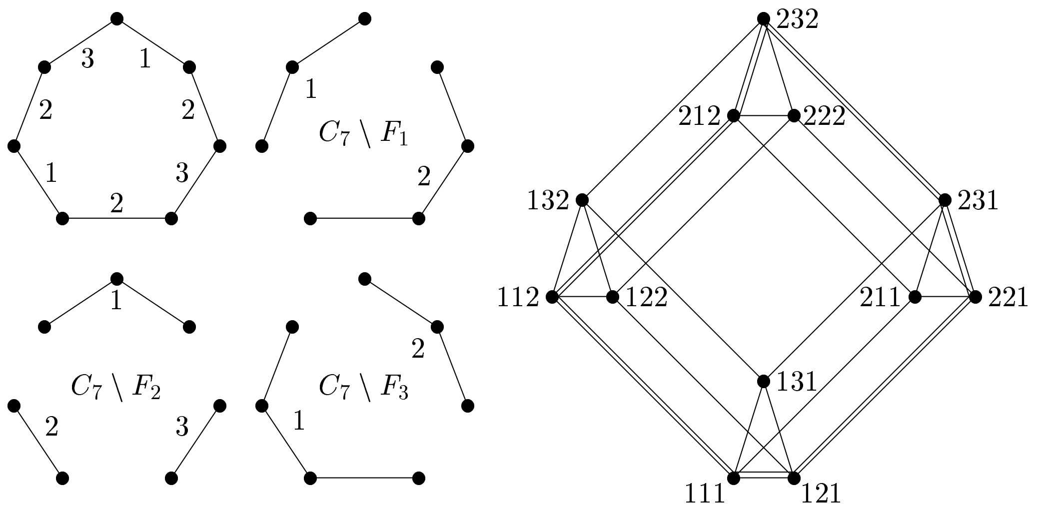

Figure 2: C_7 as an induced subgraph of $K_2 \square K_3 \square K_2$

4 Isometric subgraphs of Hamming graphs

We have already mentioned that isometric subgraphs are induced. It is well known that C_7 is an induced but not an isometric subgraph of a Hamming graph. Hence, in order to characterize partial Hamming graphs, we add another labeling condition.

Condition D. *Let G be a labeled graph. Then the labels of any shortest path are pairwise different.*

Theorem 4.1 *Let G be a connected graph. Then G is a partial Hamming graph if and only if there exists a labeling of G that fulfills Conditions B, C, and D.*

Proof. Let G be an isometric subgraph of a Hamming graph H. Consider the labeling of $E(G)$ induced by the color map c of H. By the proof of Theorem 3.3, c fulfills Conditions B and C. Moreover, as G is isometric in H, any shortest path of G is a shortest path of H, hence c fulfills Condition D as well.

Conversely, let ℓ be a labeling of G that satisfies Condition B, C, and D. Let $\mathcal{F} = \{F_1, F_2, \ldots, F_k\}$ be the partition of $E(G)$ induced by ℓ and let f be the corresponding embedding into $H = G/F_1 \,\square\, G/F_2 \,\square\, \cdots \,\square\, G/F_k$. By Conditions B and C and the proof of Theorem 3.3 we know that G is an induced subgraph of H. We claim it is also isometric. Let u and v be any vertices of G and let $P = x_1 x_2 \ldots x_r$ ($x_1 = u$, $x_r = v$) be a shortest u,v-path in G. Then as the embedding is induced, $f(x_i)f(x_{i+1})$ is an edge of H and hence $d_H(f(u), f(v)) \leq d_G(u, v)$. Moreover, by Condition D, edges of

P receive pairwise different labels which implies that $f(u)$ and $f(v)$ differ in at least $d_G(u,v)$ coordinates. Hence $d_H(f(u)f(v)) \geq d_G(u,v)$ and so $d_H(f(u)f(v)) = d_G(u,v)$.

To complete the proof we show that G/F_i is a complete graph for $i = 1, 2, \ldots, k$. Let C and C' be connected components of $G \setminus F_i$ and assume there is no edge in F_i connecting a vertex of C with a vertex of C'. Then a shortest path between a vertex of C and a vertex of C' contains at least two edges of label i, a contradiction with Condition D. $\square$

An isometric embedding $\beta : G \to H_1 \square H_2 \square \cdots \square H_n$ is called *irredundant* if $|H_i| \geq 2$ for all i and if every vertex $u \in \cup_{i=1}^{n} H_i$ occurs as a coordinate value of the image of some $w \in G$. In [12] (cf. also [15]) it is proved that any isometric irredundant embedding of a graph G into a product of complete graphs is the canonical isometric embedding. Hence:

Corollary 4.2 *Let G be a connected graph equipped with a labeling that fulfills Conditions B, C, and D. Then this labeling coincides with the partition induced by Θ^*.*

Acknowledgment

We thank Boštjan Brešar and Alenka Lipovec for several useful remarks that helped to improve the presentation and shorten some arguments.

References

[1] F. Aurenhammer, M. Formann, R.M. Idury, A.A. Schäffer, and F. Wagner, Faster isometric embedding in products of complete graphs, Discrete Appl. Math. 52 (1994) 17–28.

[2] H.-J. Bandelt, Characterization of Hamming graphs, manuscript, August 1992.

[3] H.J. Bandelt, H.M. Mulder and E. Wilkeit, Quasi-median graphs and algebras, J. Graph Theory 18 (1994) 681–703.

[4] B. Brešar, Partial Hamming graphs and expansion procedures, Discrete Math. 237 (2001) 13–27.

[5] B. Brešar, On subgraphs of Cartesian product graphs and S-primeness, submitted.

[6] V. Chepoi, d-convexity and isometric subgraphs of Hamming graphs, Cybernetics 1 (1988) 6–9.

[7] D. Djoković, Distance preserving subgraphs of hypercubes, J. Combin. Theory Ser. B 14 (1973) 263–267.

[8] T. Feder, Product graph representations, J. Graph Theory 16 (1992) 467–488.

[9] T. Feder, Stable networks and product graphs, Mem. Amer. Math. Soc. 116(555) (1995) 1–223.

[10] R.L. Graham and P.M. Winkler, On isometric embeddings of graphs, Trans. Amer. Math. Soc. 288 (1985) 527–536.

[11] W. Imrich, Embedding graphs into Cartesian products, Ann. New York Acad. Sci. 576 (1989) 266–274.

[12] W. Imrich and S. Klavžar, A simple $O(mn)$ algorithm for recognizing Hamming graphs, Bull. Inst. Comb. Appl. 9 (1993) 45–56.

[13] W. Imrich and S. Klavžar, On the complexity of recognizing Hamming graphs and related classes of graphs, European J. Combin. 17 (1996) 209–221.

[14] W. Imrich and S. Klavžar, Recognizing Hamming graphs in linear time and space, Inform. Process. Lett. 63 (1997) 91–95.

[15] W. Imrich and S. Klavžar, Product Graphs: Structure and Recognition (John Wiley & Sons, New York, 2000).

[16] S. Klavžar, A. Lipovec and M. Petkovšek, On subgraphs of Cartesian product graphs, Discrete Math. 244 (2002) 223–230.

[17] R.H. Lamprey and B.H. Barnes, A new concept of primeness in graphs, Networks 11 (1981) 279–284.

[18] R.H. Lamprey and B.H. Barnes, A characterization of Cartesian-quasiprime graphs, Congressus Numer. 109 (1995) 117–121.

[19] M. Mollard, Two characterizations of generalized hypercube, Discrete Math. 93 (1991) 63–74.

[20] H.M. Mulder, The Interval Function of a Graph (Mathematical Centre Tracts 132, Mathematisch Centrum, Amsterdam, 1980).

[21] H.M. Mulder, Interval-regular graphs, Discrete Math. 41 (1982) 253–269.

[22] R. Squier, B. Torrence and A. Vogt, The number of edges in a subgraph of a Hamming graph, Appl. Math. Lett. 14 (2001) 701–705.

[23] E. Wilkeit, Isometric embeddings in Hamming graphs, J. Combin. Theory Ser. B 50 (1990) 179–197.

[24] P. Winkler, Isometric embeddings in products of complete graphs, Discrete Appl. Math. 7 (1984) 221–225.