A note on group actions and invariant partitions

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Abstract

Let a group G act on a set V. Then a G-invariant partition of V is the orbit-set of some subgroup in G if and only if it is the orbit-set of some normal subgroup in G. This simple fact is recalled and an application to group actions (G, V), where every G-invariant partition of V is the orbit-set of some subgroup of G, is presented.

Math. Subj. Class. (2000): 05C25, 20B25, 20E15.

Keywords: group action, invariant partition, normal partition, normally imprimitive action.

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[0*] Supported in part by "Ministrstvo za šolstvo, znanost in šport Republike Slovenije", proj.no. *******.

[0**] Supported in part by "Ministrstvo za šolstvo, znanost in šport Republike Slovenije", proj.no. Z1-3124.

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1 Introduction

The purpose of this short note is to invite the mathematical community interested in group actions to recall the following simple and elementary fact.

Proposition 1 *Let G be a group acting transitively on a set V and leaving invariant a partition $\mathcal{P}$ of V. Then $\mathcal{P}$ is the set of orbits of a subgroup in G if and only if it is the set of orbits of some normal subgroup in G.*

This result, whose proof is straightforward, should perhaps be classified as "folklore". However, the absence of such a basic observation in a number of textbooks and research papers, and the belief in its potential conceptual strength, motivated the authors to write these lines. We proceed by first covering the fundamental facts about group actions and invariant partitions that are necessary to appreciate Proposition 1. An application to group actions where every invariant partition is the orbit-set of some subgroup is given in Proposition 2.

2 Invariant partitions

It is well known that, given a transitive action (G, V) of a group G on a set V and a normal subgroup $N \triangleleft G$, the orbits of N on V give rise to a partition of V which is invariant under the action of G, that is, the *orbit-set* of N is a *G-invariant partition* of V [1, Theorem 1.6A(i)]. However, not all G-invariant partitions of V arise in this way. The smallest counterexample is provided by the natural action of the symmetric group S_3 of order six on the set of six ordered pairs of distinct points from $\{1, 2, 3\}$: this action is transitive and leaves invariant the partition $\{\{12, 21\}, \{13, 31\}, \{23, 32\}\}$, as well as the partition $\{\{12, 23, 31\}, \{21, 32, 13\}\}$. Since the unique proper nontrivial normal subgroup in S_3, the alternating group A_3, has as orbit-set the latter of the two partitions, the former does not consist of orbits of a normal subgroup in S_3. This motivates the introduction of the notion of a *normal partition*. According to Praeger [3], a partition $\mathcal{P}$ of the set V is *normal* (also *G-normal*) if $\mathcal{P}$ is the set of orbits on V of some normal subgroup of G.

At this point, a natural question arises: given a transitive action (G,V), what may be said about the situation where the orbit-set of a (non-normal) subgroup H of G is invariant with respect to the action of G? And furthemore, is there a link to normal partitions? Let us now verify that G-invariant orbit-sets of subgroups are the same as normal partitions.

PROOF OF PROPOSITION 1. The "if" part is obvious. Conversely, suppose that the partition $\mathcal{P}$ of V is G-invariant and that $\mathcal{P}$ consists of orbits of a subgroup H of G:

$$\mathcal{P} = \{Hv \mid v \in V\}.$$

We have to prove that $\mathcal{P}$ is the orbit-set of a normal subgroup of G. To this end observe that every conjugate of H in G has orbit-set $\mathcal{P}$. Now, the subgroup K generated by all conjugates of H in G is normal in G, and the orbit-set of K is $\mathcal{P}$. ■

We remark that in the above proof, *the kernel* of the action induced by G on the partition $\mathcal{P}$, that is, the set

$$N = \{g \in G \mid gP = P \text{ for all } P \in \mathcal{P}\}$$

consisting of all elements in G fixing every part of the partition $\mathcal{P}$, is the largest normal subgroup in G having orbit-set $\mathcal{P}$.

An example of a G-invariant partition $\mathcal{P}$ of V arising as the orbit-set of a non-normal subgroup H in G is, for instance, the bipartition $\mathcal{P} = \{\{1,2,3\},\{4,5,6\}\}$ of the set $V = \{1,2,\ldots,6\}$, where G is the subgroup of permutations of the symmetric group S_6 preserving $\mathcal{P}$, and H is the subgroup of permutations of G fixing the two parts of $\mathcal{P}$ and having an even restriction on the second part. The group G is the wreath product of S_3 by S_2. The subgroup H is isomorphic to $S_3 \times \mathbb{Z}_3$, and H is not normal in G.

3 Group actions where every invariant partition is normal

We call a transitive action (G,V) *normally imprimitive* if *every* G-invariant partition of V is normal. Thus we focus on group actions sharing an aspect of invariant partitions that is opposite to the *quasiprimitive* one. Following Praeger [2], a transitive action (G,V) is called *quasiprimitive* if there are *no* normal G-invariant partitions of V besides the two trivial ones. (As a matter of fact, she defines a transitive permutation group G to be quasiprimitive if each of its nontrivial normal subgroup is transitive.) Straightforward examples of quasiprimitive actions are supplied, for instance, by transitive actions of simple groups. Quasiprimitive actions and related quasiprimitive graphs have been investigated by Praeger [2, 3, 4, 5]. The problem of studying normally imprimitive group actions and related graphs was raised in [7, Problem 98], motivated by a class of vertex-transitive graphs having a normally imprimitive group of automorphisms on the set of vertices [6, Theorem 9.1].

Proposition 2 below indicates that faithful normally imprimitive actions which are minimal with respect to inclusion in the full symmetric group of the underlying set, are of particular interest.

Proposition 2 *Let G be a group acting transitively on a set V, and H be a transitive subgroup in G. Then, if a G-invariant partition is H-normal, it is also G-normal. In particular, if the action of H is normally imprimitive on V, then G is normally imprimitive on V.*

PROOF. Let $\mathcal{P}$ be an arbitrary G-invariant partition of V. Suppose that $\mathcal{P}$ is H-normal. Then there is a normal subgroup K in H with $\mathcal{P}$ as its orbit-set. Since K is also a subgroup of G, Proposition 1 shows that $\mathcal{P}$ is a G-normal partition. The statement about normally imprimitive actions now follows using the fact that every G-invariant partition is also H-invariant. ∎

Thus, being normally imprimitive is an *upward hereditary property* in the lattice of subgroups of the full symmetric group of the underlying set. Note that by our definition, primitive subgroups are normally imprimitive. This obvious contradictio in terminis is acceptable since, in the faithful case, being normally imprimitive and being quasiprimitive are antipodal properties, intersecting in the property of being primitive.

EXAMPLES. It is clear that every transitive action of an abelian group is normally imprimitive. Proposition 2 then implies that every group action containing a transitive abelian subgroup is normally imprimitive.

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