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# THE LAH IDENTITY AND THE ARGONAUTS

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# The Lah Identity and the Argonauts

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According to a legend LJUBLJANA, the capital of modern Slovenia, known as EMONA in the Roman times, was founded by JASON AND THE ARGONAUTS when they were fleeing on the ship ARGO with the GOLDEN FLEECE from the BLACK SEE via the DANUBE RIVER, upstream SAVA RIVER and finally reaching the dead-end at the source of LJUBLJANICA RIVER. The dragon that Jason supposedly slaughtered when camping on the bank of Ljubljanica is still the dominant symbol in the city coat-of-arms.

The legend tells us that the Argonauts took their ship Argo apart and carried it over the SLOVENIAN CARST to the ADRIATIC SEA. Since there are numerous trails leading from Ljubljana to the coast we may assume that each part of Argo was carried for the security reasons along a separate trail thereby minimizing the possibility of losing the Golden Fleece to the pursuing party.

Much later, in 1955 Slovenian mathematician IVO LAH (Štrukljeva vas, 1896 - Ljubljana, 1979) proved in [1] for any positive integer $n$ the following identity between the two $n$-degree polynomials in $x$:

$$\prod_{i=0}^{n-1}(x+i) = \sum_{k=1}^{n} \frac{n!}{k!} \begin{pmatrix} n-1 \\ k-1 \end{pmatrix} \prod_{j=0}^{k-1}(x-j) \text{[Lah Identity]}$$

The coefficients

$$L(n,k) = \tfrac{n!}{k!} \begin{pmatrix} n-1 \\ k-1 \end{pmatrix} \quad \text{[Unsigned Lah Numbers]}$$

are called the unsigned LAH NUMBERS [2]. Here we give a combinatorial proof of the Lah identity. Note that is suffices to prove it for sufficiently many values of $x$. In fact we will prove it for all integers $x \geq n$. The story with the Argonauts will help us visualize the proof.

Let $n$ be the number of Argonauts. Say, each Argonaut has a number assigned. The numbers are: $1, 2, ..., n$.

Let $x \geq n$ be the number of trails from Ljubljana to the coast of Adriatic Sea.

Finally, let $k \leq n$ be the number of indistinguishable parts that the ship Argo is divided into for transportation.

We further assume that each part is carried by some positive number of linearly ordered Argonauts along a separate trail.

Let us take a special example. Let $x = 10, n = 5$ and $k = 3$. The parts are carried as follows: (3,1) take route 1. (2) takes route 2.(4,5) take route 10.

This can be encoded as a vector:

$$[31 - 2 - - - - - - - - 45].$$

There are $(x-1) = 9$ dashes that separate trails from each other. One can choose the positions of the dashes in $\left( \begin{array}{c} n+x-1 \\ n \end{array} \right)$ ways. Combined with $n!$ independent ways to permute the $n$ Argonauts, we obtain the left hand side of the Lah identity.

The same possibilites can be counted in a different way, namely depending on $k$, the number of indistinguishing parts that the Argo is divided into. Let us take the same permutation of $n = 5$ Argonauts $< 31245 >$ and split it with two vertical lines into three nonempty parts: $[31|2|45]$. Since each part is nonempty we may first take a permutation of the form $\{= || =\}$ and extend it by introducing in each part an extra symbol (without increasing the number of possibilities) $\{== | = | ==\}$. There are obviously

$$\left( \begin{array}{c} n-k+k-1 \\ k-1 \end{array} \right) = \left( \begin{array}{c} n-1 \\ k-1 \end{array} \right)$$

possibilites. Every one of them has to be multiplied by $n!$ since there are $n!$ permutations of Argonauts and divided by $k!$ since the order of parts is irrelevant. The result is actually the Lah number $L(n,k)$. This number is to be multiplied by $x(x-1)...(x-k+1)$ since this is the number of ways to assign trails to the parts, and the Lah identity follows when we sum over all possible values of $k$.

## References


[1] Ivo Lah, Eine neue Art von Zahlen, ihre Eigenschaften und Anwendung in der mathematischen Statistik Mitteilungsbl. Math. Statist. 7 (1955), 203–212.

[2] John Riordan, Introduction to Combinatorial Analysis, Wiley, New York, 1958.