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Contents

Introduction

In 1967 Gunning and Narasimhan [GN] proved that every open (noncompact) Riemann surface admits a holomorphic function without critical points, thus giving an affirmative answer to a long standing question. Their proof was an ingenious application of the approximation methods introduced by Behnke and Stein.

A complex manifold is called Stein (after K. Stein [S], 1951) if it is biholomorphic to a closed complex submanifold of a complex Euclidean space $\mathbb{C}^N$. Open Riemann surfaces are precisely Stein manifolds of complex dimension one. In this paper we prove the following basic result:

Theorem A. *Every Stein manifold admits a holomorphic function without critical points.*

The question answered by Theorem A has been open since the 1967 work [GN] and was stated explicitly on page 70 of Gromov's monograph [Gro1] (1986). Our proof, which also applies to open Riemann surfaces, is conceptually different from the one in [GN] which does not generalize to higher dimensions. For a more precise result, with interpolation on subvarieties and approximation on holomorphic convex subsets, see Theorem 1.1 in Sect. 1.

Theorem A implies that *every Stein manifold admits a nonsingular holomorphic foliation by closed complex hypersurfaces* (take the level sets of a noncritical holomorphic function). By Corollary 1.2 *we may choose this foliation* *to be transverse to any closed complex submanifold of X.* Furthermore, any closed nonsingular complex hypersurface $V \subset X$ with trivial normal bundle is a leaf (or a union of leaves if V is disconnected) of a nonsingular holomorphic foliation of X by closed complex hypersurfaces (Corollary 1.3). In particular, *any smooth complex curve C in a Stein surface X is a leaf of a nonsingular holomorphic foliation of X by complex curves* (since in this case any complex bundle on C is trivial). Combining Theorem A with the Oka-Grauert principle we also show that *if $E \subset TX$ is a complex subbundle of codimension one in the tangent bundle TX such that the quotient bundle TX/E is trivial then E is homotopic to an integrable holomorphic subbundle of TX* (Corollary 1.4').

Theorem A gives noncritical holomorphic functions on a wider class of complex manifolds. For instance, if $g: Y \to X$ is a holomorphic submersion to a Stein manifold X then for every noncritical function f on X we get a noncritical function $f \circ g$ on Y. In particular, *any Riemann domain over a Stein manifold admits a noncritical function.* However, the stronger version with interpolation on closed complex subvarieties (Theorem 1.1) fails on certain non-Stein domains in Stein manifolds.

A noncritical holomorphic function on X can also be considered as a *holomorphic submersion $X \to \mathbb{C}$.* A natural problem is to find the largest interger $q = q(X) \leq \dim X$ for which there exists a holomorphic submersion $f = (f_1, \ldots, f_q): X \to \mathbb{C}^q$. The differentials $df_1, \ldots, df_q$ span a trivial holomorphic subbundle of rank q of the holomorphic cotangent bundle T^*X which gives a necessary condition for the existence of such a submersion. Is this necessary condition also sufficient? The methods developed in this paper provide the following basic step in this direction. We say that a pair of compact sets $A \subset B$ in X is a *noncritical strongly pseudoconvex extension* in X if there is a strongly plurisubharmonic function without critical points in a neighborhood $U \supset \overline{B \backslash A}$ such that $A \cap U = \{\rho \leq 0\}$ and $B \cap U = \{\rho \leq 1\}$.

Theorem B. *Suppose that $A \subset B$ is a noncritical strongly pseudoconvex extension in a Stein manifold X. If $q < \dim X$ then any holomorphic submersion from a neighborhood of A to $\mathbb{C}^q$ can be approximated uniformly on A by holomorphic submersions from a neighborhood of B to $\mathbb{C}^q$.*

Corollary C. *If X is a Stein manifold which admits a strongly plurisubharmonic exhaustion function with only one critical point then there exists a holomorphic submersion $X \to \mathbb{C}^q$ for every $q < \dim X$. Hence X admits nonsingular holomorphic foliations with closed leaves of any rank $k = 1, 2, \ldots, n-1$.*

For a precise statement see Theorem 1.5 and its corollaries. If X is itself a noncritical extension of a subset $A \subset\subset X$ then submersions to $\mathbb{C}^q$ can be approximately extended from A to X. The level sets of a submersion $X \to \mathbb{C}^q$ form a nonsingular holomorphic foliation of X of rank $n - q$.

Our construction of noncritical holomorphic functions depends on two main ingredients. The first is a method of approximating such functions on polynomially convex subsets $K \subset\subset \mathbb{C}^n$ by entire noncritical functions. For $n > 1$ this depends on results from the theory of holomorphic automorphisms of complex Euclidean spaces developed in [AL] and [FR]. The second ingredient is a new method for patching holomorphic functions (or maps) which preserves a certain first order differential condition (in our case the nonvanishig of the differential or, more generally, the maximal rank condition). This is based on a *compositional splitting of the biholomorphic transition map* between two such functions or maps on certain configurations of domains which we call *special Cartan pairs* (Theorem 3.1). Such splitting is obtained by a *rapidly converging iteration of Nash-Moser type*. The corresponding *linearized problem* is solved by the $\overline{\partial}$-theory, and the usual *smoothing operators* are replaced by shrinking of the domains which insures the Cauchy estimates. The construction is globalized as in the Oka-Grauert theory.

A similar approach is used to construct submersions $X \to \mathbb{C}^q$ with $1 < q < n = \dim X$, except that we are unable to prove the required approximation result for submersions $\mathbb{C}^n \to \mathbb{C}^q$ directly by using automorphisms. Instead we prove a weaker approximation result (Proposition 2.3) which still suffices to prove Theorem B above. Unfortunately our approach breaks down in the critical case $q = \dim X$ due to a *Picard type obstruction* (see the Remark following Lemma 2.4). The main problem is the following:

Problem. Let B be an open convex set in $\mathbb{C}^n$. Is it possible to approximate every holomorphic immersion $B \to \mathbb{C}^n$ uniformly on compacts in B by entire immersions $\mathbb{C}^n \to \mathbb{C}^n$?

The same question may be asked for holomorphic immersions with constant Jacobian, and this may be related to the *Jacobian problem* [BN, p. 21]. Of course it would suffice to approximate by an immersion on a large ball and apply induction. An affirmative solution would give a holomorphic immersion $X \to \mathbb{C}^n$ for every n-dimensional Stein manifolds X which admits a strongly plurisubharmonic exhaustion function with a single critical point.

The main question in the critical case $q = \dim X$ is whether every Stein manifold with the trivial tangent bundle admits a holomorphic immersion $X \to \mathbb{C}^n$ with $n = \dim X$. This well known problem, which has been stated at several places in the literature (see e.g. [BN, p. 18], [Gro1, p. 70]), was one of the motivations for the current work. For $n = 1$ this was proved by Gunning and Narasimhan [GN] in 1967. Not much is known for $n > 1$. A good test case in dimension two may be the complements $X_C = \mathbb{CP}^2 \backslash C$ of smooth cubic curves $C \subset \mathbb{CP}^2$ [BN]. It is known that X_C is parallelizable for every such C and it immerses in $\mathbb{C}^2$ for almost all C. For a further discussion see Sect. 5.

&1. Description of the main results.

Let X be a Stein manifold (for the general theory of such manifolds we refer to [GR] and [Hör]). Recall that a compact set $K \subset X$ is said to be $\mathcal{O}(X)$-*convex* (or *holomorphically convex* in X) if for any point $x \in X \backslash K$ there is

$f \in \mathcal{O}(X)$ satisfying $|f(x)| > \max_K |f|$. We denote by $j^r_x(f)$ the r-jet of f at the point $x \in X$. The following is our main result.

1.1 Theorem. *Let X be a Stein manifold, $X_0 \subset X$ a closed complex subvariety of X and $K \subset X$ a compact $\mathcal{O}(X)$-convex subset of X. Let $U \subset X$ be an open set containing $X_0 \cup K$ and let $f \in \mathcal{O}(U)$ be a holomorphic function with $df \neq 0$ on U. Then for every $\epsilon > 0$ and $r \in \mathbb{N}$ there exists a function $\widetilde{f} \in \mathcal{O}(X)$ such that $d\widetilde{f} \neq 0$ on X, $|f(x) - \widetilde{f}(x)| < \epsilon$ for all $x \in K$, and $j^r_x(f) = j^r_x(\widetilde{f})$ for all $x \in X_0$.*

Theorem 1.1 is proved in section 4. The same proof gives the following more general result. For $f \in \mathcal{O}(U)$ we write $\mathrm{Crit}(f; U) = \{x \in U: df_x = 0\}$ and call it *the critical set* of f in U.

1.1' Theorem. *Let X, X_0, K and U be as in Theorem 1.1. Assume that $f \in \mathcal{O}(U)$ satisfies $\Sigma := \mathrm{Crit}(f; U) \subset K$. Then for any $\epsilon > 0$ and $r \in \mathbb{N}$ there is an $\widetilde{f} \in \mathcal{O}(X)$ satisfying $\mathrm{Crit}(\widetilde{f}; X) = \Sigma$, $|f(x) - \widetilde{f}(x)| < \epsilon$ for all $x \in K$, and $j^r_x(f) = j^r_x(\widetilde{f})$ for all $x \in X_0 \cup \Sigma$.*

Note that Σ, being a subvariety of a compact set $K \subset X$, is necessarily finite. Conversely, if $\mathrm{Crit}(f; U)$ is finite, we can always include it in K (by enlarging K if necessary). Theorem 1.1' allows us to approximately extend a holomorphic function from a neighborhood of K to a global function on X without enlarging its critical set. We now give several corollaries.

1.2 Corollary. *Every Stein manifold X admits a nonsingular holomorphic foliation by closed connected complex hypersurfaces. We can choose such a foliation to be transverse to any given closed complex submanifold of X.*

Proof. Any closed complex submanifold V of a Stein manifold X is itself a Stein manifold and hence it admits a noncritical function $g \in \mathcal{O}(V)$. By Theorem 1.1 we can extend g to a noncritical holomorphic function $f \in \mathcal{O}(X)$. Each level set of f is a smooth closed complex hypersurface in X interesecting V transversely (since $d(f|_V) = dg \neq 0$ on V). ♠

1.3 Corollary. *Let V be a nonsingular closed complex hypersurface in a Stein manifold X. If the normal bundle $N = TX|_V/TV \to V$ is trivial then V is a (disjoint) union of leaves in a nonsingular holomorphic foliation of X by closed complex hypersurfaces. In particular, any smooth connected complex curve C in a complex surface X is a leaf in a nonsingular holomorphic foliation of X by closed complex curves.*

Proof. Choose a holomorphic trivialization $N \simeq V \times \mathbb{C}$. The projection $h: N \to \mathbb{C}$ on the second coordinate is a noncritical holomorphic function on N and $\{h = 0\}$ equals the zero section $N_0 \subset N$. The Docquier-Grauert theorem provides an open neighborhood $\Omega \subset X$ of V and a biholomorphic map ϕ

of Ω onto an open neighborhood of N_0 in N which maps V onto N_0. The composition $f = h \circ \phi$ is noncritical on Ω and $\{f = 0\} = V$. Applying Theorem 1.1 to f (with $X_0 = V$) we obtain a noncritical function $\widetilde{f} \in \mathcal{O}(X)$ on X which vanishes on V precisely to order one. The foliation $\mathcal{F} = \{\widetilde{f} = \mathrm{const}\}$ then satisfies the conclusion of Corollary 1.3. If C is a complex curve in a Stein manifold then C is open (noncompact) and hence any holomorphic vector bundle on C is trivial (since $\mathrm{Pic}(C) \simeq H^2(C; Z) = 0$). ♠

Remark. Applying Theorem 1.1' instead of Theorem 1.1 in the proof of Corollary 1.3 gives the following extension of the latter result: *Let $V \subset X$ be a closed complex hypersurface with at most finitely many singularities. Assume that f is a holomorphic function in a neighborhood $U \subset X$ of V whose germ f_x generates the ideal $\mathcal{J}_x^V \subset \mathcal{O}_x$ of V at any point $x \in V$. Then X admits a holomorphic foliation $\mathcal{F}$ by closed complex hypersurfaces such that V is a union of leaves of $\mathcal{F}$ and all other leaves of $\mathcal{F}$ are smooth (nonsingular).* ♠

From Theorem 1.1 and the Oka-Grauert principle [Gra1, Gra2, HL] it follows that, *on any Stein manifold X, holomorphic functions with nonvanishing differential satisfy the holomorphic h-principle, as well as the ordinary h-principle* in the sense of Gromov [Gro1, p. 66]. To explain what is meant we consider the inclusion

$$\iota \colon \Omega^1_*(X) \hookrightarrow \mathcal{C}^{(1,0)}_*(X) \tag{$*$}$$

of the space $\Omega^1_*(X)$ of all nonvanishing holomorphic one-forms on X into the space $\mathcal{C}^{(1,0)}_*(X)$ of all nonvanishing $(1,0)$-forms on X with continuous coefficients. These may be considered as holomorphic (resp. continuous) sections of the holomorphic cotangent bundle T^*X with the zero section removed. The Oka-Grauert principle applies to such sections [Gra1, HL] and shows that the inclusion (*) is a *weak homotopy equivalence*, i.e., ι induces an isomorphism of the respective homotopy groups of the two spaces. In particular, their path connected components are in one-to-one correspondence. By Theorem 1.1 $\Omega^1_*(X)$ contains an *exact* holomorphic one-form $\theta = df$. This implies

1.4 Corollary. *Every $\theta_0 \in \mathcal{C}^{(1,0)}_*(X)$ can be connected by a path in $\mathcal{C}^{(1,0)}_*(X)$ to an exact holomorphic 1-form $\theta_1 = df \in \Omega^1_*(X)$. If $\theta_0 \in \Omega^1_*(X)$ then the path may be chosen to be contained in $\Omega^1_*(X)$.*

Proof. Let $\theta_0 \in \mathcal{C}^{(1,0)}_*(X)$. Since (*) is a weak homotopy equivalence, we can connect θ_0 by a path in $\mathcal{C}^{(1,0)}_*(X)$ to a holomorphic one-form $\theta' \in \Omega^1_*(X)$.

If $\dim X = 1$, the theorem of [GN] gives a function $g \in \mathcal{O}(X)$ such that the nonvanishing one-form $\theta = e^g \theta' \in \Omega^1_*(X)$ is exact holomorphic, i.e., $\theta = df$ for some $f \in \mathcal{O}(X)$. The path $\phi_t = e^{tg}\theta'$ ($t \in [0,1]$) connects $\phi_0 = \theta'$ to $\phi_1 = e^g \theta' = df$ within $\Omega^1_*(X)$ which proves the result in this case.

Suppose now that $n = \dim X \geq 2$. By the Lefschetz theorem [AF] X admits a homotopy retraction onto a subpolyhedron $P \subset X$ of real dimension at most n. From this we see by induction over the skeleta of P that the space $\mathcal{C}_*^{(1,0)}(X)$ is connected. Since (*) is a weak homotopy equivalence, $\Omega^1_*(X)$ is also connected. Hence we may connect θ' within $\Omega^1_*(X)$ to any exact 1-form df furnished by Theorem 1.1. ♠

Remark. The same conclusion holds if we replace $\mathcal{C}_*^{(1,0)}(X)$ by the space of nonvanishing $(1,0)$-forms with *smooth* coefficients on X. ♠

Corollary 1.4 has a dual formulation in terms of complex vector subbundles of the holomorphic tangent bundle TX. A nowhere vanishing $(1,0)$-form θ on X determines a complex subbundle $E = \ker\theta \subset TX$ of codimension one whose normal bundle $N_E = TX/E \to X$ is trivial. Indeed, denoting by $\tau: TV \to N_E$ the quotient projection with kernel E, the form $\theta: TX \to \mathbb{C}$ passes down to $\widetilde\theta: N_E \to \mathbb{C}$ which provides a trivialization of N_E. If $\theta \in \Omega^1_*(X)$ is a nonvanishing holomorphic one-form on X then $E = \ker\theta$ is a holomorphic subbundle of TX. Such a subbundle is *integrable* if for any pair of sections $v, w: X \to E$ (vector fields on X which are tangent to E) their commutator is also tangent to E: $\tau([v,w]) = 0$. In such case E is the tangent bundle of a smooth foliation of X whose leaves are complex hypersurfaces in X. Holomorphic 1-forms correspond to holomorphic subbundles of TX. Corollary 1.4 implies

1.4' Corollary. *Every complex subbundle $E \subset TX$ of codimension one with trivial normal bundle TX/E is homotopic to an integrable holomorphic subbundle of TX.*

We also prove the following result on holomorphic submersions to affine spaces. We say that a pair of compact sets $A \subset \widetilde A$ in a complex manifold X is a *noncritical strongly pseudoconvex extension in* X if there is a smooth strongly plurisubharmonic function ρ in an open set $U \supset \widetilde A \backslash \mathrm{int}A$, with $d\rho \neq 0$, such that $A \cap U = \{x \in U: \rho(x) \le 0\}$ and $\widetilde A \cap U = \{x \in U: \rho(x) \le 1\}$.

1.5 Theorem. *Let X be a Stein manifold and let $(A, \widetilde A)$ be a noncritical strongly pseudoconvex extension in X. Let $q < \dim X$. If $f: U \to \mathbb{C}^q$ is a holomorphic submersion in an open set $U \supset A$ then for every $\epsilon > 0$ there are an open set $\widetilde U \supset \widetilde A$ and a holomorphic submersion $\widetilde f: \widetilde U \to \mathbb{C}^q$ with $\sup_A |f - \widetilde f| < \epsilon$. If $\rho: X \to \mathbb{R}$ is a strongly plurisubharmonic exhaustion function such that $A = \{\rho \le 0\}$ and $d\rho \neq 0$ on $\{\rho \ge 0\} = X\backslash\mathrm{intA}$ then every submersion $f: U \to \mathbb{C}^q$ in a neighborhood $U \supset A$ can be approximated, uniformly on A, by holomorphic submersions $\widetilde f: X \to \mathbb{C}^q$.*

The following corollaries follow immediatelly from Theorem 1.5.

1.6 Corollary. *If X is a Stein manifold of dimension $n \ge 2$ which admits a strongly plurisubharmonic exhaustion function with only one critical point then for any $q = 1, 2, \ldots, n-1$ there exists a holomorphic submersion $f: X \to \mathbb{C}^q$. Consequently for any $k = 1, 2, \ldots, n-1$ the manifold X admits a nonsingular holomorphic foliation by closed complex submanifolds of dimension k.*

1.7 Corollary. *If $K \subset \mathbb{C}^n$ is a compact convex set then any holomorphic submersion $f: U \to \mathbb{C}^q$ $(q < n)$ on a neighborhood $U \supset K$ can be approximated, uniformly on K, by entire submersions $\mathbb{C}^n \to \mathbb{C}^q$.*

By Morse theory any manifold in Corollary 1.6 is diffeomorphic to a ball and hence its tangent bundle is trivial. We do not know whether such X also immerses in $\mathbb{C}^n$ or whether Corollary 1.7 holds in the critical case $q = n$.

In the remainder of this section we give an outline of the construction of noncritical holomorphic functions. It depends on two main ingredients which are both developed in this paper.

The first is a technique for approximating a noncritical holomorphic function f on a compact polynomially convex subset $K \subset \mathbb{C}^n$ by entire noncritical functions. We begin by choosing a preliminary approximation of f by a generic polynomial h with a finite critical set $\Sigma \subset \mathbb{C}^n$. Since f and hence h is noncritical near K, we have $K \cap \Sigma = \emptyset$. We then find an injective map $\phi: \mathbb{C}^n \to \mathbb{C}^n \backslash \Sigma$ (a Fatou-Bieberbach map) which is very close to the identity map on K. The construction of such ϕ uses methods of the Andersen-Lempert theory of holomorphic automorphisms of $\mathbb{C}^n$ for $n \geq 2$ [A, AL], in the form developed by Rosay and the author in [FR]. Then $\widetilde{f} = h \circ \phi \in \mathcal{O}(\mathbb{C}^n)$ is noncritical on $\mathbb{C}^n$ and it approximates f on K.

The second tool is a new method for patching holomorphic functions (or maps) which preserves a certain first order differential condition (in our case the nonvanishig of the differential or, more generally, the maximal rank condition). It could be appropriately described as *patching in the domain.* The main point is the following. Suppose that (A, B) is a pair of compact sets in a complex manifold X such that $D = A \cup B$ has a basis of Stein neighborhoods and the two sets are separated in the sense that $\overline{A \backslash B} \cap \overline{B \backslash A} = \emptyset$. For any biholomorphic (=injective holomorphic) map $\gamma: U \to X$ on an open neighborhood $U \subset X$ of C which is sufficiently uniformly close to the identity map we obtain a splitting $\gamma = \beta \circ \alpha^{-1}$, where α (resp. β) is a biholomorphic map close to the identity in a neighborhood of A (resp. of B); see Theorem 3.1. If in addition γ preserves a nonsingular holomorphic foliation $\mathcal{F}$ of X then we can choose α and β such that they also preserve $\mathcal{F}$.

This result is used in the following way. Suppose that f is a noncritical function in a neighborhood of a compact set A in a Stein manifold X. Let B be another compact set in X such that the pair (A, B) has the properties stated above. Suppose furthemore that f can be approximated uniformly on a neighborhood of $C = A \cap B$ by a noncritical holomorphic function g defined in a neighborhood of B. (This is possible for instance if C is Runge in B and B is biholomorphic to a polynomially convex subset of $\mathbb{C}^n$.) Then we obtain a *biholomorphic transition map* $\gamma: U \to X$ in a neighborhood of C which is close to the identity map and satisfies $f = g \circ \gamma$. Splitting $\gamma = \beta \circ \alpha^{-1}$ as above we obtain $f \circ \alpha = g \circ \beta$ near C. The two sides define a noncritical holomorphic function in a neighborhood of $A \cup B$ which approximates f on A.

We globalize the construction by the scheme developed by Henkin and Leiterer in their proof of the Oka-Grauert principle [HL2]. (This scheme has been used in several recent papers, see e.g. [Gro2], [FP1], [FP2]). We exhaust a Stein manifold X by an increasing sequence $A_0 \subset A_1 \subset A_2 \subset \cdots \subset \cup_{k=1}^{\infty} A_k = X$ of compact holomorphically convex sets such that A_0 is a small neighborhood of the set K on which the initial function $f = f_0$ is defined, and for each $k \geq 0$ we have $A_{k+1} = A_k \cup B_k$, where (A_k, B_k) is a special Cartan pair (see above). This enables us to approximate a noncriticial function f_k on a neighborhood of A_k, obtained in the k-th step, by a noncritical function f_{k+1} on a neighborhood of A_{k+1}. The limit $\widetilde{f} = \lim_{k\to\infty} f_k$ is a noncritical function on X provided that the convergence is sufficiently fast.

We believe that Theorem 3.1 (the factorization lemma for biholomorphic maps) will be useful in other constructions of holomorphic objects satisfying a differential condition (holomorphic submersions, maps which are regular on leaves of a holomorphic foliation, etc.). For this reason we prove the result in a more general form then needed for present purposes, namely for *biholomorphic maps preserving a nonsingular holomorphic foliation.* The splitting $\gamma = \beta \circ \alpha^{-1}$ is obtained by a *rapidly convergent iteration* in which the distance from the identity map decreases quadratically at each step, except for a small term in the denominator which comes from the Cauchy estimates (when passing from the sup-norm of a function to its $\mathcal{C}^1$-norm on a smaller set). The linearized problem is solved by a *bounded solution operator for the $\overline{\partial}$-equation*, with uniform estimates on small $\mathcal{C}^2$ perturbations of a given strongly pseudoconvex domain.

The paper is organized as follows. In Sect. 2 we prove the required approximation results for noncritical holomorphic functions and submersions. In Sect. 3 prove Theorem 3.1 on splitting biholomorphic maps close to the identity. In Sect. 4 we prove Theorems 1.1 and 1.5. In Sect. 5 we discuss the homotopy principle for holomorphic submersions of Stein manifolds to Euclidean spaces and describe some related open problems.

&2. Approximation of noncritical functions and submersions.

In this section we obtain approximation results for noncritical functions and submersions on subsets of $\mathbb{C}^n$ which will be needed in the proofs of our main theorems. The results of this section depend in an essential way on the Andersén-Lempert theory of holomorphic automorphisms of $\mathbb{C}^n$ [A, AL] as developed in [FR, F1, F2, BF].

Recall that a compact set $K \subset \mathbb{C}^n$ is polynomially convex if for every $z \in \mathbb{C}^n \backslash K$ there exists an $f \in \mathcal{O}(\mathbb{C}^n)$ with $|f(z)| > \sup_K |f|$.

2.1 Theorem. *Let K be a compact polynomially convex subset of $\mathbb{C}^n$. Let f be a holomorphic function in an open set $U \supset K$ satisfying $df \neq 0$. Then for every $\epsilon > 0$ there exists an entire function $g \in \mathcal{O}(\mathbb{C}^n)$ satisfying $\sup_K |f-g| < \epsilon$ and $dg \neq 0$ on $\mathbb{C}^n$.*

Proof. Choose a compact polynomially convex set $L \subset U$ with smooth boundary and containing K in its interior. Such L may be obtained as a regular sublevel set of a strongly plurisubharmonic exhaustion function on $\mathbb{C}^n$ which is negative on K and positive on $\mathbb{C}^n \backslash U$ [Hör, Theorem 2.6.11].

Consider first the case $n = 1$. Since $L \subset \mathbb{C}$ is smoothly bounded and polynomially convex, it is a union $L = \cup_{j=1}^{m} L_j$ of finitely many compact, connected and simply connected sets L_j. Since $f'(z) \neq 0$ for $z \in U$, there is a holomorphic function h in a neighborhood of L such that $f'(z) = e^{h(z)}$ for each z.

For every $j = 2, \ldots, m$ we connect L_1 to L_j by a simple smooth arc $C_j \subset \mathbb{C}$ contained in $\mathbb{C} \backslash L$ except for its endpoints, one in L_1 and the other in L_j. Furthermore we choose the arcs C_j to be pairwise disjoint. Set $S = L \cup C_2 \cup \cdots \cup C_m$. Then the sets S and $\mathbb{C} \backslash S$ are connected, and h extends to a smooth function on S which is holomorphic near L.

By Mergelyan's theorem we can approximate h uniformly on S as close as desired by a holomorphic polynomial $\widetilde{h}$. Choose a point $a \in L_1$ and define $g(z) = \int_a^z \exp(\widetilde{h}(\zeta))\, d\zeta$. The integral does not depend on the choice of the path and hence g is an entire function on $\mathbb{C}$, with $g'(z) = \exp(\widetilde{h}(z)) \neq 0$ for each $z \in \mathbb{C}$. If $z \in L$, we can choose the path of integration from a to z entirely contained in S and with length bounded from above independently of z. (If $z \in L_j$ for $j > 1$, we include the arc C_j in the path of integration.) Hence g approximates f uniformly on $S \supset L$ which completes the proof for $n = 1$.

Assume now $n \geq 2$. Since L is polynomially convex, there exists for any $\epsilon > 0$ a holomorphic polynomial h on $\mathbb{C}^n$ satisfying $\sup_L |f - h| < \epsilon/2$. If ϵ is chosen sufficiently small then $dh \neq 0$ on K. For a generic choice of h its critical set $\Sigma = \{z \in \mathbb{C}^n : dh(z) = 0\} \subset \mathbb{C}^n \backslash K$ is finite (since it is given by n polynomial equations $\partial h / \partial z_j = 0$, $j = 1, \ldots, n$). To complete the proof we need the following.

2.2 Proposition. *Let K be a compact polynomially convex subset of $\mathbb{C}^n$ for some $n \geq 2$ and let Σ be a finite set contained in $\mathbb{C}^n \backslash K$. Then for any $\delta > 0$ there exists a biholomorphic map ϕ of $\mathbb{C}^n$ onto a subset $\Omega \subset \mathbb{C}^n \backslash \Sigma$ such that $|\phi(z) - z| < \delta$ for all $z \in K$.*

Recall that a biholomorphic of $\mathbb{C}^n$ onto a proper subset of $\mathbb{C}^n$ is called a *Fatou-Bieberbach map*. Thus ϕ in the proposition is a Fatou-Bieberbach map close to the identity on K whose range avoids Σ.

Assume for a moment that Proposition 2.2 holds. Let $g = h \circ \phi \in \mathcal{O}(\mathbb{C}^n)$. Then $dg(z) = dh(\phi(z)) \cdot d\phi(z) \neq 0$ for every $z \in \mathbb{C}^n$ (since $\phi(z) \in \Omega \subset \mathbb{C}^n \backslash \Sigma$ and $\{dh = 0\} = \Sigma$). Let $c = \sup_{z \in L} |dh(z)|$. Choose $\delta < \min\{\mathrm{dist}(K, \mathbb{C}^n \backslash L), \epsilon/2c\}$. Then we have for every $z \in K$

$$|g(z) - h(z)| = |h(\phi(z)) - h(z)| \leq c|\phi(z) - z| < c\delta < \epsilon/2$$

and hence $|g(z) - f(z)| < \epsilon$ for $z \in K$. This proves Theorem 2.1. ♠

Proof of Proposition 2.2. Choose $\epsilon \in (0,1)$. Let B denote the closed unit ball centered at the origin in $\mathbb{C}^n$ and rB its dilation by $r > 0$. Let $|z|$ denote the Euclidean norm of $z \in \mathbb{C}^n$. Choose a compact set $L \subset \mathbb{C}^n \backslash \Sigma$ containing K in its interior. Let $r_1 > 1$ be chosen such that $L \subset (r_1 - 1)B$. Set $r_k = r_1 + k - 1$ and $\epsilon_k = 2^{-k-1}\epsilon$ for $k = 1, 2, 3, \ldots$.

Consider the holomorphic flow on a neighborhood of $L \cup \Sigma$ in $\mathbb{C}^n$ which rests near L and moves the finite set Σ out of the ball $r_1 B$ within $\mathbb{C}^n \backslash L$. Since the trace of this flow is polynomially convex, the time-one map can be approximated uniformly on L by holomorphic automorphisms of $\mathbb{C}^n$ according to Theorem 1.1 in [FR]. This gives a holomorphic automorphism ψ_1 of $\mathbb{C}^n$ satisfying $|\psi_1(z) - z| < \epsilon_1$ for $z \in L$ and $\psi_1(\Sigma) \cap r_1 B = \emptyset$. (That is, we push Σ out of the ball $r_1 B$ by a holomorphic automorphism of $\mathbb{C}^n$ which is ϵ_1-close to the identity map on L.)

Set $\Sigma_1 = \psi_1(\Sigma)$. By the same argument there is an automorphism ψ_2 of $\mathbb{C}^n$ satisfying $|\psi_2(z) - z| < \epsilon_2$ for $z \in r_1 B$ and $\psi_2(\Sigma_1) \cap r_2 B = \emptyset$. Continuing inductively we obtain a sequence of automorphisms ψ_k of $\mathbb{C}^n$ such that $|\psi_k(z) - z| < \epsilon_k$ on $r_{k-1}B$ and $\psi_k(\Sigma_{k-1}) \cap r_k B = \emptyset$ for each $k = 2, 3, \ldots$. By Proposition 5.1 in [F2] (whose proof is entirely elementary) the sequence of compositions $\psi_k \circ \psi_{k-1} \circ \cdots \circ \psi_1$ converges as $k \to \infty$ to a biholomorphic map $\psi\colon \Omega \to \mathbb{C}^n$ from an open set $\Omega \subset \mathbb{C}^n$ onto $\mathbb{C}^n$. By construction we have $L \subset \Omega \subset \mathbb{C}^n \backslash \Sigma$ and $|\psi(z) - z| < \epsilon$ for $z \in L$. The inverse map $\phi = \psi^{-1}\colon \mathbb{C}^n \to \Omega$ maps $\mathbb{C}^n$ biholomorphically onto $\Omega \subset \mathbb{C}^n \backslash \Sigma$ and is uniformly close to the identity on K. Choosing ϵ sufficiently small we can insure that $|\phi(z) - z| < \delta$ for $z \in K$. ♠

Next we prove a result on approximating holomorphic submersions on certain subsets of Euclidean spaces. We eventually obtain a much stronger result, stated as Corollary 1.7 in Sect. 1 above. However, we don't see a way to prove Corollary 1.7 directly by using automorphisms. Instead we first prove a weaker result which we then use in the globalization scheme developed in Sections 3 and 4.

2.3 Proposition. *Let $x = (z, w)$ be the coordinates on $\mathbb{C}^n = \mathbb{C}^k \times \mathbb{C}^l$. Assume that $D \subset \mathbb{C}^k$ and $K \subset \mathbb{C}^n$ are compact polynomially convex subsets such that $D \times \{0\}^l \subset K \subset D \times \mathbb{C}^l$ and the fiber $K_z = \{w \in \mathbb{C}^l\colon (z, w) \in K\}$ is convex for every $z \in D$. If $l \geq n - q + 1 \geq 2$ (hence $q < n$) then for every holomorphic submersion $f = (f_1, \ldots, f_q)\colon U \to \mathbb{C}^q$ in an open set $U \supset K$, for every $\epsilon > 0$, and for every compact set $L \subset D \times \mathbb{C}^l$ there exists a holomorphic submersion $g = (g_1, \ldots, g_q)\colon V \to \mathbb{C}^q$ in an open set $V \supset L$ satisfying $\sup_K |f - g| < \epsilon$.*

Proof. Let $\pi\colon \mathbb{C}^n \to \mathbb{C}^k$ denote the projection $\pi(z, w) = z$. Since K is polynomially convex, we can approximate f uniformly on a neighborhood of K by a polynomial map $h = (h_1, \ldots, h_q)\colon \mathbb{C}^n \to \mathbb{C}^q$. A generic choice of h insures that the algebraic subvariety $\Sigma = \{x \in \mathbb{C}^n\colon \mathrm{rank}\, dh_x < q\}$ (which does not intersect K) has dimension $q - 1 \leq k$ and the projection $\pi|_\Sigma\colon \Sigma \to \mathbb{C}^k$ is proper. We may asume that $L = D \times B$ where $B \subset \mathbb{C}^l$ is a closed ball. To complete the proof it suffices to take $g = h \circ \psi$ where ψ is a furnished by the following lemma.

2.4 Lemma. (Hypotheses as above.) *For every $\delta > 0$ there exists a holomorphic automorphism ψ of $\mathbb{C}^n$ of the form $\psi(z,w) = (z, \alpha(z,w))$ such that $\psi(L) \cap \Sigma = \emptyset$ and $\sup_{x \in K} |\psi(x) - x| < \delta$.*

Remark. If $\dim \Sigma = n-1$ (which happens when $q = n$) then in general there exists no ψ with the stated properties. Moreover, due to the possible hyperbolicity of $\mathbb{C}^n \backslash \Sigma$, there need not exist any entire map $\psi \colon \mathbb{C}^n \backslash \Sigma$ which approximates the identity map on a given set $K \subset \mathbb{C}^n \backslash \Sigma$. It is at this precise point where our approach fails in the critical case $q = n$.

Proof of Lemma 2.4. We shall need a version of Theorem 1.1 (or Theorem 2.1) from [FR] with holomorphic dependence on parameters. Recall that a vector field is said to be *complete* if its flow exists for all times and all initial conditions. We shall consider holomorphic vector fields on $\mathbb{C}^n$ of the form

$$V(z,w) = \sum_{j=1}^{l} a_j(z,w) \frac{\partial}{\partial w_j}, \tag{2.1}$$

where the a_j's are entire (or polynomial) functions on $\mathbb{C}^n = \mathbb{C}^k \times \mathbb{C}^l$. Its flow preserves the foliation $\{z = const\}$, and V is complete on $\mathbb{C}^n$ if and only if the field $V(z, \cdot\,)$ is complete on $\{z\} \times \mathbb{C}^l$ for each $z \in \mathbb{C}^k$.

2.5 Proposition. *If $l \geq 2$ then every polynomial vector field (2.1) on $\mathbb{C}^k \times \mathbb{C}^l$ is a finite sum of complete polynomial fields of the same form.*

Proof. By Lemma 1.3 in [FR] (which is in fact due to Andersén and Lempert [AL]; see the Appendix in [F1] for a short proof) *every polynomial holomorphic vector field on $\mathbb{C}^l$ for $l \geq 2$ is a finite sum of complete polynomial fields.* We can write a polynomial field (2.1) in the form $V(z,w) = \sum_\alpha z^\alpha V_\alpha(w)$ where $V_\alpha(w) = \sum_{j=1}^{l} a_{\alpha,j}(w) \frac{\partial}{\partial w_j}$. By the quoted result each field $V_\alpha(w)$ is a finite sum of complete fields in the w-variable. The product of such a complete field with z^α remains complete on $\mathbb{C}^n$ which proves the result. ♠

From Proposition 2.5 it follows that the time-t map of any entire holomorphic vector field of the form (2.1) can be approximated, uniformly on any compact set on which it exists, by holomorphic automorphisms of $\mathbb{C}^n$ of the form $(z,w) \to \big(z, \varphi(z,w)\big)$ (see Lemma 1.4 in [FR]). The same holds for time t-map of a time-dependent entire holomorphic vector field of the form (2.1) with smooth dependence on the time variable. From this one obtains the following parametric analogue of [FR, Theorem 2.1]:

2.6 Corollary. *Assume that* $\phi_t: \Omega_0 \to \Omega_t$ $(t \in [0,T])$ *is a smooth isotopy of biholomorphic maps between domains in* $\mathbb{C}^n$, *with* ϕ_0 *the identity map on* Ω_0, *where* $n = k + l$, $l \ge 2$, *and each* ϕ_t *is of the form*

$$\phi_t(z,w) = \big(z, \varphi_t(z,w)\big) \qquad (z \in \mathbb{C}^k,\ w \in \mathbb{C}^l).$$

If $M \subset \Omega_0$ *is a compact polynomially convex set such that* $\phi_t(M)$ *is polynomially convex for every* $t \in [0,T]$ *then* ϕ_T *can be approximated uniformly on* M *by automorphisms of* $\mathbb{C}^n$ *of the form* $(z,w) \to \big(z, \varphi(z,w)\big)$.

We can now conclude the proof of Lemma 2.4: Let Σ, K and L be as in the lemma. The set $\Sigma' = \Sigma \cap (D \times \mathbb{C}^l)$ is polynomially convex. Since $\Sigma \cap K = \emptyset$, the union $M = K \cup \Sigma'$ is also polynomially convex. Let $\theta_t(z,w) = (z, e^t w)$. Since the fibers K_z $(z \in D)$ are convex and contain the origin, the subvariety $\theta_t(\Sigma) \subset \mathbb{C}^n$ is disjoint from K for all $t \ge 0$. It follows that the set $M_t = K \cup \theta_t(\Sigma')$ is polynomially convex for every $t \ge 0$. Clearly $\theta_T(\Sigma) \cap L = \emptyset$ for a sufficiently large $T > 0$.

Consider the holomorphic flow ϕ_t which rests on a neighborhood of K and equals θ_t on a neighborhood of Σ'. Its time-t map takes M biholomorphically onto M_t. Since the latter set is polynomially convex for every $t \ge 0$, Corollary 2.6 gives an automorphism $\phi(z,w) = \big(z, \varphi(z,w)\big)$ which approximates ϕ_T uniformly on a neighborhood of M. Thus ϕ is close to the identity on a neighborhood of K and it maps Σ' out of L. Since ϕ preserves the planes $\{z = \mathit{const}\}$, it follows that $\phi(\Sigma) \cap L = \emptyset$. Its inverse $\psi = \phi^{-1}$ then satisfies the conclusion of Lemma 2.4. ♠

&3. Splitting biholomorphic mappings close to the identity.

We begins with some terminology that will be used in the remainder of the paper. Let X be a complex manifold. An injective holomorphic map $\gamma: U \to X$ from an open set $U \subset X$ into X will be called *biholomorphic*. (Precisely, γ is a biholomorphic map of U onto its image $\gamma(U) \subset X$.) We say that γ *preserves a holomorphic foliation* $\mathcal{F}$ of X if $\gamma(x) \in \mathcal{F}_x$ for each $x \in U$, where $\mathcal{F}_x$ denotes the leaf of $\mathcal{F}$ through x. A biholomorphic map $\gamma: U \to X$ preserving a foliation $\mathcal{F}$ of X will be called an $\mathcal{F}$*-map on* U.

3.1 Theorem. *Let* $A, B \subset X$ *be compact sets in a complex manifold* X *such that* $D = A \cup B$ *has a basis of Stein neighborhoods in* X *and* $\overline{A\backslash B} \cap \overline{B\backslash A} = \emptyset$. *Given an open set* $\widetilde{C} \subset X$ *containing* $C = A \cap B$ *there exist open sets* $A' \supset A$, $B' \supset B$, $C' \supset C$, *with* $C' \subset A' \cap B' \subset \widetilde{C}$, *satisfying the following. For every biholomorphic map* $\gamma: \widetilde{C} \to X$ *which is sufficiently uniformly close to the identity there exist biholomorphic maps* $\alpha: A' \to X$, $\beta: B' \to X$, *uniformly close to the identity on their respective domains and satisfying*

$$\gamma = \beta \circ \alpha^{-1} \quad \text{on } C'.$$

If γ preserves a nonsingular holomorphic foliation $\mathcal{F}$ of X then we can choose α and β to preserve $\mathcal{F}$ as well. If X_0 is a closed complex subvariety of X such that $X_0 \cap C = \emptyset$, we can choose α and β as above such that in addition they are tangent to the identity map to any given finite order along X_0.

Theorem 3.1 is one of the main technical results of the paper and is a key ingredient in the proof of Theorem 1.1. The reader may notice a remote analogy with the classical *Cartan lemma on product splitting* of a holomorphic map with values in a complex Lie group. The essential difference is that we are splitting a biholomorphic map into a *composition* of biholomorphic maps. While Cartan's lemma can be proved using a bounded linear solution operator on strongly pseudoconvex domains and the implicit function theorem, here we cannot avoid iterations since the domains shrink at each step.

We begin with some preparatory results. Let d be a metric on X induced by a smooth hermitean metric on TX. Given a map $\gamma: U \subset X \to X$ we shall write

$$||\gamma - id||_U = \sup_{x \in U} d(\gamma(x), x).$$

We shall say that γ is *ϵ-close to the identity on U* if $||\gamma - id||_U < \epsilon$. Given a relatively compact set $K \subset\subset X$ and $r > 0$ we write

$$K_r = \{x \in X: d(x, K) = \inf_{y \in K} d(x, y) \le r\},$$

$$K(r) = \{x \in X: d(x, y) < r \text{ for some } y \in K\}.$$

Note that $K(r)$ is open while K_r is closed (and hence compact for all sufficiently small $r > 0$).

3.2 Lemma. *For every pair of subsets $A, B \subset\subset X$ we have $(A \cup B)_r = A_r \cup B_r$ for all $r > 0$. If we assume in addition $\overline{A \backslash B} \cap \overline{B \backslash A} = \emptyset$ then for all sufficiently small $r > 0$ we also have $(A \cap B)_r = A_r \cap B_r$ and $\overline{A_r \backslash B_r} \cap \overline{B_r \backslash A_r} = \emptyset$. The analogous properties hold for the sets $A(r)$ and $B(r)$.*

Proof. The inclusion $A_r \cup B_r \subset (A \cup B)_r$ is trivial. If $x \in (A \cup B)_r$ then there is a sequence of points $y_j \in A \cup B$ such that $\lim_{j \to \infty} d(x, y_j) \le r$. Infinitely many terms of the sequence y_j belong to one of the two sets A, B which implies that $x \in A_r$ or $x \in B_r$. This proves $(A \cup B)_r \subset A_r \cup B_r$ and hence the two sets are equal. To prove the second property we write $A = (A \backslash B) \cup C$, $B = (B \backslash A) \cup C$, where $C = A \cap B$. By the first property we have $A_r = (A \backslash B)_r \cup C_r$ and $B_r = (B \backslash A)_r \cup C_r$. The separation condition $\overline{A \backslash B} \cap \overline{B \backslash A} = \emptyset$ insures that $(A \backslash B)_r \cap (B \backslash A)_r = \emptyset$ for all sufficiently small $r > 0$. This gives $A_r \cap B_r = C_r$, $A_r \backslash B_r = (A \backslash B)_r \backslash C_r$, $B_r \backslash A_r = (B \backslash A)_r \backslash C_r$, and $\overline{A_r \backslash B_r} \cap \overline{B_r \backslash A_r} = \emptyset$. Similarly one proves the corresponding properties for $A(r)$ and $B(r)$. ♠

The next lemma shows that it suffices to prove Theorem 3.1 in the case when the union $D = A \cup B$ is the closure of a smooth strongly pseudoconvex domain in a Stein manifold.

3.3 Lemma. *Let $A, B \subset X$ be compact sets in a complex manifold X satisfying $\overline{A\backslash B} \cap \overline{B\backslash A} = \emptyset$ and such that $A \cup B$ has a basis of Stein neighborhoods in X. Given open sets $\widetilde{A} \supset A$, $\widetilde{B} \supset B$, $\widetilde{C} \supset C = A \cap B$, $\widetilde{D} \supset A \cup B$, there exist compact sets $A', B' \subset X$ satisfying the following properties:*

(a) $A \subset A' \subset \widetilde{A}$, $B \subset B' \subset \widetilde{B}$, $A' \cap B' \subset \widetilde{C}$,

(b) $\overline{A'\backslash B'} \cap \overline{B'\backslash A'} = \emptyset$,

(c) the set $D' = A' \cup B' \subset \widetilde{D}$ is the closure of a smoothly bounded strongly pseudoconvex Stein domain in X.

Proof. If $r > 0$ is chosen sufficiently small then $A_r \subset \widetilde{A}$, $B_r \subset \widetilde{B}$, and $A_r \cap B_r = C_r \subset \widetilde{C}$ (in the last equality we used Lemma 3.2). The set $A_r \cup B_r = (A \cup B)_r$ contains a neighborhood of $A \cup B$ and hence by assumption there is a closed smooth strongly pseudoconvex Stein domain D' with $A \cup B \subset D' \subset A_r \cup B_r$. The sets $A' = A_r \cap D'$ and $B' = B_r \cap D'$ then satisfy the stated properties. ♠

Assumption. Replacing X by a suitable neighborhood of $D = A \cup B$ in X we shall assume in the remainder of this section that X is a Stein manifold and $D = \{\rho \leq 0\} \subset X$, where $\rho: X \to \mathbb{R}$ is a smooth strongly pseudoconvex exhaustion function with $d\rho \neq 0$ on $bD = \{\rho = 0\}$.

Let $\mathcal{F}$ be a nonsingular holomorphic foliation of X. Denote by $F \subset TX$ the tangent bundle of $\mathcal{F}$ with fibers $F_x = T_x\mathcal{F}_x$ $(x \in X)$. By Cartan's Theorem A there exist finitely many holomorphic vector fields $L_1, L_2, \ldots, L_m$ on X such that the vectors $L_j(x)$ $(j = 1, \ldots, m)$ span F_x for every x in a neighborhood of $D = A \cup B$. By shrinking X around D we may assume that the L_j's span F over X.

Denote by $\theta^j_t(x)$ the flow of L_j for time $t \in \mathbb{C}$, solving $\frac{\partial}{\partial t}\theta^j_t(x) = L_j(\theta^j_t(x))$ and $\theta^j_0(x) = x$. The map θ^j is defined and holomorphic for (x, t) in an open neighborhood of $X \times \{0\}$ in $X \times \mathbb{C}$. Let

$$\theta(x, t) = \theta(x, t_1, \ldots, t_m) = \theta^m_{t_m} \circ \cdots \circ \theta^2_{t_2} \circ \theta^1_{t_1}(x).$$

This is a holomorphic map from an open neighborhood $U \subset X \times \mathbb{C}^m$ of the zero section $X \times \{0\}^m$ with values in X, satisfying $\theta(x, t) \in \mathcal{F}_x$ for all $(x, t) \in U$ (since the fields L_j are tangent to the leaves of $\mathcal{F}$) and

$$\theta(x, 0) = x, \qquad \frac{\partial}{\partial t_j}\theta(x, t)|_{t=0} = L_j(x) \quad (1 \leq j \leq m,\ x \in X).$$

In particular, the differential $\partial_t\theta|_{t=0}$ maps the trivial bundle $X \times \mathbb{C}^m$ surjectively onto $F = T\mathcal{F}$. We can split $X \times \mathbb{C}^m = E \oplus H$ where H is the kernel of $\partial_t\theta|_{t=0}$ and E is some complementary holomorphic subbundle. Thus $\partial_t\theta|_{t=0}: E \to F$

is an isomorhism of holomorphic vector bundles. In any holomorphic vector bundle chart on E we have a Taylor expansion

$$\theta(x, t_1, \ldots, t_m) = x + \sum_{j=1}^{m} t_j\, L_j(x) + O(|t|^2) \tag{3.1}$$

where the constant in the remainder $O(|t|^2)$ can be chosen uniform on any compact subset of the base set.

Choose a hermitean metric $|\cdot|_E$ on E. Given an open set $V \subset X$ and a section $c: V \to E|_V$ we shall write $||c||_V = \sup_{x\in V} |c(x)|_E$. Recall that an $\mathcal{F}$-map in X is an injective holomorphic map preserving the leaves of $\mathcal{F}$. The following lemma follows immediatelly from the implicit function theorem.

3.4 Lemma. *For every open relatively compact set $V \subset\subset X$ there exist constants $M_1 \geq 1$ and $\epsilon_0 > 0$ satisfying the following property. For any $\mathcal{F}$-map $\gamma: V \to X$ satisfying $||\gamma - id||_V < \epsilon_0$ there is a unique holomorphic section $c: V \to E$ of $E|_V \to V$ such that for every $x \in V$ we have $\theta(x, c(x)) = \gamma(x)$ and*

$$M_1^{-1}|c(x)| \leq d(\gamma(x), x) \leq M_1|c(x)|.$$

Remark. Of course $\mathcal{F}$ may be the trivial foliation with X itself as the only leaf; in such case we have $E \simeq TX$, and we represent every biholomorphic map $\gamma: V \to X$ sufficiently close to the identity map as $\gamma(x) = \theta(x, c(x))$ for some holomorphic section $c: V \to TV$. ♠

We shall write the composition $\gamma \circ \alpha$ simply as $\gamma\alpha$. From now on all our sets in X will be assumed to be contained in a fixed relatively compact set for which Lemma 3.4 holds with a fixed constant M_1.

3.5 Lemma. *Let $V \subset\subset X$. There are constants $\delta_0 > 0$ (small) and $M_2 > 0$ (large) with the following property. Let $0 < \delta < \delta_0$ and $0 < 4\epsilon < \delta$. Assume that $\alpha, \beta, \gamma: V(\delta) \to X$ are $\mathcal{F}$-maps which are ϵ-close to the identity on $V(\delta)$. Then $\widetilde{\gamma} := \beta^{-1}\gamma\alpha: V \to X$ is a a well defined $\mathcal{F}$-map on V. Write*

$$\alpha(x) = \theta(x, a(x)), \quad \beta(x) = \theta(x, b(x)),$$
$$\gamma(x) = \theta(x, c(x)), \quad \widetilde{\gamma}(x) = \theta(x, \widetilde{c}(x)),$$

where a, b, c are sections of $E|_{V(\delta)} \to V(\delta)$ and $\widetilde{c}$ is a section of $E|_V \to V$ given by Lemma 3.4. Then we have

$$||\,\widetilde{c} - (c + a - b)||_V \leq M_2\delta^{-1}\epsilon^2. \tag{3.2}$$

If $c = b - a$ on $V(\delta)$ then $||\widetilde{c}||_V \leq M_2\delta^{-1}\epsilon^2$ and hence $||\widetilde{\gamma} - id||_V \leq M_1M_2\delta^{-1}\epsilon^2$.

Proof. The conditions imply that the composition $\gamma\alpha$ maps V biholomorphically onto a subset of $V(2\epsilon)$. Since β is ϵ-close to the identity map on $V(\delta)$, the degree theory shows that its range contains $V(\delta-\epsilon)$. Hence β^{-1} is defined on $V(\delta-\epsilon)$ and is ϵ-close to the identity on this set. Since $4\epsilon<\delta$, it follows that the composition $\widetilde{\gamma}=\beta^{-1}\gamma\alpha$ is defined on V and maps V biholomorphically onto a subset of $V(3\epsilon)\subset\subset V(\delta)$.

To prove the estimate (3.2) we choose a holomorphic vector bundle chart on $\pi\colon E\to X$ over an open set $\widetilde{U}\subset X$ and let $U\subset\subset\widetilde{U}$. We shall use the expansion (3.1) for θ on $\pi^{-1}(U)\subset E$. This will suffice since $\overline{V(\delta)}$ can be covered by finitely many such sets U. We replace the fiber variable t in (3.1) by one of the functions $a(x), b(x)$, or $c(x)$. These are bounded on $V(\delta)$ by a $M_1\epsilon$ where M_1 is the constant from Lemma 3.4. This gives for $x\in U\cap V(\delta)$:

$$\alpha(x)=x+\sum_{j=1}^m a_j(x)L_j(x)+O(\epsilon^2),$$
$$\beta(x)=x+\sum_{j=1}^m b_j(x)L_j(x)+O(\epsilon^2),$$
$$\gamma(x)=x+\sum_{j=1}^m c_j(x)L_j(x)+O(\epsilon^2),$$

The remainder $O(\epsilon^2)$ is uniform with respect to $x\in U\cap V(\delta)$. For $x\in U\cap V$ this gives

$$\begin{aligned}\gamma(\alpha(x))&=\alpha(x)+\sum_{j=1}^m c_j(\alpha(x))L_j(\alpha(x))+O(\epsilon^2)\\ &=x+\sum_{j=1}^m\left(a_j(x)+c_j(x)\right)L_j(x)\\ &\qquad+\sum_{j=1}^m\big(c_j(\alpha(x))L_j(\alpha(x))-c_j(x)L_j(x)\big)+O(\epsilon^2).\end{aligned}$$

To estimate the terms in the last sum we fix a j and write $g(x)=c_j(x)L_j(x)$ for $x\in U\cap V(\delta)$. Since $||c_j||_{V(\delta)}<M_1\epsilon$ and $4\epsilon<\delta$, the Cauchy estimates imply $||dc_j||_{U\cap V(\epsilon)}=O(\epsilon/\delta)$. (Here dc_j denotes the differential of c_j.) Since L_j is holomorphic in a fixed neighborhood of $\overline{V(\delta)}$ in X, we may assume that its expression in the local coordinates on U is uniformly bounded and has uniformly bounded differential. This gives $||dg||_{U\cap V(\epsilon)}=O(\epsilon/\delta)$. Since $d(x,\alpha(x))<\epsilon$, there is a smooth arc $\lambda\colon[0,1]\to U$ of length comparable to ϵ such that $\lambda(0)=x$ and $\lambda(1)=\alpha(x)$. Then

$$|g(\alpha(x))-g(x)|\le\int_0^1|dg(\lambda(\tau)|\cdot|\lambda'(\tau)|\,d\tau\le O(\delta^{-1}\epsilon^2)$$

(the extra ϵ comes from the length of the arc λ). This gives for $x \in U \cap V$

$$\gamma(\alpha(x)) = x + \sum_{j=1}^{m} (a_j(x) + c_j(x))\, L_j(x) + O(\delta^{-1}\epsilon^2)$$

We can repeat this argument for the composition of several maps provided that ϵ is sufficiently small in comparison to δ; the error term remains of order $O(\delta^{-1}\epsilon^2)$.

It remains to find the Taylor expansion of β^{-1} on the set $U \cap V(2\epsilon)$ where U is a local chart as above. Set $\widetilde{\beta}(x) = x - \sum_{j=1}^{m} b_j(x) L_j(x)$ for $x \in U \cap V(\delta)$. Assuming that $\beta(x) \in U \cap V(2\epsilon)$ we obtain

$$\begin{aligned}\widetilde{\beta}(\beta(x)) &= \beta(x) - \sum_{j=1}^{m} b_j(\beta(x) L_j(\beta(x)) \\ &= \quad x + \sum_{j=1}^{m} \big(b_j(x)L_j(x) - b_j(\beta(x))L_j(\beta(x))\big) + O(\epsilon^2) \\ &= \quad x + O(\delta^{-1}\epsilon^2).\end{aligned}$$

We have estimated the terms in the parentheses on the middle line by $O(\delta^{-1}\epsilon^2)$ in exactly the same way as above, using the Cauchy estimates and integrating over a path of length comparable to ϵ. Writing $\beta(x) = y \in U \cap V(2\epsilon)$, $x = \beta^{-1}(y)$, the above gives $\widetilde{\beta}(y) = \beta^{-1}(y) + O(\delta^{-1}\epsilon^2)$ and therefore

$$\beta^{-1}(y) = y - \sum_{j=1}^{m} b_j(y) L_j(y) + O(\delta^{-1}\epsilon^2).$$

The same argument as before gives

$$\widetilde{\gamma}(x) = (\beta^{-1}\gamma\alpha)(x) = x + \sum_{j=1}^{m} \big(c_j(x) + a_j(x) - b_j(x)\big) L_j(x) + O(\delta^{-1}\epsilon^2)$$

for $x \in U \cap V$. This proves the estimate (3.2). ♠

3.6 Lemma. *Let $E \to X$ be a holomorphic vector bundle over a Stein manifold X. Let $U, V \subset X$ be open sets such that $\overline{U \backslash V} \cap \overline{V \backslash U} = \emptyset$ and $D = U \cup V$ is a relatively compact, smoothly bounded, strongly pseudoconvex domain in X. Set $W = U \cap V$. There is a constant $M_3 \geq 1$ such that for every bounded holomorphic section $c: W \to E$ there exist bounded holomorphic sections $a: U \to E$, $b: V \to E$ satisfying*

$$c = b|_W - a|_W, \quad ||a||_U < M_3 ||c||_W, \quad ||b||_V < M_3 ||c||_W.$$

Such a and b are given by bounded linear operators between the spaces of bounded holomorphic sections of E on the respective sets. The constant M_3 can be chosen uniform for all such pairs (U, V) in X close to an initial pair (U_0, V_0) provided that $D = U \cup V$ is sufficiently $\mathcal{C}^2$-close to $D_0 = U_0 \cup V_0$.

Proof. This is a standard application of the solvability of the $\overline{\partial}$-equation. We give a brief sketch for the sake of completeness.

Condition (b) insures that there is a smooth function $\chi: X \to [0, 1]$ which equals zero in a neighborhood of $\overline{U \backslash V}$ and equals one in a neighborhood of $\overline{V \backslash U}$. Since $D = U \cup V$ is a relatively compact strongly pseudoconvex domain in X, there exists a bounded linear solution operator T for the $\overline{\partial}$-equation associated to sections of $E \to X$ over D. Precisely, for any bounded $\overline{\partial}$-closed E-valued $(0, 1)$-form g on D we have $\overline{\partial}(Tg) = g$ and $||Tg||_D \leq \text{const}||g||_D$. Furthermore, the constant may be chosen uniformly for all domains in X which are sufficiently $\mathcal{C}^2$-close to an initial strongly pseudoconvex domain. (For functions this result can be found on p. 82 in [HL1]. The problem for sections can be reduced to that for functions by embedding$E$ as a subbundle of a trivial bundle over X.)

Observe that χc extends to a bounded smooth section of E over U and $(\chi - 1)c$ extends to a bounded section over V. Since $\text{supp}\,(\overline{\partial}\chi) \cap D \subset W = U \cap V$, the bounded $(0, 1)$-form $g = \overline{\partial}(\chi c) = \overline{\partial}((\chi - 1)c) = c\overline{\partial}\chi$ on W extends to a bounded $(0, 1)$-form on D which is zero outside of W. It is immediate that the pair of sections

$$a = \chi c - T(g)|_U, \qquad b = (\chi - 1)c - T(g)|_V$$

satisfies Lemma 3.6. ♠

The part of Theorem 3.1 with the additional interpolation along a subvariety $X_0 \subset X$ will require the following modification of Lemma 3.6.

3.6' Lemma. (Assumptions as in Lemma 3.6.) *If X_0 is a closed complex subvariety of X with $X_0 \cap \overline{W} = \emptyset$ then for every $s \in \mathbb{N}$ we can choose the sections a and b as in Lemma 3.6 such that, in addition, they vanish to order s on X_0.*

Proof. For functions this is Lemma 3.2 in [FP2]. The same proof applies to sections of $E \to X$ by embedding E into a trivial bundle over X. ♠

3.7 Lemma. *Let $A, B \subset X$ be compact subsets of X such that $\overline{A \backslash B} \cap \overline{B \backslash A} = \emptyset$ and their union $D = A \cup B$ is a closed, smoothly bounded, strongly pseudoconvex domain in X. Let $\mathcal{F}$ be a nonsingular holomorphic foliation of X and let X_0 be a closed complex subvariety of X with $X_0 \cap C = \emptyset$, where $C = A \cap B$. Then there are constants $r_0 > 0, \delta_0 > 0$ (small) and $M_4, M_5 > 1$ (large) satisfying the following. Let $0 < r \leq r_0$, $0 < \delta \leq \delta_0$ and $s \in \mathbb{N}$. For every $\mathcal{F}$-map $\gamma: C(r + \delta) \to X$ satisfying $4M_4||\gamma - id||_{C(r+\delta)} < \delta$ there exist*

$\mathcal{F}$-maps $\alpha: A(r+\delta) \to X$ and $\beta: B(r+\delta) \to X$, tangent to the identity map to order s along X_0, such that $\widetilde{\gamma} := \beta^{-1}\gamma\alpha$ is an $\mathcal{F}$-map on $C(r)$ satisfying

$$||\widetilde{\gamma} - id||_{C(r)} < M_5 \delta^{-1} ||\gamma - id||^2_{C(r+\delta)}. \tag{3.3}$$

Proof. If r_0 and δ_0 are chosen sufficiently small, the set $D(t)$ is a small $\mathcal{C}^2$-perturbation of the strongly pseudoconvex domain $D = A \cup B$ for every $t \in [0, r_0 + \delta_0]$ and hence we can use the same constant as a bound on the sup-norm of an operator solving the $\overline{\partial}$-problem on $D(t)$.

Let $\epsilon = ||\gamma - id||_{C(r+\delta)}$. By Lemma 3.4 we have $\gamma(x) = \theta(x, c(x))$ for a holomorphic section c of the bundle $E \to X$ over $C(r+\delta)$ with $||c||_{C(r+\delta)} \leq M_1\epsilon$. (Here we can use the constant M_1 for the set $D(r_0+\delta_0)$.) Write $c = a - b$ where a is a section of E over $A(r+\delta)$ and b is a section over $B(r+\delta)$ furnished by Lemma 3.6 or 3.6'. The sup-norms of a and b on their respective domains are bounded by $M_1 M_3 \epsilon$, where the constant M_3 from Lemma 3.5 can be chosen independent of r and δ. Set

$$\alpha(x) = \theta(x, a(x)) \qquad (x \in A(r+\delta)),$$
$$\beta(x) = \theta(x, b(x)) \qquad (x \in B(r+\delta)).$$

By Lemma 3.4 we have $||\alpha - id||_{A(r+\delta)} < M_1^2 M_3 \epsilon$ and $||\beta - id||_{B(r+\delta)} < M_1^2 M_3 \epsilon$. Set $M_4 = M_1^2 M_3$. If $0 < 4M_4\epsilon < \delta$ then by Lemma 3.5 the composition $\widetilde{\gamma} = \beta^{-1}\gamma\alpha$ is an $\mathcal{F}$-map on $C(r)$ satisfying the estimate (3.3) with $M_5 = M_2 M_4^2 = M_1^4 M_2 M_3^2$. ♠

Proof of Theorem 3.1. By Lemma 3.3 we may assume that $D = A \cup B$ is the closure of a smooth strongly pseudoconvex domain in X and $\overline{A \backslash B} \cap \overline{B \backslash A} = \emptyset$. Choose a sufficiently small number $0 < r_0 < 1$ such that the initial $\mathcal{F}$-map γ is defined on the set $C_0 = C(r_0)$ and Lemma 3.7 holds for all $\delta, r > 0$ with $\delta + r \leq r_0$. For each $k = 0, 1, 2, \ldots$ we set

$$r_k = r_0 \prod_{j=1}^{k} (1 - 2^{-j}), \quad \delta_k = r_k - r_{k+1} = r_k 2^{-k-1}.$$

The sequence $r_k > 0$ is decreasing, $r^* = \lim_{k\to\infty} r_k > 0$, $\delta_k > r^* 2^{-k-1}$ for all k, and $\sum_{k=0}^{\infty} \delta_k = r_0 - r^*$. Set $A_k = A(r_k)$, $B_k = B(r_k)$, $C_k = C(r_k)$. We choose $r_0 > 0$ sufficiently small such that $C_k = A_k \cap B_k$ for all k according to Lemma 3.2.

Let $\epsilon_0 := ||\gamma - id||_{C_0}$. Assuming that $4M_4\epsilon_0 < \delta_0 = r_0/2$, Lemma 3.7 gives $\mathcal{F}$-maps $\alpha_0: A_0 \to X$ and $\beta_0: B_0 \to X$ such that $\gamma_1 = \beta_0^{-1}\gamma\alpha_0: C_1 \to X$ is an $\mathcal{F}$-map satisfying

$$||\gamma_1 - id||_{C_1} < M_5 \delta_0^{-1} \epsilon_0^2 < 2M\epsilon_0^2$$

where we set $M = M_5/r^*$. Define $\epsilon_1 = ||\gamma_1 - id||_{C_1}$, so $\epsilon_1 < 2M\epsilon_0^2$. Assuming for a moment that $4M_4\epsilon_1 < \delta_1$ we can apply Lemma 3.7 to obtain a pair of $\mathcal{F}$-maps $\alpha_1: A_1 \to X$, $\beta_1: B_1 \to X$ such that $\gamma_2 = \beta_1^{-1}\gamma_1\alpha_1: C_1 \to X$ is an $\mathcal{F}$-map satisfying

$$\epsilon_2 := ||\gamma_2 - id||_{C_2} < M_5\delta_1^{-1}\epsilon_1^2 < 2^2M\epsilon_1^2.$$

Continuing inductively we construct sequences of $\mathcal{F}$-maps

$$\alpha_k: A_k \to X, \quad \beta_k: B_k \to X, \quad \gamma_k: C_k \to X$$

such that $\gamma_{k+1} = \beta_k^{-1}\gamma_k\alpha_k: C_{k+1} \to X$ is an $\mathcal{F}$-map satisfying

$$\epsilon_{k+1} := ||\gamma_{k+1} - id||_{C_{k+1}} < M_5\delta_k^{-1}\epsilon_k^2 < 2^{k+1}M\epsilon_k^2. \tag{3.4}$$

The necessary condition for the induction step is that $4M_4\epsilon_k < \delta_k$ holds for each k. Since $\delta_k > r^*2^{-k-1}$, it suffices to have

$$4M_4\epsilon_k < r^*2^{-k-1} \qquad (k = 0, 1, 2, \ldots). \tag{3.5}$$

In order to obtain convergence of this process we need the following.

3.8 Lemma. *Let $M, M_4 \geq 1$. Let the sequence $\rho_k > 0$ be defined recursively by $\rho_0 = \epsilon_0 > 0$ and $\rho_{k+1} = 2^{k+1}M\rho_k^2$ for $k = 0, 1, \ldots$. If $\epsilon_0 < r^*/32MM_4$ then $\rho_k < (4M\epsilon_0)^{2^k} < (1/8)^{2^k}$ and $4M_4\rho_k < r^*2^{-k-1}$ for all $k = 0, 1, 2, \ldots$.*

Assuming Lemma 3.8 we complete the proof of Theorem 3.1 as follows. From (3.4) we see that $\epsilon_k \leq \rho_k$ where ρ_k is the sequence from Lemma 3.8. From the assumption $\epsilon_0 < r^*/32MM_4$ we obtain $q := 4M\epsilon_0 < r^*/8M_4 < 1/8$ (since $0 < r^* < 1$ and $M_4 \geq 1$). Hence the sequence $\epsilon_k = ||\gamma_k - id||_{C_k} < q^{2^k} < (1/8)^{2^k}$ converges to zero very rapidly as $k \to \infty$. The second estimate on ρ_k in Lemma 3.8 insures that (3.5) holds and hence the induction described above works.

Let $\widetilde{\alpha}_k = \alpha_0\alpha_1\cdots\alpha_k: A_k \to X$ and $\widetilde{\beta}_k = \beta_0\beta_1\cdots\beta_k: B_k \to X$. By our construction we have $\gamma = \widetilde{\beta}_k\gamma_{k+1}\widetilde{\alpha}_k^{-1}$ on C_{k+1}. The construction also insures that the sequences $\widetilde{\alpha}_k$ resp. $\widetilde{\beta}_k$ converge, uniformly on $A(r^*)$ resp. on $B(r^*)$, to $\mathcal{F}$-maps $\alpha: A(r^*) \to X$ resp. $\beta: B(r^*) \to X$ as $k \to \infty$. Furthermore, the sequence γ_k converges uniformly on $C(r^*)$ to the identity map according to (3.4) and Lemma 3.8. In the limit we obtain $\gamma = \beta\alpha^{-1}$ on $C(r^*)$. ♠

Proof of Lemma 3.8. The sequence is of the form $\rho_k = 2^{a_k}M^{b_k}\epsilon_0^{c_k}$ where the sequences of exponents satisfy the recursive relations

$$\begin{aligned} a_{k+1} &= 2a_k + k + 1, & a_0 &= 0; \\ b_{k+1} &= 2b_k + 1, & b_0 &= 0; \\ c_{k+1} &= 2c_k, & c_0 &= 0. \end{aligned}$$

The solutions are $a_k = 2^k \sum_{j=1}^k j2^{-j} < 2^{k+1}$, $b_k = 2^k - 1$, $c_k = 2^k$. Thus

$$\rho_k < 2^{2^{k+1}} M^{2^k} \epsilon_0^{2^k} = (4M\epsilon_0)^{2^k}$$

which proves the first required estimate. From the assumption $\epsilon_0 < r^*/32MM_4$ we get $q := 4M\epsilon_0 < r^*/8M_4 < 1/8$. Hence $\rho_k < q^{2^k} < (1/8)^{2^k}$ and

$$4M_4\rho_k < (4M_4 q)q^{2^k-1} = (4M_4 \cdot 4M\epsilon_0)(1/8)^{2^k-1} < (r^*/2)2^{-k} = r^* 2^{-k-1}$$

for all $k \geq 0$ which proves the second estimate. ♠

Remark. Our proof shows that the pair (α, β) solving $\gamma = \beta\alpha^{-1}$ is given by nonlinear operators $\alpha = \mathcal{A}(\gamma)$, $\beta = \mathcal{B}(\gamma)$, defined on the set of $\mathcal{F}$-maps γ which are sufficiently uniformly close to the identity on a neighborhood of C. This yields the analogous result for parametrized families $\{\gamma_p : p \in P\}$ of $\mathcal{F}$-maps which depend continuously on a parameter p in a compact Hausdorff space P and which are sufficiently close to the identity map on a neighborhood of C.

&4. Proofs of the main theorems.

Let X be a complex manifold. A compact set $K \subset X$ is said to be a *Stein compactum* if it has a basis of open Stein neighborhoods. Let d be a metric on X induced by a smooth hermitean metric on TX. We shall use the terminology introduced in Sect. 3. In particular, an injective holomorphic map $\gamma: V \subset X \to X$ is said to be biholomorphic, and it is called an $\mathcal{F}$-map when it preserves a foliation $\mathcal{F}$ of X. We let $||\gamma - id||_V = \sup_{x\in V} d(\gamma(x), x)$.

4.1 Lemma. *Let K be a Stein compactum in a complex manifold X. Let $U \subset X$ be an open set containing K and $f: U \to \mathbb{C}^q$ a holomorphic submersion for some $q \leq \dim X$. Then there exist constants $\epsilon_0 > 0, M > 0$ and an open set $V \supset K$ with the following property. For any ϵ with $0 < \epsilon < \epsilon_0$ and for any holomorphic submersion $g: U \to \mathbb{C}^q$ with $\sup_{x\in U} |f(x) - g(x)| < \epsilon$ there is a biholomorphic map $\gamma: V \to X$ satisfying $f = g \circ \gamma$ on V and $||\gamma - id||_V < M\epsilon$.*

Proof. We may assume that the set U is Stein. Then $TX|_U = \ker df \oplus E$ for some holomorphic vector subbundle $E \subset TX|_U$ of rank q. The tangent map of f maps E isomorphically onto $f^*(T\mathbb{C}^q) \simeq U \times \mathbb{C}^q$. Choose q holomorphic vector field on U which are linearly independent and span E at each point (for instance, we may take the df-preimages of the standard coordinate vector fields on $\mathbb{C}^q$). Denote by $\theta(x, t_1, \ldots, t_q)$ the composition of their flows (see the construction of θ (3.1) in Sect. 3 above). The map θ is defined in an open neighborhood $\Omega \subset U \times \mathbb{C}^q$ of the zero section in $U \times \mathbb{C}^q$. For $x \in U$ write $\Omega_x = \{t \in \mathbb{C}^q : (x,t) \in \Omega\}$. After shrinking Ω we may assume that the fibers Ω_x are connected and $F_x = \{\theta(x,t) : t \in \Omega_x\} \subset X$ is a local complex submanifold of X which intersects the level set $f = f(x)$ transversely at x.

Furthermore, the map $t \in \Omega_x \to f(\theta(x,t)) \in \mathbb{C}^q$ maps Ω_x biholomorphically onto a neighborhood of the point $f(x)$ in $\mathbb{C}^q$. The same holds for the map $t \in \Omega_x \to g(\theta(x,t))$ provided that $g\colon U \to \mathbb{C}^q$ is sufficiently uniformly close to f and we restrict x to a compact subset of U. It follows that, if $V \subset\subset U$ and g is sufficiently close to f on U, there is for every $x \in V$ a unique point $c(x) \in \Omega_x$ such that $g(\theta(x,c(x))) = f(x)$. Clearly $c\colon V \to \mathbb{C}^q$ is holomorphic and the map $\gamma(x) = \theta(x,c(x)) \in X$ $(x \in V)$ satisfies Lemma 4.1. ♠

Remark. The map γ obtained above is already of the form given by the conclusion of Lemma 3.4. However, we keep Lemma 3.4 for independent interest and future applications. ♠

The following notion of a *special Cartan pair* is a small variation of the corresponding notions in [HL2] and [FP1].

Definition. *An ordered pair of compact sets (A,B) in a Stein manifold X is said to be a special Cartan pair in X if*

(i) the sets A, B, $C = A \cap B$, $D = A \cup B$ are holomorphically convex in X,

(ii) $\overline{A\backslash B} \cap \overline{B\backslash A} = \emptyset$, and

(iii) there is a biholomorphic map from an open neighborhood of B in X onto a domain in $\mathbb{C}^n$, $n = \dim X$, such that B is mapped onto a polynomially convex subset of $\mathbb{C}^n$.

The following is the main step in the proof of Theorem 1.1.

4.2 Proposition. *If (A,B) is a special Cartan pair in a Stein manifold X then every noncritical holomorphic function f in a neighborhood of A can be approximated, uniformly on A, by noncritical holomorphic functions $\widetilde f$ defined in a neighborhood of $A \cup B$. If X_0 is a closed complex subvariety of X with $X_0 \cap B = \emptyset$ and $s \in \mathbb{N}$, we can choose $\widetilde f$ such that, in addition to the above, $\widetilde f - f$ vanishes to a given order $r \in \mathbb{N}$ on X_0.*

Proof. By property (iii) of a special Cartan pair there is a biholomorphic map $\psi\colon U \to \psi(U) \subset \mathbb{C}^n$ from an open set $U \supset B$ such that $B' = \psi(B)$ is a polynomially convex subset of $\mathbb{C}^n$. Since C is holomorphically convex in X and hence in U, the set $C' = \psi(C)$ is holomorphically convex in B' and hence is polynomially convex in $\mathbb{C}^n$.

If f is a noncritical holomorphic function in a neighborhood of A then $f' = f \circ \psi^{-1}$ is a noncritical holomorphic function in an open set $\widetilde C \supset C'$ in $\mathbb{C}^n$. Choose a compact polynomially convex set K in $\mathbb{C}^n$ with $C' \subset \mathrm{int} K \subset K \subset \widetilde C$. By Theorem 2.1 we can approximate f' uniformly on K by a noncritical holomorphic function $g' \in \mathcal{O}(\mathbb{C}^n)$. Thus $g = g' \circ \psi$ is a noncritical holomorphic function in a neighborhood of B which approximates f uniformly on a neighborhood of C in X.

By Lemma 4.1 there is a biholomorphic map γ in a neighborhood of C in X such that $f = g \circ \gamma$ and γ is uniformly close to the identity map on C. By Theorem 3.1 we can split $\gamma = \beta \circ \alpha^{-1}$ where α is a biholomorphic map close to the identity in a neighborhood of A and β is a map with the analogous properties in a neighborhood of B. (In the presence of the subvariety X_0 we insure in addition that α and β agree with the identity map to order r along X_0.) From $f = g \circ \gamma = g \circ \beta \circ \alpha^{-1}$ (which holds in a neighborhood of C) we obtain $f \circ \alpha = g \circ \beta$. Thus the two sides define a noncritical holomorphic function $\widetilde{f}$ in a neighborhood of $D = A \cup B$ which approximates f on A and which agrees with f to order r on X_0. ♠

Proof of Theorem 1.1. It suffices to globalize the construction in Proposition 4.2. To this end we recall from [HL2, Sect. 2] that every Stein manifold X can be exhausted by an increasing sequence of compact, holomorphically convex subsets $A_0 \subset A_1 \subset \cdots \subset \cup_{k=0}^{\infty} A_k = X$ such that for every k we have $A_{k+1} = A_k \cup B_k$ where (A_k, B_k) is a special Cartan pair in X (see the definition above; these pairs are realized in [HL2, Corollary 2.8] as *special pseudoconvex bumps*). We can choose the initial set A_0 such that $K \subset A_0 \subset U$ where U is an open set on which the initial noncritical function $f = f_0$ is defined. Applying Proposition 4.2 inductively we obtain for each $k = 0, 1, 2, \ldots$ a noncritical holomorphic function f_k in a neighborhood of A_k such that f_{k+1} approximates f_k as well as desired on A_k. If the approximation is sufficiently close at each step then the sequence f_k converges uniformly on compacts in X to a noncritical holomorphic function $\widetilde{f} = \lim_{k\to\infty} f_k$ on X which approximates f on K.

The requirement that $\widetilde{f}$ interpolate f on X_0 requires the following modification in the proof. Let f be noncritical on an open set $U \supset X_0 \cup K$. Choose a compact holomorphically convex set L in X containing K in its interior. By Lemma 8.4 in [FP1] there is a finite sequence $A_0 \subset A_1 \subset \cdots \subset A_j$ of compact holomorphically convex subsets of X such that for each $k = 0, 1, \ldots, j-1$ we have $A_{k+1} = A_k \cup B_k$, where (A_k, B_k) is a special Cartan pair in X and the following hold:

(i) $K \cup (X_0 \cap L) \subset A_0 \subset\subset U$,

(ii) for $k = 1, \ldots, j$ we have $B_j \cap X_0 = \emptyset$, and

(iii) $A_j = L$.

Note that our present notation differs from [FP1]: the set A_k in [FP1] is the same as B_{k-1} in this paper, while the set A_k in this paper is the same as $\cup_{l=0}^{k} A_l$ in [FP1].

We proceed as before, using Proposition 4.2. After the first k steps (with $k < j$) we have a noncritical function f_k in a neighborhood of A_k. By Proposition 4.2 we can approximate f_k uniformly on A_k by a noncritical function f_{k+1} in a neighborhood of A_{k+1} such that $f_{k+1} - f_k$ vanishes to order r on X_0 (this is possible since $B_k \cap X_0 = \emptyset$). After j steps we obtain a noncritical function f_j in a neighborhood of L which agrees with f to order r on X_0. By

Cartan theory we can approximate f_j uniformly in a neighborhood of L by a holomorphic function $\widetilde{f}_j \in \mathcal{O}(X)$ such that $\widetilde{f}_j - f$ vanishes to order r on X_0. Hence $\widetilde{f}_j$ is noncritical in a neighborhood of $X_0 \cup L$ and we can proceed. We complete the proof as before by exhausting X by a sequence of compact, holomorphically convex subsets and applying induction. ♠

Proof of Theorem 1.1'. We only need to make the following small adjustment to the proof of Theorem 1.1. By assumption we have $\Sigma := \mathrm{Crit}(f;U) \subset K$. Replacing the subvariety $X_0 \subset X$ by $X_0 \cup \Sigma$ we may assume that f is noncritical off X_0. We shall use the same notation as in the proof of Theorem 1.1. As in that proof we construct a sequence $f_k \in \mathcal{O}(A_k)$ such that $\mathrm{Crit}(f_k; A_k) = \Sigma$ and all other properties are satisfied. The main point is that at k-th step of the induction we have $A_{k+1} = A_k \cup B_k$, where the set B_k does not intersect X_0. It follows that f_k is noncritical in a neighborhood of $C_k = A_k \cap B_k$. This allows us to proceed exactly as before: first approximate f_k uniformly in a neighborhood of C_k by a noncritical function g_k defined on a neighborhood of B_k, and then patch f_k and g_k into a function $f_{k+1} \in \mathcal{O}(A_{k+1})$ with $\mathrm{Crit}(f_{k+1}; A_{k+1}) = \Sigma$. When extending f_{k+1} (by approximation) to a neighborhood of $X_0 \cup A_{k+1}$ we can insure that we interpolate f_{k+1} to a sufficiently high order at each point of Σ so that we don't increase its critical set. ♠

Proof of Theorem 1.5. The construction is the same as in the proof of Theorem 1.1 except that we replace Theorem 2.1 by Proposition 2.3. Thus, let $(A, \widetilde{A})$ be a noncritical strongly pseudoconvex extension in a Stein manifold X (see the paragraph preceding Theorem 1.5). As in [HL2] we find a finite sequence of compact, smoothly bounded, strongly pseudoconvex domains $A = A_0 \subset A_1 \subset \ldots \subset A_j = \widetilde{A}$ such that each set A_k is holomorphically convex in X and $A_{k+1} = A_k \cup B_k$, where B_k is a small *convex bump* (small in the sense that it may be contained in an element of a given open covering of X). More precisely, using Narasimhan's lemma to the effect that, in suitable local holomorphic coordinates, any strongly pseudoconvex domain becomes strongly convex, we can choose the sequence of bumps $B_0, \ldots, B_{j-1}$ to satisfy the following. For each $k = 0, \ldots, j-1$ there are holomorphic coordinates $z = (z_1, \ldots, z_n)$ in an open set $U_k \subset\subset X$ containing B_k such that, in these coordinates, U_k is the polydisc $\{|z_l| < 1;\ 1 \le l \le n\}$ and $A_k \cap U_k$ is of the form $\Im z_n \le h(z_1, \ldots, z_{n-1}, \Re z_n)$ for some strongly concave smooth function h with values in $(0,1)$ (hence the above set is geometrically convex).

For every $k = 0, \ldots, j-1$ we choose a slightly smaller polydisc $U'_k \subset\subset U_k$ such that $B_k \subset U'_k$. Observe that the pair of sets $K_k = A_k \cap \overline{U'_k}$, $L_k = \overline{U'_k}$ (considered as subsets of $\mathbb{C}^n$) satisfies the hypothesis of Proposition 2.3. Thus, given a holomorphic submersion $f = (f_1, \ldots, f_q) \colon U \to \mathbb{C}^q$ in a neighborhood of $A = A_0$, Proposition 2.3 gives a submersion from a neighborhood of $\overline{U'_0}$ to $\mathbb{C}^q$ which approximates f uniformly on a neighborhood of K_0, and hence in a neighborhood of $C_0 = A_0 \cap B_0 \subset K_0$. By Lemma 4.1 there is a biholomorphic transition map γ between these submersions which is close to the identity in a neighborhood of C_0. Splitting $\gamma = \beta \circ \alpha^{-1}$ as in Theorem 3.1 and proceeding as in the proof of Theorem 1.1 above we obtain a submersion f_1 from a neighborhood of $A_1 = A_0 \cup B_0 \subset X$ to $\mathbb{C}^q$ which approximates f in a neighborhood of A_0. Continuing inductively we obtain in j steps a submersion from a neighborhood of $\widetilde{A}$ to $\mathbb{C}^q$ which approximates f on A. ♠

&5. Further results and some open problems.

The results discussed in this paper lead to the following

Problem 1. *Let X be a Stein manifold of complex dimension $n > 1$. For which values values of $q \le n$ does there exist a holomorphic submersion $f = (f_1, \ldots, f_q)\colon X \to \mathbb{C}^q$?*

Equivalently, we are looking for $f_1, \ldots, f_q \in \mathcal{O}(X)$ such that $df_1 \wedge \cdots \wedge df_q \neq 0$ on X. Such a q-tuple $df_1, \ldots, df_q$ spans a trivial rank q holomorphic subbundle of the holomorphic cotangent bundle T^*X (by duality the tangent bundle TX then also contains a trivial rank q subbundle). Is this necessary condition also sufficient? Theorem 1.1 gives an affirmative answer for $q = 1$ (and in this case the necessary condition is always satisfied). To formulate the question more precisely we denote by $\mathcal{C}^q_*(X)$ the space of all q-tuples $\theta = (\theta_1, \cdots, \theta_q)$ of $(1,0)$-forms on X with continuous coefficients which are $\mathbb{C}$-linearly independent at every point of X.

Problem 2. *Is it possible to connect any $\theta^0 \in \mathcal{C}^q_*(X)$ by a path $\theta^t \in \mathcal{C}^q_*(X)$ ($t \in [0,1]$) to a q-tuple $\theta^1 = (df_1, \ldots, df_q) \in \mathcal{C}^q_*(X)$, where $f_1, \ldots, f_q \in \mathcal{O}(X)$ are holomorphic functions with independent differentials? In particular, assuming that TX is trivial, does there exist a holomorphic immersion $f\colon X \to \mathbb{C}^n$ with $n = \dim X$? Is every trivialization of TX homotopic to the tangent map of a holomorphic immersion $f\colon X \to \mathbb{C}^n$?*

In Gromov's terminology the question is *whether holomorphic submersions from Stein manifolds to Euclidean spaces satisfy the (basic) h-principle.* Corollary 1.4 gives an affirmative answer for $q = 1$. The critical case $q = \dim X$ has been of special interest ever since the 1967 result of Gunning and Narasimhan [GN]. Our Theorem 1.5 provides an essential step in this direction for $q < n$ (i.e., we can extend submersions across noncritical strongly pseudoconvex extensions). We expect that our methods can be used to give an affirmative answer for $q < n$. However, when $q = n$, the proof of our Proposition 2.3 fails due to a possible Picard-type obstruction.

For submersions of *smooth manifolds* to Euclidean spaces the h-principle was proved by Phillips [P] and Gromov [Gro1, p. 26], but the proof does not apply in the holomorphic case. One of the principle motivations in the development of the h-principle was the *Smale-Hirsch theory of smooth immersions* $X \to \mathbb{R}^q$ for $q > \dim_{\mathbb{R}} X$. For *holomorphic immersions of Stein manifolds*

$X \to \mathbb{C}^q$ with $q > \dim X$ the h-principle has been proved by Eliashberg and Gromov [Gro1, pp. 65-75]. Such immersions can be constructed by stratifying the manifold by a descending finite chain of suitably chosen complex subvarieties and solving an interpolation problem involving the second order jet of the map along each stratum (the so called *elimination of singularities method*). No such reduction seems possible for holomorphic submersions.

Combining the Lefschetz theorem [AF] (to the effect that a Stein manifold of dimension n has the homotopy type of a CW-complex or real dimension at most n) with the Oka-Grauert principle [Gra1, Gra2, HL2, FP1] one sees that *every holomorphic vector bundle $\pi: E \to X$ of rank $m \geq [n/2] + 1$ over an n-dimensional Stein manifold admits a nowhere vanishing holomorphic section*, and hence it admits a trivial holomorphic line subbundle. We can apply this argument inductively to conclude that the cotangent bundle T^*X admits a trivial holomorphic subbundle of rank $q = [(n+1)/2]$. This justifies

Conjecture. *Every Stein manifold of dimension n admits a holomorphic submersion to $\mathbb{C}^q$ with $q = [(n+1)/2]$.*

Here is another related problem whose solution would yield the h-principle for submersions $X \to \mathbb{C}^q$ in all dimensions $q \leq \dim X$. Let $\mathcal{O}^*(X)$ denote the set of all nonvanishing holomorphic functions on X.

Problem 3. *Let L be a nowhere vanishing holomorphic vector field on a Stein manifold X. Does there exist $f \in \mathcal{O}(X)$ such that $Lf \neq 0$ on X? Can we choose f such that Lf is homotopic in $\mathcal{O}^*(X)$ to any given $g \in \mathcal{O}^*(X)$?*

Problem 3 has been stated in [Gro1, p. 70]. Not much seems to be known besides the solution on open Riemann surfaces [GN]. On $\mathbb{C}^n$ the problem has the following equivalent formulation: *Given entire functions $a_1, \ldots, a_n \in \mathcal{O}(\mathbb{C}^n)$ without common zeros, find $f \in \mathcal{O}(\mathbb{C}^n)$ such that*

$$Lf(z) = \sum_{j=1}^{n} a_j(z) \frac{\partial f}{\partial z_j}(z) \neq 0 \qquad (z \in \mathbb{C}^n). \tag{5.1}$$

This problem is open even for vector fields L on $\mathbb{C}^2$ with polynomial coefficients. It is clear that L, considered as an operator on $\mathcal{O}(\mathbb{C}^n)$, does not have any nice properties (such as closed range) since the equation $Lf = g$ in general cannot be solved even on the individual leaves of L (these are open Riemann surfaces). However, all we need is a nonvanishing function in the range of L.

Our efforts to solve this problem have not been successful so far. Using Theorem 3.1 one can reduce Problem 3 to a local approximation problem for functions satisfying $Lf \neq 0$. We can solve this problem at *non-characteristic boundary points* of a given domain $A \subset X$ on which f is defined (these are points where L is transverse to the boundary of A), but the characteristic points present a serious problem.

5.1 Proposition. *If Problem 3 is solvable on all complex Euclidean spaces then it is solvable on all Stein manifolds.*

Proof. Let L be a nonvanishing holomorphic vector field on a Stein manifold X. We embed X as a closed complex submanifold of some $\mathbb{C}^N$ and extend L to a nonvanishing holomorphic vector field on an open set $U \supset X$ in $\mathbb{C}^N$. By [Ha] there is a closed neighborhood $A \subset U$ of X in $\mathbb{C}^N$ such that the pair $(A, \mathbb{C}^N)$ is homotopically equivalent to a relative CW-complex of real dimension at most N. Hence L extends from A to a nonvanishing continuos vector field of type $(1,0)$ on $\mathbb{C}^N$. By the Oka-Grauert principle [Gra1, Gra2, HL2] the extension can be homotopically deformed to a nonvanishing entire vector field $\widetilde{L}$ on $\mathbb{C}^N$ by a homotopy that is fixed on X. If $\widetilde{f} \in \mathcal{O}(\mathbb{C}^N)$ satisfies $\widetilde{L}\widetilde{f} \neq 0$ on $\mathbb{C}^N$ then $f = \widetilde{f}|_X$ satisfies $Lf \neq 0$ on X. ♠

The relevance of Problem 3 is shown by

5.2 Proposition. *If Problem 3 is solvable for all nonvanishing holomorphic vector fields on a given Stein manifold X then for every trivial rank q holomorphic vector subbundle E of the tangent bundle TX then there exists a holomorphic submersion $f\colon X \to \mathbb{C}^q$ such that $d_x f$ maps E_x isomorphically onto $T_{f(x)}\mathbb{C}^q \simeq \mathbb{C}^q$ for every $x \in X$. In particular, if TX trivial then (under the stated assumption) there exists a holomorphic immersion $X \to \mathbb{C}^n$ with $n = \dim X$.*

Proof. Induction on $q = \mathrm{rank} E$. When $q = 1$, E is spanned by a single holomorphic vector field $L \neq 0$ and any $f \in \mathcal{O}(X)$ with $Lf \neq 0$ satisfies the conclusion. Assume that the result holds for subbundles of rank $< q$ and let $E \subset TX$ be a trivial subbundle of rank q. Choose a nowhere vanishing section $L_1\colon X \to E$ such that the quotient bundle $E/(L_1)$ is trivial, where (L_1) denotes the line bundle spanned by L_1. Choose $f_1 \in \mathcal{O}(X)$ such that $L_1 f_1 \neq 0$ on X and let $E' = E \cap \ker df_1$. Then $E' \simeq E/(L_1)$ is a trivial bundle of rank $q-1$. By the induction hypothesis there is a submersion $f' = (f_2, \ldots, f_q)\colon X \to \mathbb{C}^{q-1}$ such that df' maps E' isomorphically onto $X \times \mathbb{C}^{q-1}$. It is immediate that $f = (f_1, f_2, \ldots, f_q)\colon X \to \mathbb{C}^q$ satisfies the conclusion of Proposition 5.2. ♠

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