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APPLICATIONS OF HERMITE TRANSFORM IN COMPUTER ALGEBRA

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Applications of Hermite transform in computer algebra

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Abstract

Let L be a linear differential operator with polynomial coefficients. We show that there is an isomorphism of differential operators $\mathcal{D}_\alpha$ and an integral transform $\mathcal{H}_\alpha$ (called the Hermite transform) on functions for which $(\mathcal{D}_\alpha L)f(x) = 0$ implies $L\mathcal{H}_\alpha(f)(x) = 0$. We present an algorithm that computes the Hermite transform of a rational function and use it to find $n+1$ linearly independent solutions of $Ly = 0$ when $(\mathcal{D}_\alpha L)y = 0$ has a rational solution with n distinct finite poles.

Key words: linear differential equation, hypergeometric polynomials, Hermite transform, rational functions

1 Introduction

One of the well-known methods for solving linear differential equations is Frobenius' method of power series. In symbolic computation, other types of polynomial series have also been considered [4,10]. Rebillard [10] in his thesis considers Fourier series of families of hypergeometric polynomials and defines a transformation $\mathcal{R}_\mathcal{P}$ which for a given hypergeometric polynomial family $\mathcal{P} = \{P_n\}$ to every linear differential operator L assigns a linear recurrence operator $\mathcal{L}$ such that for a formal series $y(x) = \sum_{n=0}^{\infty} c_n P_n(x)$ satisfying $Ly = 0$ its coefficient sequence $c = (c_n)$ satisfies $\mathcal{L}c = 0$. Therefore the problem of finding formal series solutions of a differential equation reduces to the problem of solving a recurrence relation for the coefficient sequence. In Section 2 we recall some facts on hypergeometric polynomials and describe the transformation $\mathcal{R}_\mathcal{P}$.

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In the rest of the paper we restrict ourselves to families of Hermite-related polynomials $H^{(\alpha)} = \{H_n^{(\alpha)}(x)\}$ (with $\alpha = 2$ giving the classical Hermite polynomials $H_n(x)$). It turns out that in this case the transformation $\mathcal{R}_{H^{(\alpha)}}$ is a monomorphism, which is not the case when a general hypergeometric family is involved. The same is also true of the analogous transformation $\mathcal{R}_P$ corresponding to the power family $P = \{x^n\}$. Since the ranges of both transformations, $\mathcal{R}_{H^{(\alpha)}}$ and $\mathcal{R}_P$, are the same we obtain an automorphism $\mathcal{D}_\alpha = \mathcal{R}_P^{-1} \circ \mathcal{R}_{H^{(\alpha)}}$ of the Weyl algebra $K[x, D]$ with the following property: $\sum_{n=0}^{\infty} c_n H_n^{(\alpha)}(x)$ belongs to Ker$L$ iff $\sum_{n=0}^{\infty} c_n x^n$ belongs to Ker$\mathcal{D}_\alpha L$. The transformation $\mathcal{D}_\alpha$ is defined in Section 3 (see also [11]). In Section 4 we define a 1-parametric family of integral transforms $\mathcal{H}_\alpha$ (called the Hermite transform) on functions for which $(\mathcal{D}_\alpha L)f(x) = 0$ implies $L\mathcal{H}_\alpha(f)(x) = 0$. Thus any solution of $(\mathcal{D}_\alpha L)y = 0$ gives rise to a solution of $Ly = 0$ provided that $\mathcal{H}_\alpha(f)(x)$ exists. In Section 5 we describe an algorithm that computes the Hermite transform of a rational function. We use the algorithm in Section 6 to construct $n + 1$ linearly independent solutions of $Ly = 0$ when the corresponding equation $(\mathcal{D}_\alpha L)y = 0$ has a rational solution with n distinct poles. We also specify a polynomial equation to determine the values of α for which the transformed equation can have a rational solution.

Throughout the paper, K denotes an arbitrary field of characteristic zero and $\mathbb{N}$ the set of nonnegative integers. We use E to denote the shift operator acting on sequences over K. Similarly, we denote by $K[n, E, E^{-1}]$ the algebra of recurrence operators and by $K[x, D]$ the Weyl algebra of linear differential operators with polynomial coefficients where D denotes the usual differential operator $Dp(x) = \frac{d}{dx}p(x)$. If we write an element $L \in K[x, D]$ in the monomial form

$$L = \sum_{i,k} a_{ik} x^i D^k,$$

then $\max_{a_{ik} \neq 0} k$ is the *order of* L and $\max_{a_{ik} \neq 0}(i + k)$ is the *total degree of* L. By L_a we denote

$$L_a = \sum_{i,k} a_{ik} (x + a)^i D^k.$$

Note that $L_a f(x) = 0$ iff $Lf(x - a) = 0$.

2 Hypergeometric polynomials

We introduce here hypergeometric polynomials and describe a procedure due to Rebillard [10] which to a linear differential operator L of order r assigns a linear recurrence operator $\mathcal{L}$ such that $L(\sum_{n=0}^{\infty} c_n P_n(x)) = \sum_{n=0}^{\infty} (\mathcal{L}c)_n P_n^{(r)}(x)$. For more details on hypergeometric polynomials see [8,10].

A polynomial family $\mathcal{P} = \{P_n(x)\}_{n=0}^{\infty}$ is called *hypergeometric* if each P_n is of degree exactly n and satisfies the hypergeometric differential equation

$$\sigma(x)y'' + \tau(x)y' + \lambda_n y = 0 \tag{1}$$

where σ and τ are polynomials of degree at most two, resp. one, and $\lambda_n = -n\left(\frac{n-1}{2}\sigma''(x) + \tau'(x)\right)$. With σ and τ the family is determined up to a multiplicative constant. To get a complete family – this means that for every $n \in \mathbb{N}$ the equation (1) has a polynomial solution of degree n – the function $n \mapsto \lambda_n$ has to be injective. Thus τ has to be of degree one, and if $\sigma'' \neq 0$ then $2\tau'/\sigma'' \notin \{0, -1, -2, \ldots\}$.

For every hypergeometric family there exist recurrence operators $\mathbf{X}, \mathbf{B}$ and $\mathbf{S}$ in $K[n, E, E^{-1}]$ such that for each P_n we have

$$
\begin{aligned}
xP_n(x) = \mathbf{X}P_n(x) &= x_1(n)P_{n+1}(x) + x_0(n)P_n(x) + x_{-1}(n)P_{n-1}(x), && (2)\\
P_n(x) = \mathbf{B}P_n'(x) &= \beta_1(n)P_{n+1}'(x) + \beta_0(n)P_n'(x) + \beta_{-1}(n)P_{n-1}'(x), && (3)\\
\sigma(x)P_n'(x) = \mathbf{S}P_n(x) &= s_1(n)P_{n+1}(x) + s_0(n)P_n(x) + s_{-1}(n)P_{n-1}(x), && (4)
\end{aligned}
$$

where $x_i, \beta_i, s_i \in K(n)$ and depend only on σ and τ [7,8,10]. If σ is of degree zero (with no loss of generality we may assume that $\sigma(x) = 1$) then

$$P_n'(x) = \mathbf{S}P_n(x) = s_{-1}(n)P_{n-1}(x),$$

hence k-th derivative $P_n^{(k)}$ can be expressed by P_{n-k}.

Let us recall that for a recurrence operator $\mathbf{F} \in K[n, E, E^{-1}]$

$$\mathbf{F} = \sum_{k=r}^{s} f_k(n)E^k, \quad r, s \in \mathbb{Z},$$

its *adjoint operator* $\mathcal{F}$ is defined by [10,3]

$$\mathcal{F} = \sum_{k=-s}^{-r} f_{-k}(n+k)E^k.$$

If $(a_n)_{n=-\infty}^{\infty}$ and $(b_n)_{n=-\infty}^{\infty}$ are two sequences of functions, then

$$\sum_{n=-\infty}^{\infty} a_n \mathbf{F}(b_n) = \sum_{n=-\infty}^{\infty} \mathcal{F}(a_n) b_n,$$

where the two sums are treated as formal series of functions.

Consider a linear differential operator of order r

$$L = \sum_{k=0}^{r} p_k(x)D^k$$

with polynomial coefficients. In order to apply L to a formal series of hypergeometric polynomials $\sum_{n=0}^{\infty} c_n P_n(x)$ we need to know how L acts on P_n. By (2) and (3) we obtain a recurrence operator $\mathbf{L}$ such that $LP_n = \mathbf{L}P_n^{(r)}$ and

$$L\left(\sum_{n=0}^{\infty} c_n P_n(x)\right) = \sum_{n=0}^{\infty} (\mathcal{L}c)_n P_n^{(r)}(x),$$

where $\mathcal{L}$ is the adjoint operator of $\mathbf{L}$. In this way every hypergeometric family $\mathcal{P} = \{P_n(x)\}_{n=0}^{\infty}$ gives rise to the transformation $\mathcal{R}_{\mathcal{P}} : K[x, D] \to K[n, E, E^{-1}]$ having the following property: if a formal series $\sum c_n P_n(x)$ satisfies $Ly = 0$ then its coefficient sequence $c = (c_n)$ satisfies $\mathcal{R}_{\mathcal{P}}(L)(c) = 0$.

When $\sigma(x) = 1$, as in the case of Hermite polynomials (see below), we use (2) and (4) to construct a recurrence operator $\mathbf{L} = \sum_{k=0}^{r} \mathbf{S}^k p_k(\mathbf{X})$ so that $LP_n = \mathbf{L}P_n$. The transformation $\mathcal{R}_{\mathcal{P}}$ is then defined by

$$\mathcal{R}_{\mathcal{P}} : L = \sum_{k=0}^{r} p_k(x)D^k \mapsto \mathcal{L} = \sum_{k=0}^{r} p_k(\mathcal{X})\mathcal{S}^k.$$

Note that $L(\sum c_n P_n) = 0$ iff $\mathcal{R}_{\mathcal{P}}(L)(c) = 0$. Therefore if we use algorithms of [2], [1], or [9], resp., to find polynomial, rational, or hypergeometric solutions, resp., of $\mathcal{R}_{\mathcal{P}}(L)(c) = 0$ we can find all Hermite series solutions of $Ly = 0$ with polynomial, rational, or hypergeometric coefficients, resp.

3 The isomorphism $\mathcal{D}_\alpha$

We introduce in this section the Hermite-related polynomials $H^{(\alpha)}$ and show that the transformation $\mathcal{R}_{H^{(\alpha)}}$ is a monomorphism from $K[x, D]$ to $K[n, E, E^{-1}]$. The same is also true for the transformation $\mathcal{R}_P$ corresponding to the power family $\{x^n\}$. Since the ranges of both transformations are the same, we define an automorphism $\mathcal{D}_\alpha = \mathcal{R}_P^{-1} \circ \mathcal{R}_{H^{(\alpha)}}$ of $K[x, D]$ with the following property: $\sum_{n=0}^{\infty} c_n H_n^{(\alpha)}(x)$ belongs to Ker$L$ iff $\sum_{n=0}^{\infty} c_n x^n$ belongs to Ker$\mathcal{D}_\alpha L$.

Let us focus on the hypergeometric differential equation

$$y''(x) - \alpha x y'(x) + \alpha n y(x) = 0 \qquad (5)$$

depending on a complex parameter $\alpha \neq 0$. For $\alpha = 2$ this differential equation is satisfied by the classical Hermite polynomial $H_n(x)$. As $\sigma(x) = 1$ and $\tau(x) = -\alpha x$, Eq. (5) has, by Rodrigues' type formula [8], a polynomial solution for every $n \in \mathbb{N}$

$$H_n^{(\alpha)}(x) = \frac{B_n}{\rho} \frac{d^n}{dx^n} \rho(x),$$

where B_n is a normalization coefficient and ρ is a solution of the Pearson differential equation

$$(\sigma\rho)' = \tau\rho,$$

therefore $\rho(x) = \mathrm{e}^{-\alpha x^2/2}$.

Definition 1 *Let* $\alpha \in \mathbb{C} - \{0\}$*. The family of* Hermite-related polynomials $H^{(\alpha)} = \{H_n^{(\alpha)}; n \in \mathbb{N}\}$ *is defined by*

$$H_n^{(\alpha)}(x) = (-1)^n \, \mathrm{e}^{\frac{\alpha}{2}x^2} \, \frac{d^n}{dx^n} \, \mathrm{e}^{-\frac{\alpha}{2}x^2} \, .$$

Applying the formulas from [7,10] to the family $H^{(\alpha)}$ we obtain the corresponding operators $\mathbf{X}_\alpha$ and $\mathbf{S}_\alpha$

$$\mathbf{X}_\alpha = \frac{1}{\alpha} E + nE^{-1}, \qquad \mathbf{S}_\alpha = \alpha n E^{-1},$$

hence the transformation $\mathcal{R}_{H^{(\alpha)}} : K[x, D] \to K[n, E, E^{-1}]$ is defined by

$$\mathcal{R}_{H^{(\alpha)}} : \sum_{k=0}^{r} p_k(x) D^k \longmapsto \sum_{k=0}^{r} p_k(\mathcal{X}_\alpha) \mathcal{S}_\alpha^k,$$

where

$$\mathcal{X}_\alpha = (n+1)E + \frac{1}{\alpha} E^{-1}, \qquad \mathcal{S}_\alpha = \alpha(n+1)E$$

are the adjoints of $\mathbf{X}_\alpha$ and $\mathbf{S}_\alpha$, resp.

Proposition 2 *Let* $\mathcal{P}$ *be a hypergeometric polynomial family with* $\sigma(x) = 1$ *and let* $\mathcal{X}$ *and* $\mathcal{S}$ *be the adjoints of* $\mathbf{X}$ *and* $\mathbf{S}$*, resp., defined in (2) and (4). If* $\mathcal{X}$ *and* $\mathcal{S}$ *satisfy the commutation rule*

$$\mathcal{S}\mathcal{X} = \mathcal{X}\mathcal{S} + 1, \tag{6}$$

then the transformation $\mathcal{R}_{\mathcal{P}} : K[x, D] \to K[n, E, E^{-1}]$

$$\mathcal{R}_{\mathcal{P}} : \sum_{k=0}^{r} p_k(x)D^k \longmapsto \sum_{k=0}^{r} p_k(\mathcal{X})\mathcal{S}^k$$

is a homomorphism of K*-algebras.*

PROOF. Let $L_1 = \sum_{k=0}^{r_1} p_k(x)D^k$ and $L_2 = \sum_{j=0}^{r_2} q_j(x)D^j$. Clearly $\mathcal{R}_{\mathcal{P}}(\lambda_1 L_1 + \lambda_2 L_2) = \lambda_1 \mathcal{R}_{\mathcal{P}}(L_1) + \lambda_2 \mathcal{R}_{\mathcal{P}}(L_2)$. Using Leibniz' rule $D^j p(x) = \sum_{i=0}^{j} \binom{j}{i} p^{(i)}(x) D^{j-i}$ it follows that

$$\begin{aligned}
\mathcal{R}_{\mathcal{P}}(L_2 L_1) &= \mathcal{R}_{\mathcal{P}}\left(\sum_{j=0}^{r_2}\sum_{k=0}^{r_1} q_j(x)D^j p_k(x)D^k\right) \\
&= \mathcal{R}_{\mathcal{P}}\left(\sum_{j=0}^{r_2}\sum_{k=0}^{r_1}\sum_{i=0}^{j} \binom{j}{i} q_j(x)p_k^{(i)}(x)D^{j-i+k}\right) \\
&= \sum_{j=0}^{r_2}\sum_{k=0}^{r_1}\sum_{i=0}^{j} \binom{j}{i} q_j(\mathcal{X})p_k^{(i)}(\mathcal{X})\mathcal{S}^{j-i+k}.
\end{aligned}$$

Since operators $\mathcal{X}$ and $\mathcal{S}$ obey the same commutation rule as x and D, we can replace in Leibniz' rule x and D by $\mathcal{X}$ and $\mathcal{S}$, resp., to obtain

$$\begin{aligned}
\mathcal{R}_{\mathcal{P}}(L_2)\mathcal{R}_{\mathcal{P}}(L_1) &= \sum_{j=0}^{r_2}\sum_{k=0}^{r_1} q_j(\mathcal{X})\mathcal{S}^j p_k(\mathcal{X})\mathcal{S}^k \\
&= \sum_{j=0}^{r_2}\sum_{k=0}^{r_1}\sum_{i=0}^{j} \binom{j}{i} q_j(\mathcal{X})p_k^{(i)}(\mathcal{X})\mathcal{S}^{j-i+k},
\end{aligned}$$

hence $\mathcal{R}_{\mathcal{P}}(L_2 L_1) = \mathcal{R}_{\mathcal{P}}(L_2)\mathcal{R}_{\mathcal{P}}(L_1)$. □

Let $P = \{x^n\}$ be the power family. Since $xx^n = \mathbf{X}_P x^n = E x^n$ and $Dx^n = \mathbf{S}_P x^n = nE^{-1}x^n$, the adjoint operators of $\mathbf{X}_P$ and $\mathbf{S}_P$ are $\mathcal{X}_P = E^{-1}$ and $\mathcal{S}_P = (n+1)E$. Therefore both families, $H^{(\alpha)}$ and P, satisfy the commutation rule (6), hence $\mathcal{R}_{H^{(\alpha)}}$ and $\mathcal{R}_P$ are homomorphisms.

Furthermore, both transformations are one-to-one. Since $\mathcal{R}_P$ is linear it suffices to show that $\mathrm{Ker}\mathcal{R}_P = \{0\}$. Suppose that $\mathcal{R}_P(L) = 0$ for some nonzero $L = \sum_{i,k} a_{ik}x^i D^k$. Writing

$$L = \sum_i a_{i,l+i} x^i D^{l+i} + \sum_{k-i>l} a_{ik} x^i D^k,$$

where $l = \min_{a_{ik}\neq 0}\{k-i\}$ (so at least one $a_{i,l+i} \neq 0$), the coefficient of E^l in $\mathcal{R}_P(L)$ equals to

$$\sum_i a_{i,l+i}(n-i+1)(n-i+2)\cdots(n+l) \neq 0$$

which is in contradiction with $\mathcal{R}_P(L) = 0$.
The same argument is applied to show that $\mathcal{R}_{H(\alpha)}$ is one-to-one.

We next observe that

$$\begin{aligned} \mathcal{X}_\alpha &= \tfrac{1}{\alpha}\mathcal{X}_P + \mathcal{S}_P, & \mathcal{X}_P &= \alpha\mathcal{X}_\alpha - \mathcal{S}_\alpha, \\ \mathcal{S}_\alpha &= \alpha\mathcal{S}_P, & \mathcal{S}_P &= \tfrac{1}{\alpha}\mathcal{S}_\alpha, \end{aligned}$$

meaning that the ranges of $\mathcal{R}_{H(\alpha)}$ and $\mathcal{R}_P$ coincide, so we can define an isomorphism $\mathcal{D}_\alpha = \mathcal{R}_P^{-1} \circ \mathcal{R}_{H(\alpha)} : K[x,D] \to K[x,D]$ such that $\mathcal{D}_\alpha L$ applied to a formal power series $\sum c_n x^n$ induces the same recurrence operator as L acting on a formal Hermite series $\sum c_n H_n^{(\alpha)}(x)$, therefore

$$(\mathcal{D}_\alpha L)\left(\sum c_n x^n\right) = 0 \iff L\left(\sum c_n H_n^{(\alpha)}(x)\right) = 0.$$

To describe $\mathcal{D}_\alpha$ on $K[x,D]$, it suffices to give it on the two generators x and D

$$\begin{aligned} \mathcal{D}_\alpha : \quad x &\longmapsto \frac{1}{\alpha}x + D, \\ D &\longmapsto \alpha D. \end{aligned}$$

The inverse is given by

$$\begin{aligned} \mathcal{D}_\alpha^{-1} : \quad x &\longmapsto \alpha x - D, \\ D &\longmapsto \frac{1}{\alpha}D. \end{aligned}$$

4 The Hermite transform $\mathcal{H}_\alpha$

We define in this section a 1-parametric family of linear integral transforms $\mathcal{H}_\alpha$ on functions and show that $L\mathcal{H}_\alpha(f)(x) = 0$ as soon as $(\mathcal{D}_\alpha L)f(x) = 0$.

Let for the moment $\alpha = 2$. Then $H_n^{(2)}(x) = H_n(x)$ is the classical Hermite polynomial and we have an integral representation [5, p. 278]

$$H_n(x) = \frac{(-2\mathrm{i})^n}{\sqrt{\pi}} \,\mathrm{e}^{x^2} \int\limits_{-\infty}^{\infty} t^n \,\mathrm{e}^{-t^2} \,\mathrm{e}^{2\mathrm{i}xt} \,dt.$$

We can read this as $H_n(x) = \mathcal{H}_2(t^n)(x)$, where

$$\mathcal{H}_2(t^n)(x) = \frac{1}{\sqrt{\pi}} \int\limits_{-\infty}^{\infty} (-2\mathrm{i}t)^n \,\mathrm{e}^{-(t-\mathrm{i}x)^2} \,dt.$$

In the previous section we introduced an automorphism $\mathcal{D}_\alpha$ of $K[x, D]$ having the following property

$$L\left(\sum c_n H_n^{(\alpha)}\right) = 0 \iff (\mathcal{D}_\alpha L)\left(\sum c_n x^n\right) = 0,$$

hence

$$L\mathcal{H}_2(\sum c_n t^n)(x) = 0 \iff (\mathcal{D}_2 L)(\sum c_n x^n) = 0.$$

Since $H_n^{(\alpha)}(x) = (-1)^n \,\mathrm{e}^{\frac{\alpha}{2}x^2} \frac{d^n}{dx^n} \,\mathrm{e}^{-\frac{\alpha}{2}x^2}$ we have for arbitrary α

$$H_n^{(\alpha)}(x) = \sqrt{\frac{\alpha}{2}}^{\,n} H_n(\sqrt{\frac{\alpha}{2}}x)$$

which for $\alpha > 0$ (to ensure the convergence of the integral) yields the integral representation

$$H_n^{(\alpha)}(x) = \sqrt{\frac{\alpha}{2\pi}} \int\limits_{-\infty}^{\infty} (-\mathrm{i}\alpha t)^n \,\mathrm{e}^{-\frac{\alpha}{2}(t-\mathrm{i}x)^2} \,dt. \tag{7}$$

Definition 3 *Let f be a complex function and $\alpha > 0$. If there is a (finite or infinite) interval (a, b) such that the integral*

$$\mathcal{H}_\alpha(f)(x) = \sqrt{\frac{\alpha}{2\pi}} \int\limits_{-\infty}^{\infty} f(-\mathrm{i}\alpha t) \,\mathrm{e}^{-\frac{\alpha}{2}(t-\mathrm{i}x)^2} \,dt \tag{8}$$

is convergent for all $x \in (a, b)$, then we will call the function $\mathcal{H}_\alpha(f)(x)$ the Hermite transform *of f with parameter α.*

It is easy to see that $\mathcal{H}_\alpha$ is linear. Evidently, not every function has a Hermite transform, so we will be concerned with questions of existence. For instance, if $f(t) = \mathrm{e}^{t^4}$, it is obvious that the integral

$$\int_{-\infty}^{\infty} \mathrm{e}^{\alpha^4 t^4} \, \mathrm{e}^{-\frac{\alpha}{2}(t-\mathrm{i}x)^2} \, dt$$

does not exist for any $\alpha > 0$ and $x \in \mathbb{R}$. On the other hand, the Hermite transform of e^{t^2} exists for all positive α and it is defined on $\mathbb{R}$.

Definition 4 *Let $\alpha > 0$. A complex function f satisfies the* $\mathcal{H}_\alpha$-condition *if there exist a real number $k < \alpha/2$, a nonnegative number N and a polynomial p such that*

(1) $|f(-\mathrm{i}\alpha t)| \leq p(t)\,\mathrm{e}^{kt^2}$ for $|t| \geq N$,
(2) $f(-\mathrm{i}\alpha t)$ is absolutely integrable on $(-N, N)$.

Let f satisfy the $\mathcal{H}_\alpha$-condition. We write (8) in the form

$$\mathrm{e}^{\frac{\alpha}{2}x^2} \int_{-\infty}^{\infty} f(-\mathrm{i}\alpha t)\, \mathrm{e}^{-\frac{\alpha}{2}t^2} \, \mathrm{e}^{\mathrm{i}\alpha x t} \, dt \tag{9}$$

and split the integral in three parts

$$\int_{-\infty}^{\infty} f(-\mathrm{i}\alpha t)\, \mathrm{e}^{-\frac{\alpha}{2}t^2} \, \mathrm{e}^{\mathrm{i}\alpha x t} \, dt = I_1 + I_2 + I_3,$$

where

$$I_1 = \int_{-N}^{N} f(-\mathrm{i}\alpha t)\, \mathrm{e}^{-\frac{\alpha}{2}t^2} \, \mathrm{e}^{\mathrm{i}\alpha x t} \, dt,$$
$$I_2 = \int_{-\infty}^{-N} f(-\mathrm{i}\alpha t)\, \mathrm{e}^{-\frac{\alpha}{2}t^2} \, \mathrm{e}^{\mathrm{i}\alpha x t} \, dt,$$
$$I_3 = \int_{N}^{\infty} f(-\mathrm{i}\alpha t)\, \mathrm{e}^{-\frac{\alpha}{2}t^2} \, \mathrm{e}^{\mathrm{i}\alpha x t} \, dt.$$

Then I_1, I_2 and I_3 are bounded, hence the integral (8) is absolutely convergent on the real axis and uniformly convergent on any finite interval.

Theorem 5 *Let $\alpha > 0$ and let f satisfy the $\mathcal{H}_\alpha$-condition. Then the Hermite transform of f with parameter α is defined for every $x \in \mathbb{R}$. The integral (8) converges uniformly on any finite interval and*

$$\frac{d^k}{dx^k}\int_{-\infty}^{\infty} f(-\mathrm{i}\alpha t)\,\mathrm{e}^{-\frac{\alpha}{2}(t-\mathrm{i}x)^2}\,dt = \int_{-\infty}^{\infty} f(-\mathrm{i}\alpha t)\frac{\partial^k}{\partial x^k}\,\mathrm{e}^{-\frac{\alpha}{2}(t-\mathrm{i}x)^2}\,dt.$$

PROOF. We have to justify that

$$\int_{-\infty}^{\infty} f(-\mathrm{i}\alpha t)\,\mathrm{e}^{-\frac{\alpha}{2}t^2}\,\mathrm{e}^{\mathrm{i}\alpha x t}\,dt$$

can be repeatedly differentiated under the integral sign. Since k-th derivative of $\mathrm{e}^{\mathrm{i}\alpha x t}$ with respect to x is $(\mathrm{i}\alpha t)^k\,\mathrm{e}^{\mathrm{i}\alpha x t}$ and observing that $t^k f$ also satisfies the $\mathcal{H}_\alpha$-condition if f does, the integral

$$\int_{-\infty}^{\infty} f(-\mathrm{i}\alpha t)\,\mathrm{e}^{-\frac{\alpha}{2}t^2}\,\frac{\partial^k}{\partial x^k}\,\mathrm{e}^{\mathrm{i}\alpha x t}\,dt$$

converges uniformly on $\mathbb{R}$, which concludes the proof. $\square$

Proposition 6 *Let $\alpha > 0$ and let f satisfy the $\mathcal{H}_\alpha$-condition. Then*

(1) $\mathcal{H}_\alpha(f')(x) = \frac{1}{\alpha}D\mathcal{H}_\alpha(f)(x)$
(2) $\mathcal{H}_\alpha(tf)(x) = (\alpha x - D)\mathcal{H}_\alpha(f)(x)$

PROOF. Using integration by parts and Theorem 5, we obtain

$$\begin{aligned}\mathcal{H}_\alpha(f')(x) &= \mathrm{i}\sqrt{\frac{\alpha}{2\pi}}\int_{-\infty}^{\infty}(t-\mathrm{i}x)f(-\mathrm{i}\alpha t)\,\mathrm{e}^{-\frac{\alpha}{2}(t-\mathrm{i}x)^2}\,dt\\ &= \sqrt{\frac{\alpha}{2\pi}}\frac{1}{\alpha}\int_{-\infty}^{\infty} f(-\mathrm{i}\alpha t)\frac{\partial}{\partial x}\,\mathrm{e}^{-\frac{\alpha}{2}(t-\mathrm{i}x)^2}\,dt = \frac{1}{\alpha}D\mathcal{H}_\alpha(f)(x),\end{aligned}$$

which proves the first statement. For the second statement, we compute

$$\begin{aligned}\mathcal{H}_\alpha(tf)(x) &= \sqrt{\frac{\alpha}{2\pi}}\int_{-\infty}^{\infty}(\alpha x - \alpha x - \mathrm{i}\alpha t)f(-\mathrm{i}\alpha t)\,\mathrm{e}^{-\frac{\alpha}{2}(t-\mathrm{i}x)^2}\,dt\\ &= \alpha x\mathcal{H}_\alpha(f)(x) - D\mathcal{H}_\alpha(f)(x).\end{aligned}$$

□

Theorem 7 *Let L be a linear differential operator with polynomial coefficients, let $\alpha > 0$ and let f satisfy the $\mathcal{H}_\alpha$-condition. If $(\mathcal{D}_\alpha L)f(x) = 0$ then $L\mathcal{H}_\alpha(f)(x) = 0$.*

PROOF. Since $\mathcal{D}_\alpha$ is an isomorphism on $K[x, D]$, we can write $\mathcal{D}_\alpha L = \sum_{i,k} a_{ik} x^i D^k$, hence $L = \sum_{i,k} a_{ik}(\alpha x - D)^i \alpha^{-k} D^k$. Then using Proposition 6 and linearity of $\mathcal{H}_\alpha$, we have

$$\begin{aligned} L\mathcal{H}_\alpha(f)(x) &= \sum_{i,k} a_{ik}(\alpha x - D)^i \alpha^{-k} D^k \mathcal{H}_\alpha(f)(x) \\ &= \sum_{i,k} a_{ik}(\alpha x - D)^i \mathcal{H}_\alpha(f^{(k)})(x) \\ &= \sum_{i,k} a_{ik} \mathcal{H}_\alpha(t^i f^{(k)})(x) \\ &= \mathcal{H}_\alpha\left(\sum_{i,k} a_{ik} t^i f^{(k)}\right)(x) = 0. \end{aligned}$$

□

Example 8 *Let $L = D^2 - 4xD + 3x^2 + 2n - 1$ and $n \in \mathbb{N}$. Then*

$$\mathcal{D}_\alpha L = (\alpha^2 - 4\alpha + 3)D^2 + (-4x + 6x/\alpha)D + 3x^2/\alpha^2 + 3/\alpha + 2n - 1.$$

Choose $\alpha = 3$ to get a first order differential equation $(\mathcal{D}_3 L)y = 0$ with a solution $f(x) = x^n \,\mathrm{e}^{x^2/12}$, hence $\mathcal{H}_3(f)(x) = \sqrt{\frac{3}{2\pi}}\,\mathrm{e}^{x^2/2} H_n(x)$ satisfies $Ly = 0$. On the other hand, for $\alpha = 1$ a solution of $(\mathcal{D}_1 L)y = 0$ is $g(x) = x^{-n-1}\,\mathrm{e}^{-3x^2/4}$ which does not satisfy the $\mathcal{H}_1$-condition.

Therefore each solution f of $\mathcal{D}_\alpha Ly = 0$ that satisfies the $\mathcal{H}_\alpha$-condition gives us a solution of $Ly = 0$. However it is possible that $f(t)$ does not fulfil the $\mathcal{H}_\alpha$-condition while $f(t + \alpha d)$ does for some d. An example: $f(t) = t^{-1}$ and $d = 1$. In this case we can still obtain a solution of $Ly = 0$ knowing a solution f of $(\mathcal{D}_\alpha L)y = 0$.

Corollary 9 *Let $d \in \mathbb{C}$ be such that $f(t + \alpha d)$ satisfies the $\mathcal{H}_\alpha$-condition. If $(\mathcal{D}_\alpha L)f(x) = 0$ then $L\mathcal{H}_\alpha(f(t + \alpha d))(x - d) = 0$.*

PROOF. Note that $(\mathcal{D}_\alpha L)f(x) = 0$ implies $(\mathcal{D}_\alpha L)_c f(x + c) = 0$ for every $c \in \mathbb{C}$. Since $(\mathcal{D}_\alpha L)_c = \mathcal{D}_\alpha L_{c/\alpha}$ if follows that $(\mathcal{D}_\alpha L_{c/\alpha})f(x + c) = 0$ for every

c. Taking $c = \alpha d$ and using Theorem 7 we obtain $L_d \mathcal{H}_\alpha(f(t+\alpha d))(x) = 0$ and finally $L\mathcal{H}_\alpha(f(t+\alpha d))(x-d) = 0$. □

5 Computing the Hermite transform of rational functions

We show in this section how the Hermite transform of a rational function can easily be obtained from its partial fraction decomposition.

Proposition 10 *For $\alpha > 0$ and $k \in \mathbb{N}$ we have*

(1) $\mathcal{H}_\alpha\left(\sum_{k=0}^d a_k t^k\right)(x) = \sum_{k=0}^d a_k H_k^{(\alpha)}(x)$,
(2) $\mathcal{H}_\alpha((t+\alpha d)^k)(x-d) = \mathcal{H}_\alpha(t^k)(x)$ *for every* $d \in \mathbb{R}$.

PROOF. By (7) and (8), $\mathcal{H}_\alpha(t^n)(x) = H_n^{(\alpha)}(x)$. Therefore the first assertion follows by linearity of $\mathcal{H}_\alpha$. To prove the second statement, we integrate the analytic function $f(t) = (-\mathrm{i}\alpha t)^k \,\mathrm{e}^{-\frac{\alpha}{2}(t-\mathrm{i}x)^2}$ over the rectangle with vertices $R,\ R+\mathrm{i}d,\ -R+\mathrm{i}d,\ -R$ where $R > 0$. By the residue theorem this integral is zero for every $R > 0$. On the other hand, letting $R \to \infty$, the integrals over the vertical sides tend to zero, therefore

$$\int_{-\infty}^{\infty} f(t)dt = \int_{-\infty+\mathrm{i}d}^{\infty+\mathrm{i}d} f(t)dt.$$

The integral on the left is equal to $\sqrt{2\pi/\alpha}\mathcal{H}_\alpha(t^k)(x)$. To conclude the proof we introduce a new variable $u = t - \mathrm{i}d$ in the integral on the right to obtain

$$\begin{aligned}\mathcal{H}_\alpha(t^k)(x) &= \sqrt{\frac{\alpha}{2\pi}} \int_{-\infty}^{\infty} (-\mathrm{i}\alpha u + \alpha d)^k \,\mathrm{e}^{-\frac{\alpha}{2}(u-\mathrm{i}(x-d))^2}\, du \\ &= \mathcal{H}_\alpha((t+\alpha d)^k)(x-d).\end{aligned}$$

□

Using the partial fraction decomposition of a rational function f it is easy to see that the Hermite transform of f exists as soon as f has no poles with zero real part. Namely, if $\operatorname{Re} b \neq 0$ then $g(t) = 1/(t-b)^k$ satisfies the $\mathcal{H}_\alpha$-condition for every $\alpha > 0$, since $g(-i\alpha t)$ is bounded on $\mathbb{R}$. For simple fractions we have the following result.

Proposition 11 *Let $\alpha > 0$. For a complex number b with nonzero real part the following is true*

(1) $\mathcal{H}_\alpha(\frac{1}{t-b})(x) = \mathrm{e}^{\alpha x^2/2-bx}(C - \int_0^x \mathrm{e}^{-\alpha s^2/2+bs}\, ds)$ *where* $C = \mathcal{H}_\alpha(\frac{1}{t-b})(0)$,
(2) $\mathcal{H}_\alpha((\frac{1}{t-b})^{k+1})(x) = (-\frac{1}{\alpha})^k \frac{1}{k!} D^k \mathcal{H}_\alpha(\frac{1}{t-b})(x)$.

PROOF. Let us write $f(t) = (t-b)^{-1}$. As $\mathcal{H}_\alpha(1)(x) = 1$ and

$$\mathcal{H}_\alpha((t-b)f)(x) = \mathcal{H}_\alpha(tf)(x) - b\mathcal{H}_\alpha(f)(x),$$

by Proposition 6 we obtain that $y = \mathcal{H}_\alpha(f)(x)$ satisfies $(\alpha x - b)y - y' = 1$, therefore

$$\mathcal{H}_\alpha(f)(x) = \mathrm{e}^{\frac{\alpha}{2}x^2 - bx}\left(C - \int_0^x \mathrm{e}^{-\frac{\alpha}{2}s^2+bs}\, ds\right).$$

By putting $x = 0$ we conclude the proof of the first statement. To prove the second identity we use induction on k and Proposition 6, and we compute

$$\begin{aligned}\mathcal{H}_\alpha\left(\left(\frac{1}{t-b}\right)^{k+1}\right)(x) &= -\frac{1}{\alpha k} D\mathcal{H}_\alpha\left(\left(\frac{1}{t-b}\right)^{k}\right)(x)\\ &= \left(-\frac{1}{\alpha}\right)^k \frac{1}{k!} D^k \mathcal{H}_\alpha\left(\frac{1}{t-b}\right)(x).\end{aligned}$$

□

Corollary 12 *Let $\alpha > 0$ and let f be a rational function with a partial fraction decomposition*

$$f(t) = q(t) + \sum_{i=1}^{n}\sum_{j=1}^{m_i} \frac{c_{ij}}{(t-b_i)^j}, \tag{10}$$

where q is a polynomial and $b_i,\ i = 1, \ldots, n$, are distinct poles of f.

(1) Let $\operatorname{Re} b_i \neq 0$ for $i = 1, \ldots, n$. Then

$$\begin{aligned}\mathcal{H}_\alpha(f(t))(x) &= \mathcal{H}_\alpha(q(t))(x)\\ &\quad + \sum_{i=1}^{n}\sum_{j=1}^{m_i}\left(-\frac{1}{\alpha}\right)^{j-1} \frac{c_{ij}}{(j-1)!} D^{j-1}\left(C_i(0)F_i(x) - G_i(x)\right)\end{aligned}$$

(2) For a real number $d \neq (1/\alpha)\operatorname{Re} b_i,\ i = 1, \ldots, n$,

$$\mathcal{H}_\alpha(f(t+\alpha d))(x-d) = \mathcal{H}_\alpha(q(t))(x) \\ + \sum_{i=1}^{n} \left[\tilde{C}_i(d)u_i(x)F_i(x) - u_i(x)G_i(x) - v_i(x)\right],$$

where we have written

$$F_i(x) = \mathrm{e}^{\frac{\alpha}{2}x^2 - b_i x}, \tag{11}$$

$$G_i(x) = F_i(x)\int_0^x \frac{ds}{F_i(s)}, \tag{12}$$

$$C_i(d) = \mathcal{H}_\alpha\left(\frac{1}{t+\alpha d - b_i}\right)(0),$$

$$\tilde{C}_i(d) = \frac{C_i(d)}{F_i(d)} + \int_0^d \frac{du}{F_i(u)},$$

and u_i, v_i are some polynomials.

PROOF. The first statement is a direct consequence of linearity of $\mathcal{H}_\alpha$ and Proposition 11.

Let us prove the second statement. If $d \neq (1/\alpha)\,\mathrm{Re}\, b_i$, then $\mathcal{H}_\alpha((t+\alpha d - b_i)^{-j})(x)$ exists for all $x \in \mathbb{R}$. By Proposition 11,

$$\mathcal{H}_\alpha\left(\frac{1}{t+\alpha d - b_i}\right)(x-d) = \frac{F_i(x)}{F_i(d)}\left(C_i(d) - \int_0^{x-d} \mathrm{e}^{-\frac{\alpha}{2}s^2 + (b_i - \alpha d)s}\, ds\right),$$

where $C_i(d) = \mathcal{H}_\alpha(1/(t+\alpha d - b_i))(0)$. Introducing the variable $u = s + d$, we obtain

$$\mathcal{H}_\alpha\left(\frac{1}{t+\alpha d - b_i}\right)(x-d) = \tilde{C}_i(d)F_i(x) - G_i(x), \tag{13}$$

where we denoted

$$\tilde{C}_i(d) = \frac{C_i(d)}{F_i(d)} + \int_0^d \frac{du}{F_i(u)}.$$

One easily verifies that for a polynomial w and $k \in \mathbb{N}$

$$D^k(wF_i) = u_{ik}F_i, \tag{14}$$

$$D^k(wG_i) = u_{ik}G_i + v_{ik}, \tag{15}$$

where u_{ik} and v_{ik} are some polynomials. Therefore by linearity of $\mathcal{H}_\alpha$, Proposition 10, Proposition 11, (13), (14) and (15) we obtain

$$\mathcal{H}_\alpha(f(t+\alpha d))(x-d) = \mathcal{H}_\alpha(q(t))(x) + \sum_{i=1}^{n} \left[\tilde{C}_i(d) u_i(x) F_i(x) - u_i(x) G_i(x) - v_i(x)\right],$$

where by u_i, v_i we denoted polynomials defined by

$$u_i(x)F_i(x) = \sum_{j=1}^{m_i} \left(-\frac{1}{\alpha}\right)^{j-1} \frac{c_{ij}}{(j-1)!} D^{j-1} F_i(x), \tag{16}$$

$$u_i(x)G_i(x) + v_i(x) = \sum_{j=1}^{m_i} \left(-\frac{1}{\alpha}\right)^{j-1} \frac{c_{ij}}{(j-1)!} D^{j-1} G_i(x). \tag{17}$$

□

6 Applications to differential equations

We first establish that any rational solution of $(\mathcal{D}_\alpha L)y = 0, \alpha > 0$, having n distinct finite poles gives rise to $n+1$ linearly independent solutions of $Ly = 0$. In the rest of the section we show that the same is also true when α is a nonzero complex number and solutions are obtained by the same algorithm. Furthermore we show that $(\mathcal{D}_\alpha L)y = 0$ can have a rational solution only if α is a root of a certain polynomial.

Lemma 13 *Let $b_i, i = 1, \ldots, n$, be pairwise distinct complex numbers, let $\alpha \in \mathbb{C} - \{0\}$ and $F_i, i = 1, \ldots, n$, be as in (11), and let $w, u_i, i = 1, \ldots, n$, be arbitrary rational functions. Then*

$$S = \{u_i F_i; i = 1, \ldots, n\} \bigcup \left\{ w - \sum_{i=1}^{n} u_i F_i \int_0^x \frac{ds}{F_i(s)} \right\}$$

is linearly independent over $\mathbb{C}$.

PROOF. We prove the assertion by induction on $|S|$. If $|S| = 2$ then $u_1 F_1$ and $w - u_1 F_1 \int_0^x (1/F_1)$ are linearly independent since the first function is hyperexponential but the second is not.

Let $|S| = n+1$ and suppose that there exist $a_i \in \mathbb{C}, i = 1, \ldots, n+1$, such that

$$\sum_{i=1}^{n} a_i u_i F_i + a_{n+1} \left(w - \sum_{i=1}^{n} u_i F_i \int_0^x \frac{1}{F_i} \right) = 0. \tag{18}$$

For $i = 1, \ldots, n$, let $s_i := D(u_i F_i)/(u_i F_i) \in \mathbb{C}(x)$. Since $D(u_i F_i \int_0^x (1/F_i)) = s_i u_i F_i \int_0^x (1/F_i) + u_i$, by applying D to (18) and multiplying (18) by s_n we obtain

$$\sum_{i=1}^{n} a_i s_i u_i F_i + a_{n+1} \left(w' - \sum_{i=1}^{n} (s_i u_i F_i \int_0^x \frac{1}{F_i} + u_i) \right) = 0,$$
$$\sum_{i=1}^{n} a_i s_n u_i F_i + a_{n+1} \left(s_n w - \sum_{i=1}^{n} s_n u_i F_i \int_0^x \frac{1}{F_i} \right) = 0.$$

After subtracting the above equations we have

$$\sum_{i=1}^{n-1} a_i (s_i - s_n) u_i F_i + a_{n+1} \left(w' - \sum_{i=1}^{n} u_i - s_n w - \sum_{i=1}^{n-1} (s_i - s_n) u_i F_i \int_0^x \frac{1}{F_i} \right) = 0,$$

therefore by induction hypothesis $a_i = 0$ for $i = 1, \ldots, n-1$, and $a_{n+1} = 0$. Then from (18) it follows that $a_n = 0$, hence S is linearly independent over $\mathbb{C}$. $\square$

Theorem 14 *Let $\alpha > 0$, let f be a non-polynomial rational function as in (10) and let L be a linear differential operator with polynomial coefficients. If $(\mathcal{D}_\alpha L) f(x) = 0$ then $Ly = 0$ has $n+1$ linearly independent solutions of the form*

$$y_i(x) = \sum_{j=1}^{m_i} \left(-\frac{1}{\alpha} \right)^{j-1} \frac{c_{ij}}{(j-1)!} D^{j-1} F_i(x), \quad i = 1, \ldots, n,$$
$$y_{n+1}(x) = \mathcal{H}_\alpha(q)(x) - \sum_{i=1}^{n} \sum_{j=1}^{m_i} \left(-\frac{1}{\alpha} \right)^{j-1} \frac{c_{ij}}{(j-1)!} D^{j-1} G_i(x),$$

where $F_i(x)$ and $G_i(x)$ are as in (11) and (12).

PROOF. For every $d \neq (1/\alpha) \operatorname{Re} b_i$, $i = 1, \ldots, n$, Corollary 9 yields $L\mathcal{H}_\alpha(f(t+\alpha d))(x-d) = 0$. On the other hand, by Corollary 12,

$$L\mathcal{H}_\alpha(f(t+\alpha d))(x-d) = L\mathcal{H}_\alpha(q(t))(x) + L\left[\sum_{i=1}^{n}\left(\tilde{C}_i(d)u_i(x)F_i(x) - u_i(x)G_i(x) - v_i(x)\right)\right],$$

where u_i, v_i are defined in (16), (17). Using (14) and denoting $L(u_i(x)F_i(x)) = \tilde{u}_i(x)F_i(x)$ we obtain

$$\sum_{i=1}^{n}\tilde{C}_i(d)\tilde{u}_i(x)F_i(x) = L\left[\sum_{i=1}^{n}(u_i(x)G_i(x) + v_i(x)) - \mathcal{H}_\alpha(q(t))(x)\right]. \quad (19)$$

We shall show that $\tilde{u}_i(x)F_i(x) = 0, i = 1, \ldots, n$. Suppose that this is not true. Without loss of generality we may assume that all $\tilde{u}_i(x)F_i(x) \neq 0$. Now choose real numbers d_1 and d_2 (different from $(1/\alpha)\operatorname{Re} b_i,\ i = 1, \ldots, n$) so that $\tilde{C}_1(d_1) \neq \tilde{C}_1(d_2)$, put $d := d_1$ and $d := d_2$ in (19), then subtract the resulting equations to find

$$\sum_{i=1}^{n}\left(\tilde{C}_i(d_1) - \tilde{C}_i(d_2)\right)\tilde{u}_i(x)F_i(x) = 0.$$

As $\tilde{u}_i(x)F_i(x)$ are nonsimilar hyperexponential functions the above equation implies that $\tilde{C}_1(d_1) - \tilde{C}_1(d_2) = 0$, a contradiction, hence $\tilde{u}_1(x)F_1(x) = 0$. Repeating the above argument for $i = 2, \ldots, n$ we have $\tilde{u}_i(x)F_i(x) = 0$ for all $i = 1, \ldots, n$, so the right-hand side of (19) also equals to zero. Therefore $u_i(x)F_i(x), i = 1, \ldots, n$ and $\mathcal{H}_\alpha(q(t))(x) - \sum_{i=1}^{n}(u_i(x)G_i(x) + v_i(x))$ satisfy $Ly = 0$ and, by Lemma 13, they are linearly independent. □

Example 15 *It is known that the Weber differential equation*

$$y'' - xy' - ay = 0$$

has a formal power series solution [6]

$$y = C_1\left(1 + \sum_{k=1}^{\infty}\frac{a(a+2)\cdots(a+2k-2)}{(2k)!}x^{2k}\right) + C_2\left(x + \sum_{k=1}^{\infty}\frac{(a+1)(a+3)\cdots(a+2k-1)}{(2k+1)!}x^{2k+1}\right).$$

If $a = -n,\ n \in \mathbb{N}$, is even (odd) then the first (second) sum is a polynomial of degree n. With our method we can find explicit solutions when $a = n$. Writing $L = D^2 - xD - n$ and choosing $\alpha = 1$ we obtain $(\mathcal{D}_1 L)y = -xy' - ny = 0$

having a rational solution $y = x^{-n}$. Therefore linearly independent solutions of $Ly = 0$ are

$$y_1(x) = \frac{d^{n-1}}{dx^{n-1}} \mathrm{e}^{x^2/2} \qquad y_2(x) = \frac{d^{n-1}}{dx^{n-1}} \left(\mathrm{e}^{x^2/2} \int_0^x \mathrm{e}^{-s^2/2}\, ds \right).$$

The next step is to specify which values of α increase the likelihood that $(\mathcal{D}_\alpha L)y = 0$ has a non-polynomial rational solution. It is known [1] that this can happen only if the leading coefficient of $\mathcal{D}_\alpha L$ is nonconstant.
First let us prove two lemmas.

Lemma 16 *For $\alpha \in \mathbb{C} - \{0\}$ and $i \in \mathbb{N}$*

$$\left(\frac{1}{\alpha}x + D\right)^i = \sum_{j=0}^{i} \binom{i}{j} \left(\frac{x}{\alpha}\right)^j D^{i-j} + \text{terms of total degree} < i.$$

PROOF. We use induction on i. From $Dx = xD + 1$ it follows that $D^n x = xD^n + nD^{n-1}$, hence

$$\begin{aligned}\left(\frac{1}{\alpha}x + D\right)^{i+1} &= \sum_{j=0}^{i} \binom{i}{j} \left(\frac{x}{\alpha}\right)^{j+1} D^{i-j} + \sum_{j=0}^{i} \binom{i}{j} \left(\frac{x}{\alpha}\right)^j D^{i+1-j} + \dots \\ &= \sum_{j=0}^{i+1} \binom{i+1}{j} \left(\frac{x}{\alpha}\right)^j D^{i+1-j} + \text{terms of total degree} < i+1\end{aligned}$$

which concludes the proof. $\square$

Lemma 17 *Let $L = \sum a_{ik} x^i D^k$ be of total degree m and $\alpha \neq 0$. Then $\mathcal{D}_\alpha L$ is of order m iff $p_L(\alpha) = \sum_{i=0}^m a_{i,m-i}\alpha^{m-i} \neq 0$. Moreover, if $\mathcal{D}_\alpha L$ is of order m then it has no singular points.*

PROOF. Writing $L = \sum_{i=0}^m a_{i,m-i} x^i D^{m-i} + \sum_{i+k<m} a_{i,k} x^i D^k$ and bearing in mind Lemma 16 we obtain

$$\begin{aligned}\mathcal{D}_\alpha L &= \sum_{i=0}^{m} a_{i,m-i} \left(\frac{x}{\alpha} + D\right)^i (\alpha D)^{m-i} + \sum_{i+k<m} a_{i,k} \left(\frac{x}{\alpha} + D\right)^i (\alpha D)^k \\ &= \sum_{i=0}^{m} a_{i,m-i} \sum_{j=0}^{i} \binom{i}{j} \left(\frac{x}{\alpha}\right)^j \alpha^{m-i} D^{m-j} + \sum_{i+k<m} b_{i,k} x^i D^k \\ &= \sum_{j=0}^{m} \left(\frac{x}{\alpha}\right)^j D^{m-j} \sum_{i=j}^{m} \binom{i}{j} a_{i,m-i}\alpha^{m-i} + \sum_{i+k<m} b_{i,k} x^i D^k.\end{aligned}$$

The coefficient of D^m is $\sum_{i=0}^m a_{i,m-i}\alpha^{m-i}$ which concludes the proof. $\square$

Therefore we only have to consider a finite set of values for α – all nonzero roots of the polynomial $p_L(t) = \sum_{i=0}^m a_{i,m-i}t^{m-i}$, constructed from the terms of total degree m in L by substituting 1 for x and t for D. For $\alpha > 0$ we then use the algorithm of Abramov [1] to find rational solutions of $(\mathcal{D}_\alpha L)y = 0$ and apply Theorem 14 to get solutions of $Ly = 0$. In the rest of the section we show that the same procedure is applicable also for a complex $\alpha \neq 0$.

First we extend the definition of $\mathcal{H}_\alpha$ on polynomials by

$$\alpha \in \mathbb{C} : \quad \mathcal{H}_\alpha(t^n)(x) = H_n^{(\alpha)}(x).$$

Let $L = \sum_{i,k} a_{ik}x^i D^k$ be of total degree m and denote $L^\beta = \sum_{i,k} a_{ik}(x/\beta)^i(\beta D)^k$. Then $L^\beta y(x) = 0$ iff $Ly(\beta x) = 0$. Let α satisfy $p_L(\alpha) = 0$ and choose β such that $\beta^2 = \alpha$. Since L^β is of total degree m and

$$L^\beta = \sum_{i=0}^m a_{i,m-i}\beta^{m-2i}x^i D^{m-i} + \sum_{i+k<m} a_{ik}\beta^{k-i}x^i D^k,$$

$p_{L^\beta}(1) = 0$ and, by Lemma 17, $\mathcal{D}_1 L^\beta$ is of order less than m.

Lemma 18 *Let* $\alpha = \beta^2$. *Then* $(\mathcal{D}_\alpha L)f(x) = 0$ *iff* $(\mathcal{D}_1 L^\beta)f(\beta x) = 0$.

PROOF. We write $t = \beta x$, $g(x) = f(t)$ and

$$\begin{aligned}\mathcal{D}_\alpha L &= \sum_{i,k} a_{ik}\,(t/\alpha + D_t)^i\,(\alpha D_t)^k,\\ \mathcal{D}_1 L^\beta &= \sum_{i,k} a_{ik}\beta^{-i}(x + D_x)^i(\beta D_x)^k.\end{aligned}$$

As $D_x^k g(x) = \beta^k D_t^k f(t)$, we compute

$$\begin{aligned}(\mathcal{D}_1 L^\beta)g(x) &= \sum_{i,k} a_{ik}\beta^{k-i}(x + D_x)^i D_x^k g(x)\\ &= \sum_{i,k} a_{ik}\left(\frac{t}{\beta^2} + D_t\right)^i \beta^{2k} D_t^k f(t) = (\mathcal{D}_\alpha L)f(t).\end{aligned}$$

$\square$

Let now $(\mathcal{D}_\alpha L)y = 0$ be satisfied by a rational function f as in (10). Then $(\mathcal{D}_1 L^\beta)y = 0$ has a rational solution $g(x) = f(\beta x)$. Let $F_i(x), G_i(x)$ be as in (11), (12) and denote

$$F_i^\beta(x) = \mathrm{e}^{x^2/2 - b_i x/\beta}, \qquad\qquad G_i(x) = F_i^\beta(x) \int\limits_0^x \frac{ds}{F_i^\beta(s)},$$

hence $F_i^\beta(x) = F_i(x/\beta)$ and $G_i^\beta(x) = \beta G_i(x/\beta)$. By Theorem 14, $L^\beta y = 0$ has $n+1$ solutions

$$\begin{aligned}
y_i^\beta(x) &= \sum_{j=1}^{m_i} (-1)^{j-1} \frac{c_{ij}}{\beta^j (j-1)!} D^{j-1} F_i^\beta(x), \quad i = 1, \ldots, n, \\
y_{n+1}^\beta(x) &= \mathcal{H}_1(q(\beta t))(x) - \sum_{i=1}^{n} \sum_{j=1}^{m_i} (-1)^{j-1} \frac{c_{ij}}{\beta^j (j-1)!} D^{j-1} G_i^\beta(x),
\end{aligned}$$

and consequently $Ly = 0$ is satisfied by (note $\beta^2 = \alpha$)

$$\begin{aligned}
y_i(x) &= y_i^\beta(\beta x) = \frac{1}{\beta} \sum_{j=1}^{m_i} \left(-\frac{1}{\alpha}\right)^{j-1} \frac{c_{ij}}{(j-1)!} D^{j-1} F_i(x), \quad i = 1, \ldots, n, \\
y_{n+1}(x) &= y_{n+1}^\beta(\beta x) = \mathcal{H}_\alpha(q)(x) - \sum_{i=1}^{n} \sum_{j=1}^{m_i} \left(-\frac{1}{\alpha}\right)^{j-1} \frac{c_{ij}}{(j-1)!} D^{j-1} G_i(x).
\end{aligned}$$

In the last line we used $H_n^{(b)}(x) = \sqrt{b/a}^{\,n} H_n^{(a)}(\sqrt{b/a}\, x)$, hence for $q(t) = \sum_k a_k t^k$ we have

$$\mathcal{H}_1(q(\beta t))(\beta x) = \sum_k a_k \beta^k H_k^{(1)}(\beta x) = \sum_k a_k H_k^{(\alpha)}(x) = \mathcal{H}_\alpha(q(t))(x).$$

Example 19 *Let $L = D^3 - 4xD^2 + (4x^2 - 5)D + 8x$. Then $p_L(t) = t^3 - 4t^2 + 4t = t(t-2)^2$, $\mathcal{D}_2 L = 2(x^2+1)D + 4x$. Using the algorithm of [1] we find a rational solution*

$$f(x) = \frac{1}{2\mathrm{i}} \left(\frac{1}{x - \mathrm{i}} - \frac{1}{x + \mathrm{i}} \right),$$

of $(\mathcal{D}_2 L)y = 0$. By Theorem 14, $F_1, F_2, G_1 - G_2$ satisfy $Ly = 0$, where

$$\begin{aligned}
F_1(x) &= \mathrm{e}^{x^2 - \mathrm{i}x}, \qquad G_1(x) = F_1(x) \int\limits_0^x \frac{ds}{F_1(s)}, \\
F_2(x) &= \mathrm{e}^{x^2 + \mathrm{i}x}, \qquad G_2(x) = F_2(x) \int\limits_0^x \frac{ds}{F_2(s)}.
\end{aligned}$$

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