University of Ljubljana
Institute of Mathematics, Physics and Mechanics
Department of Mathematics
Jadranska 19, 1000 Ljubljana, Slovenia

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Blaž Zmazek Janez Žerovnik

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On the Weak Reconstruction of Strong Product Graphs*

Blaž Zmazek† and Janez Žerovnik‡
FME, University of Maribor
Smetanova 17, SI-2000 Maribor, Slovenia §

Abstract

In this paper we prove that any nontrivial connected strong product graph can be uniquely reconstructed from each of its one vertex deleted subgraphs.

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Key words: reconstruction problem, strong product, composite graphs.

Dedicated to W. Imrich on the occasion of his 60th birthday.

1 Introduction

In [11] S.M.Ulam asked the question whether a graph G is uniquely determined up to isomorphism by its deck, which is the set of all graphs $G \setminus x$ obtained from G by deleting a vertex x and all edges incident to it. While the conjecture is false for infinite graphs it still is open for finite graphs. When reconstructing a class of graphs, the problem of reconstruction partitions naturally into two subproblems, namely recognition: showing that membership in the class is determined by the deck and weak reconstruction: showing that no two nonisomorphic members of the class have the same deck.

*Partially supported by the Ministry of Education, Science and Sport of Slovenia.
†E-mail: `blaz.zmazek@uni-mb.si`
‡E-mail: `janez.zerovnik@imfm.uni-lj.si`
§Also at IMFM, Jadranska 19, SI-1111 Ljubljana, Slovenia.

Recently, the weak reconstruction of Cartesian product graphs from one single–vertex deleted subgraph was studied [7]. It was shown that both the recognition and the weak reconstruction problem can be solved from a single vertex–deleted subgraph for nontrivial, connected Cartesian product graphs. Extensions of finite and infinite connected graphs to Cartesian products were also considered in [7]. In most cases, it is not possible to extend a given connected graph H to a nontrivial Cartesian product. However, if such extensions exist, they are all isomorphic (Theorem 2). In fact, unless H has a special structure, there is exactly one such extension. (For more details see section Preliminaries.) Equivalent result for finite graphs, given in terms of semistability, appears in [10]. An algorithm for the weak reconstruction of Cartesian product graphs of time complexity $O(|E||V|(\Delta^2 + |E| \log |V|))$ is given in [4]. In [6] the case of k-vertex deleted subgraphs of Cartesian products is considered, and it is proved that one can decide whether a graph H is a k-vertex deleted subgraph of a Cartesian product G with at least $k + 1$ prime factors on at least $k + 1$ vertices each, and that inthis case H uniquely determines G.

Dörfler [2] proved the validity of Ulam's conjecture for finite nontrivial strong product graphs under the assumption that at least one factor has a nontrivial relation S.

In this paper we solve both the recognition problem and the weak reconstruction problem for nontrivial, connected strong product graphs. We prove that nontrivial, connected strong product graph can be reconstructed from each of its single vertex–deleted subgraphs. Moreover, from any single vertex–deleted subgraph there is exactly one extension. Note that this is a stronger property compared to the case of the Cartesian product where all the extensions are unique only up to isomorphism.

The paper is organized as follows. In the next section definitions and some previous results are recalled. In Section 3 we prove the main result:

Theorem 1 *A connected nontrival strong product graph is uniquely determined by each of its one vertex-deleted subgraphs.*

More precisely, we prove, that given a vertex deleted subgraph of a strong product graph there is exactly one set of vertices to which the new (i.e. the deleted) vertex has to be connected in order to reconstruct the original graph.

2 Preliminaries

We will consider only finite connected simple graphs, i.e. graphs without loops and multiple edges. The vertex set of graph G is denoted by $V(G)$ and the edge set will be denoted by $E(G)$. We write shortly uv for edge $\{u, v\}$. Two edges are *adjacent* if they have a common vertex. $G \cong H$ denotes graph isomorphism, i.e. the existence of a bijection $b : V(G) \to V(H)$ such that vertices g_1, g_2 are adjacent in G exactly if vertices $b(g_1), b(g_2)$ are adjacent in H. A maximal complete subgraph is called a *clique*. Vertices of a complete graph K_q will usually be denoted by $\{0, 1, \ldots, q-1\}$. The star graphs S_a are defined as follows: The vertex set of S_a consists of a central vertex c_0 of degree a and of a vertices $c_1, c_2, \ldots, c_a$ adjacent to c_0. The *(closed) neighborhood* of a vertex x is $N_G(x) = \{x\} \cup \{y \mid xy \in E(G)\}$.

$G \setminus x$ denotes the subgraph of G induced by the vertex set $V(G) \setminus \{x\}$.

The *strong product* $G_1 \boxtimes G_2$ of graphs G_1 and G_2 has as vertices the pairs (g, h) where $g \in V(G_1)$ and $h \in V(G_2)$. Vertices (g_1, h_1) and (g_2, h_2) are adjacent if either $\{g_1, g_2\}$ is an edge of G_1 and $h_1 = h_2$ or if $g_1 = g_2$ and $\{h_1, h_2\}$ is an edge of G_2 or if $\{g_1, g_2\}$ is an edge of G_1 and $\{h_1, h_2\}$ is an edge of G_2. A graph G is *prime* (with respect to the strong product) if it cannot be expressed as a product $G_1 \boxtimes G_2$ unless one of G_1 or G_2 is a K_1. The strong product graphs enjoy the *unique factorization property*, i.e. for every graph G there is a unique set of prime graphs $G_1, G_2, \ldots G_k$ such that $G = G_1 \boxtimes G_2 \boxtimes \ldots \boxtimes G_k$ and none of the factors is a K_1 [1, 8]. An algorithm for finding the prime factors of strong direct product graphs in polynomial time is given in [3].

We will also need the definition of Cartesian product of graphs. The *Cartesian product* of G_1 and G_2 is the graph $G_1 \Box G_2$ with vertex set $V(G_1) \times V(G_2)$ and $(x_1, x_2)(y_1, y_2) \in E(G_1 \Box G_2)$ whenever $x_1 y_1 \in E(G_1)$ and $x_2 = y_2$, or $x_2 y_2 \in E(G_2)$ and $x_1 = y_1$. The unique factorization property for Cartesian product graphs is due to Sabidussi [9].

The following relation, defined on the vertex set of G which was first defined in [1], proved to be useful in studies of the strong product. The equivalence relation S is defined as follows: xSx and for $x \neq y$, xSy if

- $xy \in E(G)$ and
- $N(x) \setminus x = N(y) \setminus y$.

In [3], vertices x and y with xSy are called *interchangeable*.

We define a graph G/S on equivalence classes of S as follows. Vertices are equivalence classes of S, $V(G/S) = \{[x] \mid x \in V(G)\}$, where $[x]$ denotes the S-equivalence class of vertex x. By definition, two vertices $[x]$, $[y] \in V(G/S)$ are adjacent if there is an edge between the representatives. In other words, $[x] \sim [y]$ if and only if there are vertices $v \in [x]$ and $u \in [y]$, which are adjacent in G. It is easy to see that then all pairs $v \in [x]$, $u \in [y]$ must be adjacent in G. It can also be seen easily that $(G/S)/S \cong G/S$.

Example: In any product $K_2 \boxtimes G$ vertices $(0, v)$ and $(1, v)$ are in relation S for any $v \in G$. $\square$

Suppose that we have factored G and its factorization is $G = G_1 \boxtimes G_2 \boxtimes \ldots \boxtimes G_k$. Then we can label the vertices of G with distinct k-tuples from $V(G_1) \times V(G_2) \times \ldots \times V(G_k)$ so that the edges are consistent with the definition of the strong product. An edge is a *Cartesian edge* if the labels of its endpoints differ in exactly one component and is a *direct edge* if the labels of its endpoints differ in more than one component. Graph induced on the Cartesian edges of graph G will be called a *Cartesian skeleton* of G. Although the factorization of G with respect to the strong product is unique, its vertex labeling may not be. There may be more than one labeled version of a given strong product graph. In particular, there are examples (see [3]) of graphs for which even the sets of Cartesian and direct edges differ in different labelings. However, (compare Lemma 1.3 of [3])

Lemma 1 *If G has no interchangeable vertices then the set of Cartesian edges is uniquely determined.*

It will be useful to remember sizes of equivalence classes of S. We therefore define the *weight function* $c : V(G/S) \to \mathbf{N}$ with $c([v]) = |[v]|$, where $|[v]|$ is the cardinality of the equivalence class of v. When needed, we will consider G/S as a *weighted graph* $(G/S, c)$.

Two weighted graphs (G, c_G) and (H, c_H) are, by definition, isomorphic if and only if

- there is an isomorphism $\pi : G/S \to H/S$ such that
- $c_G(v) = c_H(\pi(v))$ for all $v \in V(G/S)$

It can be shown (see [1]) that

Lemma 2 *Two graphs G and H are isomorphic if and only if the corresponding weighted graphs G/S and H/S are isomorphic.*

We will later refer to the following theorem on uniqueness of the reconstruction of Cartesian product graphs [7].

Theorem 2 *Let G and H be finite or infinite connected Cartesian product graphs. If the one vertex deleted subgraphs G_x and H_y, where $x \in G$ and $y \in H$, are isomorphic, then $G \simeq H$.*

A one vertex extension is obviously determined by the set of vertices, say N_x in G_x which have to be adjacent to the new vertex. In general, there may be more different subsets N_x which all yield a Cartesian product graph.

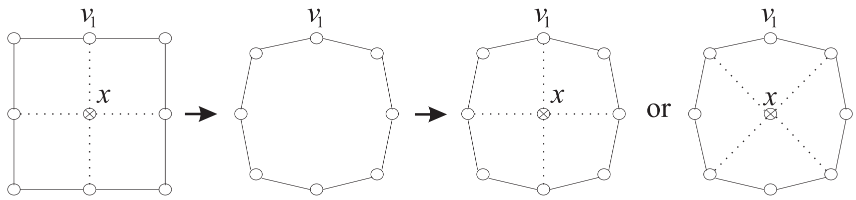

Figure 1: Nonunique extensions, example $G_x = C_8$.

The above theorem assures that in all cases, when the resulting graph is a Cartesian product, they are isomorphic. If there is exactly one such subset N_x in G_x we say that the reconstruction is *unique*. (In terminology of [10] this is equivalent to G being semistable at x.) Characterization of graphs G for which the reconstruction is always unique is given in [7]. This characterization implies that there are two different cases where the reconstruction of a Cartesian product graph is not unique:

1. $C_8 \simeq P_3 \Box P_3 \setminus \{central - vertex\}$ has two isomorphic reconstructions (see Fig. 1.)
2. Let G be a product $K_2 \Box P$ where P is a prime graph and let x_2 be a vertex of P such that P_{x_2} has at least one connected component which has at least one K_2 factor. Then the reconstruction is not unique (See example on Fig 2.)

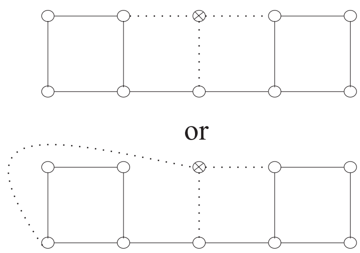

Figure 2: Nonunique extensions, example $G = K_2 \Box P_4$.

3 Reconstruction of the strong product

There are more edges in the strong product graphs. Therefore it is likely that, provided we can reconstruct the Cartesian skeleton, it will be less posibilities for valid strong reconstructions. See, for example, Fig. 3 and Fig. 6.

Lemma 3 *The relation S on $G \setminus x$ is the same relation as the relation S computed on G and restricted to $G \setminus x$. Formally,*

$$S_G|_{G \setminus x} = S_{G \setminus x}$$

Proof: Let u, v be any pair of vertices of G. If either x is in both neighborhoods $N_G(u)$ and $N_G(v)$ or in none of them, then clearly u and v are in relation S

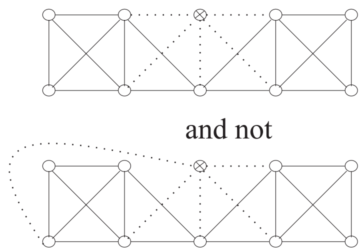

Figure 3: Unique extension, example $G = K_2 \boxtimes P_4$.

on $G \setminus x$ if and only if they are in relation S on G.

If x is in one, but not in the other neighborhood, then the only ("bad") possibility is that $N_G(u) \neq N_G(v)$ and $N_{G\setminus x}(u) = N_{G\setminus x}(v)$. It implies that $N_G(u)$ and $N_G(v)$ differ in exactly one vertex, x.

But this is not possible, as

CLAIM: two (closed) neighborhoods in a nontrivial strong product are either the same sets or they differ in at least two vertices. □

Lemma 4 *The weak reconstruction of strong product is unique.*

Proof: In the proof we will consider two cases:

CASE 1, $G = K_k \boxtimes G_2$.

In this case $(G \setminus x)/S$ as a vertex weighted graph (prime or composite) must have all weights (but one) divisible by k (for some $k \neq 1$) and one vertex weight of the form $w(v) = Ck - 1$. The reconstruction in this case is obtained by replacing a vertex of weight Ck by a clique K_{Ck}.

CASE 2, K_k is not a factor in the factorization of strong product G, hence G/S is a nontrivial strong product, say $G/S = G_1 \boxtimes G_2$.

In this case the relation S is trivial on $G_1 \boxtimes G_2$. First consider $G_1 \boxtimes G_2$ without weights. The decomposition of $G_1 \boxtimes G_2$ is unique, as S is trivial on $G \setminus S$. Hence the Cartesian skeleton is well defined and unique. Hence, the decomposition to the Cartesian and the direct edges is well defined.

In this case we have two possibilities:

SUBCASE 2.1: The equivalence class of the vertex x in G has only one vertex.

After deletion of x the graph $(G \setminus x)/S$ $(= (G/S) \setminus x$, by Lemma 3) has a well defined Cartesian skeleton which is (as unweighted graph) a single vertex deleted Cartesian product graph. It has trivial S, hence the set of Cartesian edges is well defined, hence its Cartesian skeleton can be (up to isomorphism) reconstructed by Theorem 2.

Lemma 5 *There is at most one way of reconstruction of strong product graph of factors on more than two vertices, which is consistent with the direct edges.*

Proof: The only interesting case, where the reconstruction of a Cartesian skeleton of strong product graph is not unique is the graph $P_3 \boxtimes P_3$. (The other cases are handled by Case 1. One factor of a graph of type $K_2 \boxtimes P$ has only two vertices.) The reconstruction of $P_3 \boxtimes P_3$ is unique because: In each reconstruction of Cartesian skeleton the new vertex is adjacent to at least one vertex y in the same G_1-layer and to at least one vertex z in the same G_2-layer (Fig. 4). From the definition of strong product, there exists an edge

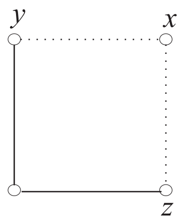

Figure 4: Cartesian skeleton

yz in the strong product graph which specifies the vertices y and z which are adjacent to the missing vertex in the strong product (Fig. 5). In particular,

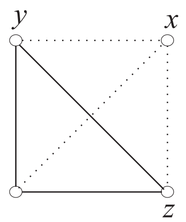

Figure 5: Direct edge yz specifies the neighbors of x.

there are two isomorphic reconstructions of Cartesian skeleton of the graph $(P_3 \boxtimes P_3) \setminus x$ (see Fig. 1), but on the other hand only one reconstruction of the strong product exists. In the second case, one has to add some edges not incident to the new vertex in order to obtain a strong product (see Fig. 6). □

SUBCASE 2.2: The equivalence class of the vertex x in G has more than one vertex.

In this case $(G \setminus x)$ has a well defined Cartesian skeleton which is (as unweighted graph) a Cartesian product graph. There must be exactly one vertex with invalid weight and x must be interchangeable with vertices from the equivalence class of the vertex x. The vertex with invalid weight can be identified as follows. For any edge xy in $E(G_1)$ consider the ratios $\frac{w(x,u)}{w(y,u)}$ of edges

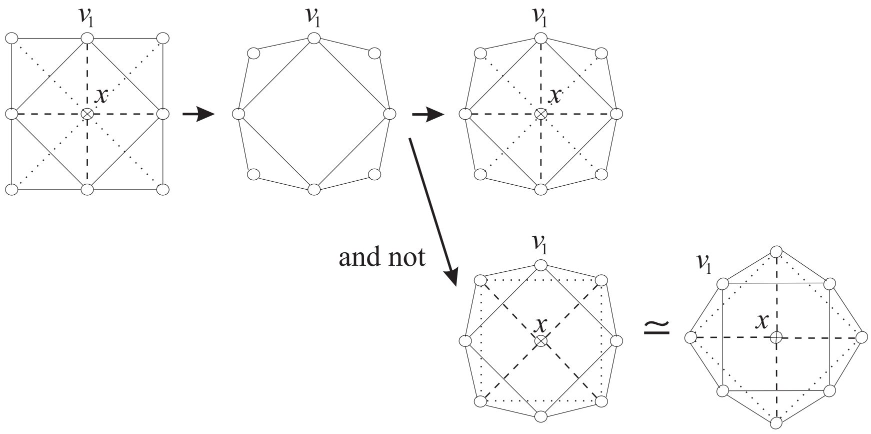

Figure 6: Unique reconstruction of $(P_3 \boxtimes P_3) \setminus x$.

$(x, u)(y, u)$. Note that there are at least three such edges. Exactly all edges with both valid weight endpoints have equal ratios $w(xy) = \frac{w(x,u)}{w(y,u)}$. If there is an edge with a different ratio, then we know one of its endpoints has invalid weight. In this way, all edges incident to the invalid weight vertex can be identified. □

Theorem 1 follows.

In conclusion, let us note that we have shown uniqueness of the reconstruction, but did not provide an algorithm. We believe that the reconstruction of a finite strong product graph from one vertex deleted subgraph is a polynomial problem, however the details of the algorithmic solution may be nontrivial.

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