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Preprint series, Vol. 40 (2002), 845

THE LJUBLJANA GRAPH

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ISSN 1318-4865

November 19, 2002

Ljubljana, November 19, 2002

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November 14, 2002

Abstract

A detailed description is given of a recently-discovered edge- but not vertex-transitive trivalent graph on 112 vertices, which turns out to be the third smallest example of such a semisymmetric cubic graph. This graph is called the *Ljubljana graph* by the authors, although it is believed that its existence may have been known by R. M. Foster. With the help of some advanced theory of covering graphs, various properties of this graph are analysed, including a connection with the Heawood graph which can be described using ideals over polynomial rings.

§1 Introduction In [3] I. Z. Bouwer wrote: "R. M. Foster (private communication) has found an edge- but not vertex-transitive cubic graph (with 112 vertices) whose girth (equal to 10) is not a multiple of 4." Neither its definition nor any further information on this graph was given in that paper. Edge- but not vertex-transitive regular graphs are now called *semisymmetric*. In 2001, during a brief visit by the first author to Ljubljana, we constructed a cubic semisymmetric graph on 112 vertices, which can be described as a regular $\mathbb{Z}_2^3$-cover of the Heawood graph. At the first author's suggestion, we named this graph *the Ljubljana graph*. A computer-based search showed that the Ljubljana graph is the only cubic semisymmetric graph on 112 vertices, and hence the one already known to Foster. Computations also revealed that the Ljubljana graph, which throughout this paper

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is denoted by $\mathcal{L}$, is the third graph in the series of cubic semisymmetric graphs (ordered by number of vertices). The smallest two members of this series are the Gray graph $\mathcal{G}$ (on 54 vertices), which was studied extensively in [10] and [11], and a biprimitive graph on 110 vertices which was described in [7]. The aim of this article is to give a precise definition of the Ljubljana graph, to give a computer-free description of its automorphism group, and to present some other interesting graph-theoretic properties of $\mathcal{L}$. The principal means of exploring the properties of $\mathcal{L}$ will be the use of advanced covering graph techniques, as developed in [8, 9].

§2 Article layout This article is split into several small paragraphs, each being a step in the analysis of the properties of the Ljubljana graph. Since the Ljubljana graph is a covering graph of the Heawood graph, §3 to §5 give a short review of the Heawood graph and its automorphism group. In order to make the article easier to read, §6 to §13 summarise the use of covering graph techniques. This theory is then applied to define the Ljubljana graph as a $\mathbb{Z}_2^3$-cover of the Heawood graph in §14 to §17, and to compute its automorphism group in §18 to §22. Finally, the hamiltonicity of $\mathcal{L}$ is dealt with in §23, two configurations associated with the Ljubljana graph are discussed in §24, and some additional properties of $\mathcal{L}$ are gathered together in §25.

§3 The Heawood graph The vertex-set of the Heawood graph $\mathcal{H}$ consists of two copies of the integer ring $\mathbb{Z}_7$, so that $V(\mathcal{H}) = \mathbb{Z}_7 \times \mathbb{Z}_2$, and then the edge-set is defined by making a vertex $(i, 0)$ adjacent to a vertex $(j, 1)$ if and only if $j - i \in \{1, 2, 4\}$. We shall abbreviate the notation by writing i in place of $(i, 0)$, and i' in place of $(i, 1)$, for all $i \in \mathbb{Z}_7$.

§4 The automorphism group of the Heawood graph Among the automorphisms of the Heawood graph, the following three are easy to find:

$$\rho\colon (i, j) \mapsto (i + 1, j), \quad \text{for } i \in \mathbb{Z}_7,\ j \in \mathbb{Z}_2,$$

associated with the left regular representation of $\mathbb{Z}_7$ on itself, and

$$\sigma\colon (i, j) \mapsto (2i, j), \quad \text{for } i \in \mathbb{Z}_7,\ j \in \mathbb{Z}_2,$$

associated with the group automorphism of $\mathbb{Z}_7$ arising from multiplication by 2, and

$$\tau\colon (i, j) \mapsto (-i, j + 1), \quad \text{for } i \in \mathbb{Z}_7,\ j \in \mathbb{Z}_2,$$

which interchanges the two sets of the biparition. These automorphisms may also be easily see as the following permutations:

$$\rho = (0, 1, 2, 3, 4, 5, 6)(0', 1', 2', 3', 4', 5', 6'),$$

$$\sigma = (0)(1,2,4)(3,6,5)(0')(1',2',4')(3',6',5'), \text{ and}$$

$$\tau = (0,0')(1,6')(2,5')(3,4')(4,3')(5,2')(6,1').$$

The group $\Gamma_{21} = \langle \rho, \sigma \rangle$ is isomorphic to the semidirect product $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$, and is edge- but not vertex-transitive on $\mathcal{H}$. The group $\Gamma_{14} = \langle \rho, \tau \rangle$ is isomorphic to the dihedral group D_7 and is vertex-transitive on $\mathcal{H}$. The group $\Gamma_{42} = \langle \rho, \sigma, \tau \rangle$ is isomorphic to the semidirect product $\mathbb{Z}_7 \rtimes \mathbb{Z}_6$ and is arc-transitive on $\mathcal{H}$. The full automorphism group $\mathrm{Aut}\,\mathcal{H}$ has order 336 and is isomorphic to $PGL(2,7)$. It is generated by the involution τ and the index 2 subgroup $\mathrm{Aut}_0\mathcal{H}$ which preserves the two sets of the bipartition. Note that $\mathrm{Aut}_0\mathcal{H}$ is isomorphic to $PSL(2,7) \cong GL(3,2)$, and that $\mathrm{Aut}_0\mathcal{H} = \langle \rho, \omega \rangle$, where

$$\omega = (1,5)(4,6)(0',5')(3',6')(0)(2)(3)(1')(2')(4').$$

Finally, observe that the normalizer $N_{\mathrm{Aut}\,\mathcal{H}}(\rho)$ of the group $\langle \rho \rangle$ in $\mathrm{Aut}\,\mathcal{H}$ is the group Γ_{42}.

§5 The Heawood graph as a Cayley graph The subgroup $\Gamma_{14} = \langle \rho, \tau \rangle \cong D_7$ of $\mathrm{Aut}\,\mathcal{H}$ acts regularly on the vertex-set of $\mathcal{H}$. This shows that the Heawood graph is a Cayley graph of the group $\langle \rho, \tau \rangle$, with respect to the generating set $\{\tau_1, \tau_2, \tau_3\}$ where

$$\tau_1 = \rho\tau = (0,1')(1,0')(2,6')(3,5')(4,4')(5,3')(6,2'),$$
$$\tau_2 = \rho^2\tau = (0,2')(1,1')(2,0')(3,6')(4,5')(5,4')(6,3'),$$
$$\tau_3 = \rho^4\tau = (0,4')(1,3')(2,2')(3,1')(4,0')(5,6')(6,5').$$

§6 Voltage assignments and derived covering graphs An ordered pair (u,v) of adjacent vertices of a graph X is called a *dart* (or an *arc*) of X. If $x = (u,v)$ is a dart of X then the pair $x^{-1} = (v,u)$ is the *inverse dart* of X. Let D be the set of all darts of a graph X and let N be a finite group. A mapping $\zeta\colon D \to N$, satisfying $\zeta(x^{-1}) = \zeta(x)^{-1}$ for every dart $x \in D$, is called a *voltage assignment* in the *voltage group* N on the graph X. A voltage assignment ζ gives rise to the *derived covering graph* $\mathrm{Cov}(X;\zeta)$, with vertex-set $V \times N$ and with adjacency relation given by $(u,a) \sim (v,b)$ if and only if u is adjacent to v in X and $a^{-1}b = \zeta(u,v)$. The graph $\mathrm{Cov}(X;\zeta)$ is also called an *N-regular cover of X*. The projection $V \times N \to V$ is clearly a graph morphism $\wp_\zeta\colon \mathrm{Cov}(X;\zeta) \to X$, called the *derived covering projection*. Voltage assignments $\zeta, \xi\colon D \to N$ are said to be *isomorphic* if there exists an automorphism $g \in \mathrm{Aut}\,X$ and an isomorphism $\tilde{g}\colon \mathrm{Cov}(X;\zeta) \to \mathrm{Cov}(X;\xi)$ such that $\wp_\xi \circ \tilde{g} = g \circ \wp_\zeta$. In particular, if g can be taken as the identity automorphism of X then ζ and ξ are said to be *equivalent*. Note that the derived covering graphs of isomorphic voltage assignments are isomorphic as graphs.

§7 Voltage assignments defined on the fundamental group Let b_0 be a vertex of a graph X, let $\pi(X,b_0)$ denote its fundamental group, and let $\zeta\colon D \to N$ be a voltage assignment on X. The extension of ζ to walks induces a group homomorphism $\zeta^*\colon \pi(X,b_0) \to N$.

Note that if $\zeta^* = \xi^*$ then the voltage assignments ζ, ξ are equivalent. Also, for an arbitrary group homomorphism $\eta\colon \pi(X, b_0) \to N$ there exists an assignment $\zeta\colon D \to N$ with $\zeta^* = \eta$. When convenient we can therefore consider voltage assignments (up to equivalence) as defined on a fundamental group. By abuse of notation we often write ζ for ζ^*. Observe that $\mathrm{Cov}(X;\zeta)$ is connected if and only of ζ^* is surjective. For the rest of this article we will restrict ourselves to connected covering graphs.

§8 Abelian voltage assignments If the voltage group N is abelian, then every group homomorphism $\zeta\colon \pi(X, b_0) \to N$ can be factored through the abelianisation of $\pi(X, b_0)$, that is, through the first homology group $H_1(H;\mathbb{Z})$. Further, if p is a prime and $N \cong \mathbb{Z}_p^k$ is an elementary abelian group, then the group homomorphism ζ induces a $\mathbb{Z}_p$-linear transformation $H_1(X;\mathbb{Z}_p) \to N$. In this case, the voltage assignment can be viewed as being defined on the $\mathbb{Z}_p$-vector space $H_1(X;\mathbb{Z}_p)$. In particular, for $p = 2$ the vector space $H_1(X;\mathbb{Z}_p)$ coincides with the cycle space $\mathcal{C}(X)$, and the voltage assignment $\mathcal{C}(X) \to N$ is then uniquely determined by the images of the elements of a basis of $\mathcal{C}(X)$.

§9 Lifting automorphisms Let $\wp_\zeta\colon \mathrm{Cov}(X;\zeta) \to X$ be the derived covering projection, and let $g \in \mathrm{Aut}\, X$ and $\tilde{g} \in \mathrm{Aut}\,\mathrm{Cov}(X;\zeta)$ be such that $\wp_\zeta \tilde{g} = g \wp_\zeta$. Then we say that g *lifts along* $\wp_\zeta$ to $\tilde{g}$, and, that $\tilde{g}$ *projects along* $\wp_\zeta$ to g. We also say that $\wp_\zeta$ (respectively ζ) is *g-admissible* in this case. More generally, if $G \le \mathrm{Aut}\, X$ then $\wp_\zeta$ (respectively ζ) is G-admissible if it is g-admissible for each $g \in G$. It can be shown that an automorphism $g \in \mathrm{Aut}\, X$ lifts along $\wp_\zeta$ if and only if it maps any closed walk with trivial voltage to a closed walk with trivial voltage. In particular, if $N \cong \mathbb{Z}_p^k$ is elementary abelian and if the voltage assignment $\zeta\colon H_1(X;\mathbb{Z}_p) \to N$ is viewed as a $\mathbb{Z}_p$-linear mapping, then g lifts along $\wp_\zeta$ if and only if $\ker\zeta$ is an invariant subspace or the induced action of g on $H_1(X;\mathbb{Z}_p)$.

§10 The induced linear representation # Let $\zeta\colon H_1(X;\mathbb{Z}_p) \to N$ be a voltage assignment on X, where $N \cong \mathbb{Z}_p^k$. Suppose that $g \in \mathrm{Aut}\, X$ lifts along the derived covering projection $\wp_\zeta\colon \mathrm{Cov}(X;\zeta) \to X$. By §9 the kernel $\ker\zeta$ is an invariant subspace for the induced action of g on $H_1(X;\mathbb{Z}_p)$. Since $N \cong H_1(X;\mathbb{Z}_p)/\ker\zeta$, there exists an induced automorphism $g^\# \in \mathrm{Aut}\, N$ defined by $g^\#(\zeta(C)) = \zeta(g(C))$, for every $C \in H_1(X;\mathbb{Z}_p)$. Moreover, it can be shown that if $\wp_\zeta$ is G-admissible, then the mapping $\#\colon g \mapsto g^\#$ is a group homomorphism from G onto a subgroup $G^\#$ of $\mathrm{Aut}\, N$.

§11 Lifting a group Let $\zeta\colon \pi(X;b) \to N$ be a G-admissible voltage assignment on X. Then the collection of lifts of all elements $g \in G$ constitutes a subgroup of $\mathrm{Aut}\,\mathrm{Cov}(X;\zeta)$ called the *lift* of G and denoted by $\tilde{G}$. The lift of the trivial group of automorphisms is known as the *group of covering transformations* or *self-equivalences* of $\wp_\zeta$, and is also denoted by $\mathrm{CT}(\wp_\zeta)$. Since $\mathrm{Cov}(X;\zeta)$ is assumed to be connected, $\mathrm{CT}(\wp_\zeta)$ acts regularly on each *fibre* of the form $\wp_\zeta^{-1}(v)$ for $v \in V$, or $\wp_\zeta^{-1}(x)$ for $x \in D$, and can be identified with the left regular representation of N on itself. The group $\mathrm{CT}(\wp_\zeta)$ is a normal subgroup of $\tilde{G}$, with each coset being the set of all lifts of one element of G. Thus, $\tilde{G}/\mathrm{CT}(\wp_\zeta) \cong G$.

The structure of $\tilde{G}$, viewed as an extension of $\mathrm{CT}(\wp_\zeta)$ by G, is difficult to analyse. In §12 we consider a very special case.

§12 The structure of the lifted group Let $\zeta\colon H_1(X;\mathbb{Z}_p) \to N$ be a G-admissible voltage assignment on a graph X, where $N \cong \mathbb{Z}_p^k$. Let Ω be the G-orbit of a base vertex b_0, and let π^Ω denote the set of all walks in X with end-vertices in Ω. Suppose further that the elements in G (or just the generators of G) map the trivial voltage walks from π^Ω to trivial voltage walks. Then the lifted group is isomorphic to the semidirect product $\tilde{G} \to N \rtimes_\# G$, where $\#\colon G \to \mathrm{Aut}\, N$ is the homomorphism defined in §10. This isomorphism is given by the rule

$$\tilde{g} \mapsto (a, g), \text{ where } a \in N \text{ is defined by } \tilde{g}(b_0, 0) = (g(b_0), a).$$

Multiplication in $N \rtimes_\# G$ is given by the rule

$$(a, g)(b, h) = (a + g^\#(b), gh) \qquad \text{for } a, b \in N,\ g, h \in G,$$

and the action of $N \rtimes_\# G$ on the vertex set $V(X) \times N$ of $\mathrm{Cov}(X;\zeta)$ is given by the rule

$$(a, g)(v, b) = (g(v), a + g^\#(b) + g^\#(\zeta(W_v)) - \zeta(g(W_v))) \quad \text{for } a, b \in N,\ g \in G,\ v \in V(X),$$

where W_v is an arbitrary walk from v to b_0. The subgroup $N \rtimes_\# \{1_G\}$ (isomorphic to N) corresponds to $\mathrm{CT}(\wp_\zeta)$ and the subgroup $\{1_N\} \rtimes_\# G$ (isomorphic to G) corresponds to the complement which maps the vertices above Ω labelled 0 to vertices labelled 0. In particular, if G is a subgroup of the stabilizer of b_0, then G lifts as a semidirect product.

§13 Decomposing a covering projection Let $\zeta\colon H_1(X;\mathbb{Z}_p) \to N$ be a G-admissible voltage assignment on X, where $N \cong \mathbb{Z}_p^k$. Let K be a subgroup of N and $q_K\colon N \to N/K$ the corresponding quotient projection. The following can be shown: If K is invariant for $G^\#$ then the voltage assignment $q_K \circ \zeta\colon H_1(X;\mathbb{Z}_p) \to N/K$ is G-admissible and there exists a covering projection $\wp_K\colon \mathrm{Cov}(X;\zeta) \to \mathrm{Cov}(X;q_K \circ \zeta)$ such that $\wp_\zeta = \wp_{(q_K \circ \zeta)} \circ \wp_K$. Conversely, if ζ' is G-admissible such that there exists $\wp'\colon \mathrm{Cov}(X;\zeta) \to \mathrm{Cov}(X;\zeta')$ with $\wp_\zeta = \wp_{\zeta'} \circ \wp'$, then there exists a $G^\#$-invariant subgroup K in N such that ζ' is equivalent to $q_K \circ \zeta$. The voltage assignments $q_K \circ \zeta$ and $q_L \circ \zeta$ are equivalent if and only if $K = L$. Finally, if $L = \alpha^\#(K)$ for some $\alpha \in \mathrm{Aut}\, X$ which lifts along $\wp_\zeta$, then the voltage assignments $q_K \circ \zeta$ and $q_L \circ \zeta$ are isomorphic.

§14 A $\mathbb{Z}_2^7$-cover of the Heawood graph Let $\mathcal{H}$ be the Heawood graph as described in §3, and let $\mathcal{B}$ be a basis of its cycle space $\mathcal{C}(\mathcal{H})$, consisting of the following cycles:

- the Hamilton cycle $C_H\colon (0, 2', 1, 3', 2, 4', 3, 5', 4, 6', 5, 0', 6, 1', 0)$, and

- the seven 6-cycles $C_i\colon (i-2, i', (i-1), (i+1)', i, (i+2)', i-2)$, for $i \in \mathbb{Z}_7$.

Let N be the additive group of the quotient ring $R = \mathbb{Z}_2[x]/\langle x^7 - 1\rangle$, viewed as a vector space of dimension 7 over $\mathbb{Z}_2$. By abuse of notation, for an arbitrary polynomial $f(x) \in \mathbb{Z}_2[x]$, we will denote the corresponding element $f(x) + \langle x^7 - 1\rangle$ of the quotient ring by $f(x)$. In view of §8, let $\zeta\colon \mathcal{C}(\mathcal{H}) \to N$ be the voltage assignment determined by the rule

$$\zeta(C_H) = 0 \quad \text{and} \quad \zeta(C_i) = x^i \ \text{ for } i \in \mathbb{Z}_7.$$

We shall denote the derived covering projection by $\tilde{\wp}\colon \tilde{\mathcal{H}} \to \mathcal{H}$. If one prefers to view the voltage assignment as being defined on darts, then ζ assigns the trivial element of N to the darts of the cycle C_H, and the element x^i to the dart $(i-2,\ (i+2)')$, for each $i \in \mathbb{Z}_7$.

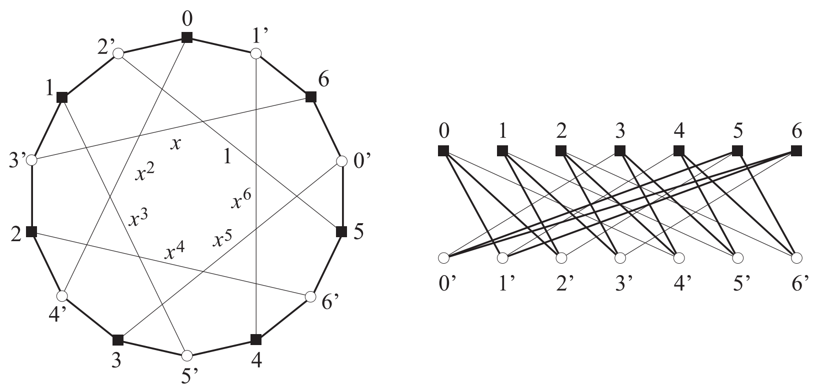

Figure 1: Two drawings of the Heawood graph with the voltage assignment ζ

§15 The lifted subgroup of $\operatorname{Aut}\tilde{\mathcal{H}}$ By §9, an automorphism g of $\mathcal{H}$ lifts along $\tilde{\wp}$ if and only if the kernel of the voltage assignment ζ is invariant for the induced action of g on the cycle space $\mathcal{C}(\mathcal{H})$. In particular, since the kernel $\ker\zeta$ is the 1-dimensional subspace of $\mathcal{C}(\mathcal{H})$ spanned by the Hamilton cycle C_H, an automorphism of $\mathcal{H}$ lifts along $\tilde{\wp}$ if and only if it maps the cycle C_H onto itself. The maximal subgroup of $\operatorname{Aut}\mathcal{H}$ that lifts along $\tilde{\wp}$ is therefore equal to $\operatorname{Aut} C_H \cap \operatorname{Aut}\mathcal{H} = \langle\rho, \tau\rangle = \Gamma_{14}$. By §10, the fact that the automorphisms ρ and τ lift along $\tilde{\wp}$ implies the existence of linear transformations $\rho^\#, \tau^\# \in \operatorname{Aut} N$ with the property that $\rho^\#\zeta = \zeta\rho$ and $\tau^\#\zeta = \zeta\tau$. Explicitly,

$$\rho^\#(f(x)) = xf(x) \quad \text{and} \quad \tau^\#(f(x)) = f(x^{-1}), \qquad \text{whenever } f(x) \in N.$$

Observe that a walk in $\mathcal{H}$ has trivial voltage if and only if it traverses each edge not contained in the Hamilton cycle C_H an even number of times. This property is clearly preserved by any element of Γ_{14}, hence by §12 the lifted group $\tilde{\Gamma}_{14}$ is isomorphic to the semidirect product $N \rtimes_\# \Gamma_{14}$, with multiplication given by the rule

$$(f(x), \rho^i\tau^k)(g(x), \rho^j\tau^l) = (f(x) + x^i g(x^{-k}), \rho^{i+(-1)^k j}\tau^{k+l}),$$

whenever $f(x)$, $g(x) \in N$, $i, j \in \mathbb{Z}_7$ and $k, l \in \mathbb{Z}_2$. The action of $N \rtimes_\# \Gamma_{14}$ on the vertex-set $V(\mathcal{H}) \times N$ of $\tilde{\mathcal{H}}$ is given by the rule

$$(f(x), \rho^i\tau^k)(v, g(x)) = ((\rho^i\tau^k)(v), f(x) + x^i g(x^{-k})),$$

whenever $f(x)$, $g(x) \in N$, $v \in V(\mathcal{H})$, $i \in \mathbb{Z}_7$ and $k \in \mathbb{Z}_2$. Since Γ_{14} acts regularly on vertices of $\mathcal{H}$, the lifted group $\tilde{\Gamma}_{14}$ acts regularly on the vertices of $\tilde{\mathcal{H}}$, and so $\tilde{\mathcal{H}}$ is a Cayley graph for the group $\tilde{\Gamma}_{14}$. Recall from §5 that a generating set for the presentation of $\mathcal{H}$ as a Cayley graph of Γ_{14} may be comprised of $\tau_1 = \rho\tau$, $\tau_2 = \rho^2\tau$ and $\tau_3 = \rho^4\tau$. It follows that the generators of $\tilde{\mathcal{H}}$ as a Cayley graph of $\tilde{\Gamma}_{14}$ are the lifts $\tilde{\tau}_1$, $\tilde{\tau}_2$ and $\tilde{\tau}_3$ satisfying

$$\tilde{\tau}_1(0,0) = (1', 0), \qquad \tilde{\tau}_2(0,0) = (2', 0), \qquad \text{and} \qquad \tilde{\tau}_3(0,0) = (4', x^2),$$

that is, the lifts that map the origin $(0,0) \in V(\tilde{\mathcal{H}})$ to its neighbours. Explicitly,

$$\tilde{\tau}_1 = (0, \tau_1) \in N \rtimes_\# \Gamma_{14}, \quad \tilde{\tau}_2 = (0, \tau_2) \in N \rtimes_\# \Gamma_{14}, \quad \text{and} \quad \tilde{\tau}_3 = (x^2, \tau_1) \in N \rtimes_\# \Gamma_{14}.$$

§16 Decomposing the covering projection $\tilde{\wp}$ Let $\tilde{\wp}\colon \tilde{\mathcal{H}} \to \mathcal{H}$ be the covering projection as in §14. Let us find all $\langle\rho\rangle$-admissible regular covering projections $\wp\colon \tilde{X} \to \mathcal{H}$, up to equivalence, for which there exists a regular covering projection $\wp'\colon \mathcal{H} \to \tilde{X}$ satisfying $\tilde{\wp} = \wp \circ \wp'$. By §13 we have to find all $\rho^\#$-invariant subspaces of the 7-dimensional $\mathbb{Z}_2$-vector space $N = \mathbb{Z}_2[x]/\langle x^7 - 1\rangle$. Recall from §15 that $\rho^\#(f(x)) = xf(x)$ for all $f(x) \in N$. This implies that a subspace of N is $\rho^\#$-invariant if and only if it is closed under multiplication by x in the ring $R = \mathbb{Z}_2[x]/\langle x^7-1\rangle$, that is, if and only if it forms an ideal of R. Since the ideals of R are generated by the divisors of the polynomial $x^7-1 = (1+x)(1+x^2+x^3)(1+x+x^3)$ in $\mathbb{Z}_2[x]$, we obtain exactly six proper non-trivial ideals of R, namely $\langle 1+x\rangle$, $\langle 1+x^2+x^3\rangle$, $\langle 1+x+x^3\rangle$, $\langle(1+x)(1+x^2+x^3)\rangle$, $\langle(1+x)(1+x+x^3)\rangle$ and $\langle(1+x+x^3)(1+x^2+x^3)\rangle$. In view of §13 there are six pairwise non-equivalent regular $\langle\rho\rangle$-admissible covering projections $\wp_{N/K}$ derived from the voltage assignments $q_K \circ \zeta$, where K is one of the above six ideals of R and $q_K\colon N \to N/K$ is the natural quotient projection. Note that the covering projection $\wp_{N/K}$ is derived from the voltage assignment mapping according to the same rule as ζ, except that its images are now computed modulo K instead of modulo $\langle x^7-1\rangle$. Since

$$\tau^\#(\langle 1+x\rangle) = \langle\tau^\#(1+x)\rangle = \langle 1+x^6\rangle = \langle 1+x\rangle, \quad \text{and then also}$$

$$\tau^\#(\langle 1+x+x^3\rangle) = \langle\tau^\#(1+x+x^3)\rangle = \langle 1+x^6+x^4\rangle = \langle 1+x^2+x^3\rangle,$$

it follows that the covering projections associated with $\langle 1+x+x^3\rangle$ and $\langle 1+x^2+x^3\rangle$ are isomorphic. The same holds for those associated with $\langle(1+x)(1+x+x^3)\rangle$ and $\langle(1+x)(1+x^2+x^3)\rangle$. As a final remark, note that $N/K \cong \mathbb{Z}_2^r$, where $r \in \{1,3,4,6\}$ is the degree of the generating polynomial of the ideal K.

§17 The Ljubljana graph as a $\mathbb{Z}_2^3$-cover of the Heawood graph With the assumptions and the notation of §16, let $\wp_{\mathcal{L}}\colon \mathcal{L} \to \mathcal{H}$ be the covering projection arising from the ideal $K = \langle 1 + x^2 + x^3 \rangle$. Recall that the voltage group $\bar{N} = N/K \cong \mathbb{Z}_2^3$ is the additive group of the quotient ring $\mathbb{Z}_2[x]/\langle 1 + x^2 + x^3 \rangle$, and the voltage assignment $\zeta_{\mathcal{L}} = q_K \circ \zeta$ is congruent to ζ modulo $1 + x^2 + x^3$ (see Figure 1). The covering graph $\mathcal{L}$ is cubic, bipartite and has order $14 \cdot 2^3 = 112$.

§18 The maximal group which lifts along $\wp_{\mathcal{L}}\colon \mathcal{L} \to \mathcal{H}$ Let M be the maximal subgroup of $\operatorname{Aut}\mathcal{H}$ which lifts along the covering projection $\wp_{\mathcal{L}}\colon \mathcal{L} \to \mathcal{H}$ as introduced in §17. We know that $\rho \in M$. By §9 an automorphism $g \in \operatorname{Aut}\mathcal{H}$ belongs to M if and only if it preserves the kernel of $\zeta_{\mathcal{L}}\colon \mathcal{C}(\mathcal{H}) \to \mathbb{Z}_2^3$. Clearly, the kernel $\ker \zeta_{\mathcal{L}}$ consists of all elements of $\mathcal{C}(\mathcal{H})$ with ζ-voltages in K, and has dimension 5. Now set $W = C_0 + C_2 + C_3$ and observe that $\zeta(\rho^i(W)) \in K$ for each $i \in \mathbb{Z}_7$. Since $\{C_H, \rho(W), \rho^2(W), \rho^3(W), \rho^3(W), \rho^4(W)\}$ is linearly independent it follows that $\ker \zeta_{\mathcal{L}}$ is spanned by C_H and $\rho^i(W)$, for $i \in \mathbb{Z}_7$. A straightforward calculation gives

$$\sigma(C_H) = \sum_{i \in \mathbb{Z}_7} \rho^i(W) \quad \text{and} \quad \sigma(\rho^i(W)) = \rho^{4i}(\sigma(W)) = C_H + \rho^{4i}(W),$$

which implies that $\sigma \in M$. On the other hand, $\tau(C_i) = C_{-i}$ for $i \in \mathbb{Z}_7$, and so $\zeta_{\mathcal{L}}(\tau(W)) = \zeta_{\mathcal{L}}(C_0 + C_1 + C_3) = 1 + x + x^3 \neq 0$. Similarly, $\zeta_{\mathcal{L}}(\omega(C_H)) = x^2 \neq 0$. Hence $\tau, \omega \notin M$. We now prove that $M = \langle \rho, \sigma \rangle = \Gamma_{21}$. As already shown, $\Gamma_{21} \leq M$, and hence $|M| = 3 \cdot 7 \cdot 2^r$, where $r \in \{0, 1, 2, 3, 4\}$. If $r = 4$ then $M = \operatorname{Aut}\mathcal{H}$, contradicting the fact that $\tau \notin M$. If $r = 3$ then M is a normal subgroup of index 2 in $\operatorname{Aut}\mathcal{H}$ and $M \cap \operatorname{Aut}_0\mathcal{H}$ is normal in $\operatorname{Aut}_0\mathcal{H} \cong PSL(2,7)$. Since this intersection is non-trivial and since $PSL(2,7)$ is simple, we find $M = \operatorname{Aut}_0\mathcal{H}$, contradicting the fact that $\omega \notin M$. Let k be the number of Sylow 7-subgroups of M. By Sylow's theorem, $k \equiv 1 \bmod 7$, and k divides $3 \cdot 2^r$. If $k > 1$, then we have $r \geq 3$, a contradiction. This shows that M is contained in the normalizer $N_{\operatorname{Aut}\mathcal{H}}(\rho)$ of $\langle \rho \rangle$ in $\operatorname{Aut}\mathcal{H}$. Next recall from §4 that $N_{\operatorname{Aut}\mathcal{H}}(\rho) = \langle \rho, \sigma, \tau \rangle = \Gamma_{42}$. Since $\tau \notin M$, the group M is a proper subgroup of Γ_{42}, and so $M = \Gamma_{21}$, as required.

§19 The girth of $\mathcal{L}$ We show that the girth of $\mathcal{L}$ is 10, and that there are exactly $3 \cdot 7 \cdot 2^3 = 168$ cycles of length 10. To this end suppose that C is a cycle in $\mathcal{L}$ of length $n \leq 10$, and let $C' = \wp_{\mathcal{L}}(C)$ be its projection. Since the girth of the Heawood graph $\mathcal{H}$ is 6, C' is also a cycle of length 6, 8 or 10. As the arcs of $\mathcal{H}$ with trivial voltages induce a Hamilton cycle (of length 14), C' contains at least one arc with non-trivial voltage. Now besides the zero polynomial 0 and the cyclotomic polynomial $1 + x + \ldots + x^6$, the ideal $\langle 1 + x^2 + x^3 \rangle$ of $\mathbb{Z}_2[x]$ contains seven elements of the form $x^i + x^{i+2} + x^{i+3}$ and seven elements of the form $x^i + x^{i+3} + x^{i+4} + x^{i+5}$, where $i \in \mathbb{Z}_7$. Since the sum of voltages of edges of C' is the trivial element of the ring $\mathbb{Z}_2[x]/\langle 1 + x^2 + x^3 \rangle$, it follows that there is some $i \in \mathbb{Z}_7$ such that the set of the non-trivial voltages of the edges of C' is either $\{x^i, x^{i+2}, x^{i+3}\}$ or $\{x^i, x^{i+3}, x^{i+4}, x^{i+5}\}$. A closer inspection of the cycle structure of $\mathcal{H}$ then reveals that the length of C' is 10, and moreover, that it belongs to one of the three orbits of the automorphism ρ containing the cycles $A_1 = (0, 4', 3, 5', 1, 2', 5, 0', 6, 1', 0)$, $A_2 = (0, 2', 5, 6', 4, 5', 1, 3', 2, 4', 0)$ and $B = (1, 5', 4, 6', 2, 4', 3, 0', 5, 2', 1)$. Since each of these cycles lifts to 2^3 disjoint 10-cycles in $\mathcal{L}$ and each of these three ρ-orbits contains seven cycles, the total number of 10-cycles in $\mathcal{L}$ is $3 \cdot 7 \cdot 2^3 = 168$.

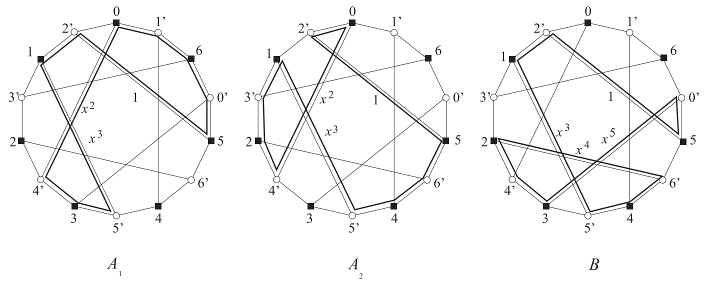

Figure 2: The three types of 10-cycles of $\mathcal{H}$ and $\mathcal{L}$

§20 Cycles of length 12 **in** $\mathcal{L}$ Let C be a cycle in $\mathcal{L}$ of length 12, and let $C' = \wp_{\mathcal{L}}(C)$ be its projection. As the girth of $\mathcal{H}$ is 6, the graph C' can be either a 6-cycle or a 12-cycle with trivial voltage. Observe that there are four ρ-orbits of 6-cycles in $\mathcal{H}$, with the following respective representatives:

$D_1 = (0, 2', 1, 3', 2, 4', 0)$,

$D_2 = (0, 4', 2, 6', 4, 1', 0)$,

$D_3 = (0, 1', 4, 5', 1, 2', 0)$,

$D_4 = (0, 2', 5, 6', 4, 1', 0)$.

Observe further that each of these cycles lifts to exactly four 12-cycles in $\mathcal{L}$, and that if C' is a 12-cycle then the voltage of C' must be trivial. By a similar argument as used in §19, we can show that all 12-cycles with trivial voltage in $\mathcal{H}$ belong to the same ρ-orbit, with representative

$E = (0, 4', 2, 3', 1, 2', 5, 0', 3, 5', 4, 1', 0)$.

A lift of a cycle in the ρ-orbits of D_i (or E) will be referred to as a cycle of type D_i (or

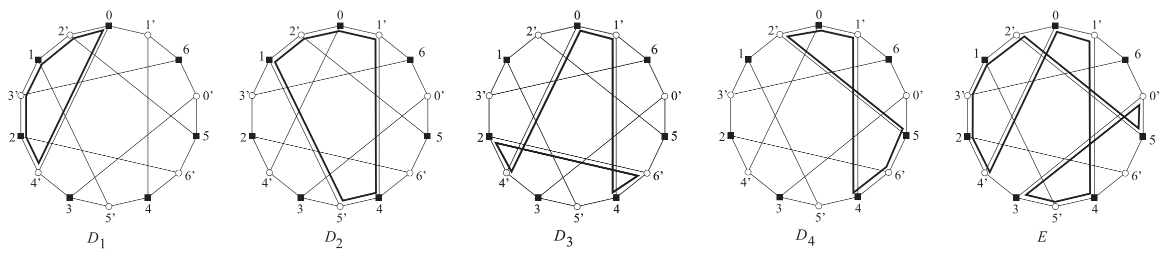

Figure 3: The five types of 12-cycles of $\mathcal{L}$

of type E, respectively). Since each 6-cycle of type D_i lifts to four 12-cycles while each

12-cycle of type E lifts to eight 12-cycles, it follows that there are $(4 \cdot 4 + 8) \cdot 7 = 168$ cycles of length 12 in the Ljubljana graph $\mathcal{L}$.

§21 Blocks of imprimitivity of $\mathcal{L}$ For a vertex u of $\mathcal{H}$, let the *fibre of* u be the set of eight vertices of $\mathcal{L}$ which project by $\wp_{\mathcal{L}}$ to u. In this Paragraph we prove that fibres are blocks of imprimitivity of the action of $\mathrm{Aut}\,\mathcal{L}$ on the vertices of $\mathcal{L}$. First observe that two antipodal vertices of a cycle of type E lie on a common 10 cycle of $\mathcal{L}$. (It suffices to check this for lifts of E; for example the lifts of vertices 0 and 5 lie on a lift of the 10 cycle $\rho(B)$.) Next observe that the 12-cycles of types D_1, D_2 and D_3 do not have this property. This implies that there are no automorphisms of $\mathcal{L}$ mapping a 12-cycle of type E to a 12-cycle of type D_1, D_2 or D_3. Let us now define an equivalence relation R on the vertex-set of $\mathcal{L}$ by the rule: $u\,R\,v$ if and only if $u = v$ or u and v are antipodal vertices of a 12-cycle of type D_i, for $i \in \{1, 2, 3\}$. The above-mentioned fact about 10-cycles shows that R is an $(\mathrm{Aut}\,\mathcal{L})$-congruence, which implies that the equivalence classes of R are blocks of imprimitivity for $\mathrm{Aut}\,\mathcal{L}$. As it is easy to see that the equivalence classes of R are exactly the fibres of the covering projection $\wp_{\mathcal{L}} \colon \mathcal{L} \to \mathcal{H}$, our claim follows.

§22 The structure and the action of $\mathrm{Aut}\,\mathcal{L}$ By §21, the fibres of $\wp_{\mathcal{L}} \colon \mathcal{L} \to \mathcal{H}$ are blocks of imprimitivity of $\mathrm{Aut}\,\mathcal{L}$, and so $\mathrm{Aut}\,\mathcal{L}$ projects along $\wp_{\mathcal{L}}$. By §18, the maximal group that lifts along $\wp_{\mathcal{L}}$ is the metacyclic group $\Gamma_{21} = \langle \rho, \sigma \rangle$. Hence $\mathrm{Aut}\,\mathcal{L} = \tilde{\Gamma}_{21}$ is an extension by Γ_{21} of the additive group $\bar{N}$ of the quotient ring $\mathbb{Z}_2[x]/\langle 1 + x^2 + x^3 \rangle$. As $|\tilde{\Gamma}_{21}| = 21$ and $\bar{N} \cong \mathbb{Z}_2^3$, the group $\mathrm{Aut}\,\mathcal{L}$ has order $2^3 \cdot 3 \cdot 7 = 168$. Moreover, since Γ_{21} acts edge- but not vertex-transitively on $\mathcal{H}$, the Ljubljana graph is semisymmetric. Next recall that there is a natural embedding $\iota \colon \bar{N} \to \mathrm{Aut}\,\mathcal{L}$, given by $\iota(a)(v, b) = (v, a + b)$, which maps $\bar{N}$ isomorphically onto the group of covering transformations. Clearly the extension $\bar{N} \to \mathrm{Aut}\,\mathcal{L} \to \Gamma_{21}$ splits, by the Zassenhaus lemma [12]. For the the sake of completeness we give a self contained proof that $\mathrm{Aut}\,\mathcal{L} \cong \bar{N} \rtimes_{\#} \Gamma_{21}$, and give its explicit action on vertex set $V(\mathcal{H}) \times \bar{N}$ of $\mathcal{L}$. First, by §10 there exists a homomorphism $\# \colon \Gamma_{21} \to \mathrm{Aut}\,\bar{N}$ satisfying $g^{\#}\zeta_{\mathcal{L}} = \zeta_{\mathcal{L}} g$ for every $g \in \Gamma_{21}$. An explicit calculation involving the generators C_i defined in §14 for $\mathcal{C}(\mathcal{H})$ shows that

$$\rho^{\#}(f(x)) = xf(x) \qquad \text{and} \qquad \sigma^{\#}(f(x)) = x^6 f(x^2) \qquad \text{whenever } f(x) \in \bar{N}.$$

Let $\tilde{\rho}$ and $\tilde{\sigma}$ be the respective lifts of ρ and σ mapping the vertex $(0, 0) \in V(\mathcal{H}) \times \bar{N}$ to vertices $(1, 0)$ and $(0, x^6)$, respectively. In view of §12, the actions of the automorphisms $\tilde{\rho}, \tilde{\sigma} \in \tilde{\Gamma}_{21}$ on $V(\mathcal{H}) \times \bar{N}$ are given by the rules

$$\tilde{\rho}(v, f(x)) = (\rho(v), xf(x)) \quad \text{and} \quad \tilde{\sigma}(v, f(x)) = (\sigma(v), x^6 + x^6 f(x^2) + \zeta_{\mathcal{L}}(\sigma(W_v))),$$

where W_v is a trivial voltage walk in $\mathcal{H}$ from v to 0. To calculate the values $\zeta_{\mathcal{L}}(\sigma(W_v))$ we can choose the walks W_v to be contained in the Hamilton cycle C_H. Since the automorphism σ maps C_H to the cycle $(0, 4', 2, 6', 4, 1', 6, 3', 1, 5', 3, 0', 5, 2', 0)$, the values $\zeta_{\mathcal{L}}(\sigma(W_v))$

are partials sums of the sequence $x^2, x^4, x^6, x^8, x^{10}, x^{12}, x^{14}(=1)$. More precisely,

$$\zeta_{\mathcal{L}}(\sigma(W_s)) = \zeta_{\mathcal{L}}(\sigma(W_{(s+1)'})) = x^2 + x^4 + \ldots + x^{2s} = x^6(1 + x^{2s}), \quad \text{for } s \in \mathbb{Z}_7.$$

Since $\sigma\rho\sigma^{-1} = \rho^2$ it follows that there exists an element $a \in \bar{N}$ such that $\tilde{\sigma}\tilde{\rho}\tilde{\sigma}^{-1} = \iota(a)\tilde{\rho}^2$. The element a can be computed using the equality $\tilde{\sigma}\tilde{\rho} = \iota(a)\tilde{\rho}^2\tilde{\sigma}$ at the vertex $(0,0)$:

First $\qquad \tilde{\sigma}(\tilde{\rho}(0,0)) = \tilde{\sigma}(1,0) = (2, x^6 + x^6 \cdot 0 + x^6(1+x^2)) = (2, x),$

and second $\qquad \iota(a)(\tilde{\rho}^2(\tilde{\sigma}(0,0))) = \iota(a)(\tilde{\rho}^2(0, x^6)) = \iota(a)(2, x).$

This shows that $a = 0$, and so $\tilde{\sigma}$ normalizes $\tilde{\rho}$. Consequently, the group $\langle \tilde{\rho}, \tilde{\sigma} \rangle$ is isomorphic to Γ_{21} and is a complement of $\iota(\bar{N}) \le \tilde{\Gamma}_{21}$. The automorphism group $\operatorname{Aut}\mathcal{L}$ is therefore an internal semidirect product $\iota(\bar{N}) \rtimes \langle \tilde{\rho}, \tilde{\sigma} \rangle$, isomorphic also to the external semidirect product $\bar{N} \rtimes_{\#} \langle \rho, \sigma \rangle$ with multiplication given by the rule

$$(f(x), \rho^i\sigma^k)(g(x), \rho^j\sigma^l) = (f(x) + x^{-2^k+i+1}g(x^{2^k}), \rho^{i+2^k j}\sigma^{k+l}).$$

§23 Hamiltonicity of $\mathcal{L}$ and its LCF code The LCF code [6] of a hamiltonian cubic graph relative to one of its Hamilton cycles $(v_0, v_1, v_2, \ldots, v_{n-1}, v_0)$ is a list $\mathtt{LCF}[a_0, a_1, \ldots, a_{n-1}]$ of elements of $\mathbb{Z}_n \setminus \{0, 1, n-1\}$ such that v_i is adjacent to v_{i+a_i} for every $i \in \mathbb{Z}_n$. In addition, if there exists a proper divisor r of n such that $a_i = a_{i+kr}$ for all $i \in \{0, 1, \ldots, r-1\}$ and $k \in \{1, 2, \ldots, \frac{n}{r}-1\}$, then the notation can be simplified to $\mathtt{LCF}[a_0, a_1, \ldots, a_{r-1}]^{\frac{n}{r}}$. In this case we say the LCF code is *periodic*, and call the integers r and $\frac{n}{r}$ the *period* and the *exponent* of the LCF code, respectively. When searching for a Hamilton cycle of $\mathcal{L}$ with a periodic LCF code, the following observations are immediate. First, the non-identity elements of $\operatorname{Aut}\mathcal{L}$ have orders 2, 3, 6 and 7. In particular, $\operatorname{Aut}\mathcal{L}$ contains no semiregular elements of order 3 or 6 (semiregular meaning that all cycles of the permutation have the same length), and therefore there is no LCF code of $\mathcal{L}$ with exponent 3 or 6. On the other hand, $\operatorname{Aut}\mathcal{L}$ does contain semiregular elements of orders 2 and 7. All elements of order 7 are conjugate however, and as is easily seen from Figure 6, the quotient graph of $\mathcal{L}$ relative to an element of order 7 is not hamiltonian. Consequently, $\mathcal{L}$ has no LCF code of exponent 7. The only remaining possibilities for the LCF codes of $\mathcal{L}$ are therefore an aperiodic LCF code, corresponding to a Hamilton cycle which is fixed by no automorphism of $\mathcal{L}$, and an LCF code of exponent 2, corresponding to a Hamilton cycle fixed by a semiregular involution, that is, an element in the group of covering transformations $\mathbb{Z}_2^3$. It transpires that both possibilities do occur. The following is an aperiodic LCF code for $\mathcal{L}$, which gives rise to the first drawing of $\mathcal{L}$ given in Figure 4:

$\mathtt{LCF}[11, 55, -23, 31, 11, -9, 55, 17, 39, -23, 31, -11, 9, -31, -23, -11, -53, 31, -47, 35,$
$11, -9, 55, 13, -17, -45, 17, 47, -29, -39, -53, -11, 13, 39, -31, 49, -13, 21, -55, 49,$
$13, -31, 25, -17, 35, -13, 11, -39, -31, -51, 21, 29, 49, -13, -35, 51, -55, -11, -21, 9,$
$21, -55, 11, 23, 29, 35, 43, -25, -9, 21, 35, -21, -39, -11, -47, 53, 23, -55, 9, -35,$

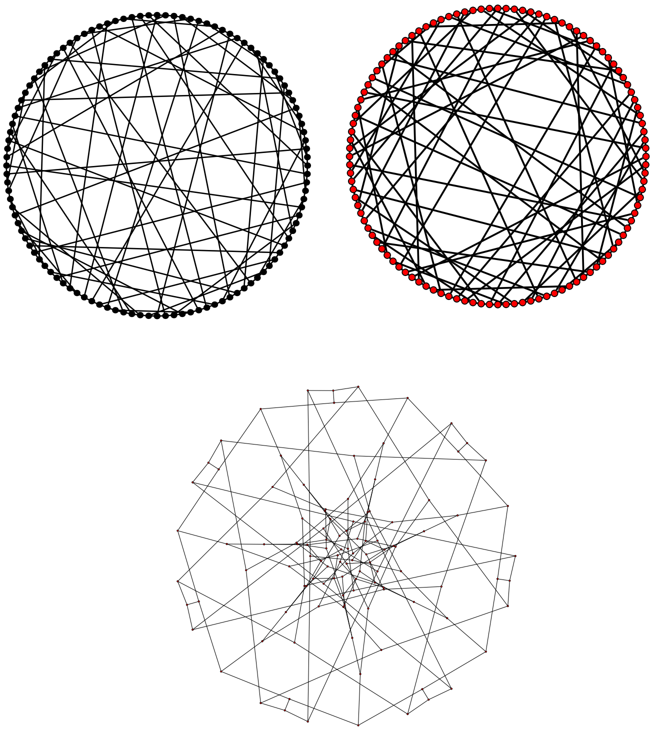

Figure 4: Three drawings of the Ljubljana graph: the first one exhibiting an asymmetric Hamilton cycle, the second exhibiting a Hamilton cycle with central symmetry, and the third exhibiting the existence of a semiregular automorphism of order 7

$$-29,-21,15,47,-49,19,-23,-9,-49,53,-21,23,45,-29,31,55,11,-15,23,-23,\\ -35,-49,39,23,-19,-35,-51,-11,9,-43,51,29]$$

On the other hand, the LCF code given below has exponent 2, and therefore gives rise to a central symmetry of $\mathcal{L}$ as shown in the second drawing in Figure 4:

$$\mathtt{LCF}[47,-23,-31,39,25,-21,-31,-41,25,15,29,-41,-19,15,-49,33,39,-35,-21,\\ 17,-33,49,41,31,-15,-29,41,31,-15,-25,21,31,-51,-25,23,9,-17,51,35,-29,\\ 21,-51,-39,33,-9,-51,51,-47,-33,19,51,-21,29,21,-31,-39]^2$$

§24 A dual pair of Ljubljana configurations A (v_3)-*configuration* is an incidence structure having v points and v lines with the following properties: each line contains exactly 3 points, each point belongs to exactly 3 lines, and any two lines intersect in at most one point. Such a configuration determines a cubic bipartite graph with a black and white vertex colouring, where black vertices correspond to points and white vertices to lines of the configuration. Further, two vertices are adjacent if and only if they correspond to an incident point-line pair. Such a graph is called the *Levi graph* of the configuration (see

[4, 5]). Conversely, each cubic bipartite graphon $2v$ vertices of girth at least 6 is the Levi graph of a dual pair of (v_3)-configurations, and these two configurations are isomorphic if and only if there is an automorphism of their Levi graph which interchanges the two bipartite sets. Since $\mathcal{L}$ is bipartite of girth 10 and semisymmetric it gives rise to a dual pair of non-isomorphic quadrangle-free configurations, shown in Figure 5. Observe that the two

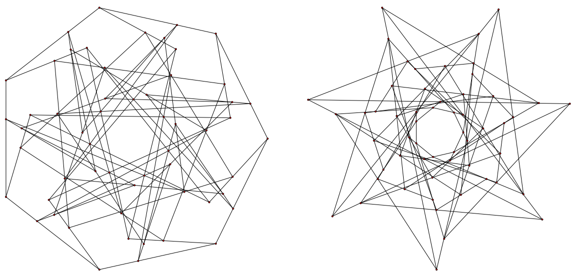

Figure 5: The Ljubljana and the Dual Ljubljana (56_3) configurations

drawings exhibit a rotational symmetry of order 7. Each was produced by applying theory of polycyclic configurations as outlined in [4]. The reason that such theory can be used in this context resides in the fact that $\mathcal{L}$ is a $\mathbb{Z}_7$-cover of a bipartite base graph (see Figure 6). Indeed as we have seen, $\mathcal{L}$ has a semiregular automorphism of order 7, and the quotient graph with respect to this is bipartite.

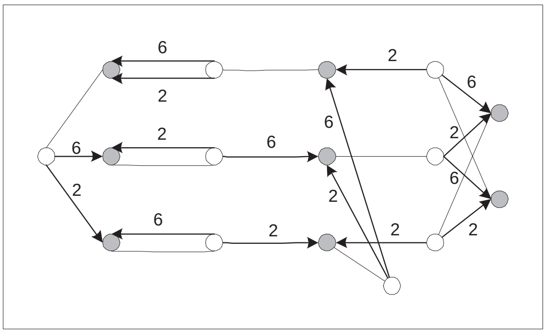

Figure 6: The Ljubljana graph $\mathcal{L}$ is a $\mathbb{Z}_7$ cover over this graph

§25 Some further properties of the Ljubljana graph Here we gather some further interesting properties of $\mathcal{L}$. First, if one wants to check that the Ljubljana graph is not vertex-transitive it suffices to show that there exist two vertices with different distance sequences (the sequences which give the numbers of vertices at successive distances from a given vertex). A direct computation shows that the vertices in one set of the bipartition have have distance sequence $(1, 3, 6, 12, 24, 34, 24, 7, 1)$, while the vertices in the other bipartition set have distance sequence $(1, 3, 6, 12, 24, 34, 25, 7)$.

Next, since the automorphism group of $\mathcal{L}$ has order $2^3.3.7 = 168$, and $\mathcal{L}$ is edge-transitive with 168 edges, it follows that the line graph $L(\mathcal{L})$ is a graphical regular representation (GRR) for the group $\mathrm{Aut}\,\mathcal{L}$. More precisely, $L(\mathcal{L})$ is a Cayley graph for the group of order 168 with presentation $\langle\, x, y \mid x^3 = y^3 = xyxy^{-1}xyx^{-1}y^{-1}x^{-1}y^{-1} = 1 \,\rangle$.

Finally, for a graph X let $X^{(2)}$ denote the graph on the same vertex-set as X, and with two vertices adjacent in $X^{(2)}$ if and only if they are at distance 2 in X. If X is connected and bipartite the graph $X^{(2)}$ has exactly two connected components, $X_1^{(2)}$ and $X_2^{(2)}$, corresponding to the two sets of the bipartition. In the case where X is the Levi graph of a pair of dual configurations $\mathcal{C}$ and $\mathcal{C}^d$, the graphs $M(\mathcal{C}) = X_1^{(2)}$ and $M(\mathcal{C}^d) = X_2^{(2)}$ are called the *Menger graphs* of $\mathcal{C}$ and $\mathcal{C}^d$, respectively.

The Menger graphs associated with the pair of Ljubljana configurations are non-isomorphic $\frac{1}{2}$-arc-transitive Cayley graphs for the group $\tilde{\Gamma}_7 \cong \mathbb{Z}_2^3 \rtimes \mathbb{Z}_7$ of order 56 (the lift of the cyclic group $\langle\rho\rangle$ as described in §22), and with automorphism group isomorphic to $\mathrm{Aut}\,\mathcal{L}$. These two graphs may be represented as Cayley graphs $\mathrm{Cay}(\tilde{\Gamma}_7, \{t, t^{-1}, t^x, (t^{-1})^x, t^{x^2}, (t^{-1})^{x^2}\})$ and $\mathrm{Cay}(\tilde{\Gamma}_7, \{t, t^{-1}, t^y, (t^{-1})^y, t^{y^2}, (t^{-1})^{y^2}\})$, where x and y are the two elements of order 3 introduced above as generators for $\mathrm{Aut}\,\mathcal{L}$, while $t = xy^{-1}$. In each of these two Menger graphs, every vertex belongs to three triangles, which are cyclically permuted by the corresponding vertex stabilizer in the automorphism group. Also since $t^x = t^y = y^{-1}x$, the two graphs contain a common 4-valent GRR of $\tilde{\Gamma}_7$, namely the graph $\mathrm{Cay}(\tilde{\Gamma}_7, \{t, t^{-1}, t^x, (t^{-1})^x\})$.

§26 Acknowledgements The authors wish to thank Marko Boben for constructing the drawings of the configurations in Figure 5, and to acknowledge the use of the MAGMA system [2] in determining some of the properties of the Ljubljana graph $\mathcal{L}$ decribed in this paper.

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