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THE GENUS OF THE GRAY GRAPH IS 7

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Dedicated to Arthur T. White for his work on the threefold connections between graphs, groups and surfaces

Abstract

Using the genus embedding of the Cartesian product of three triangles we prove one can embed the smallest cubic semisymmetric graph on 54 vertices, the so-called Gray graph, in the orientable surface of genus 7, and prove that such an embedding is optimal.

§ 1. Introduction. Using an old result on the genus of the Cartesian product of three triangles, compare [19, 6, 7], we prove that the genus of the Gray graph is indeed 7.

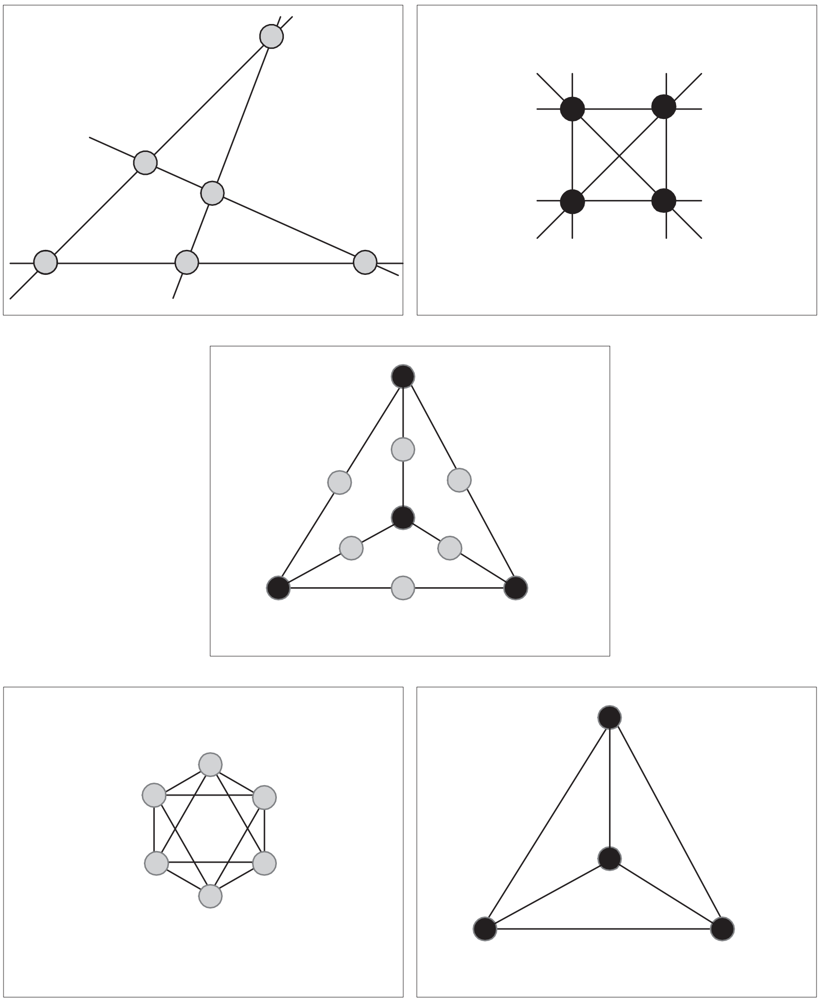

Figure 1: The Pasch configuration, its dual, their Levi graph and their Menger graphs

§ 2. Incidence Structures and Configurations. An *incidence structure* $\mathcal{C}$ is a triple $\mathcal{C} = (P, L, I)$ where P is the set of points, L is the set of lines, and $I \subseteq P \times L$ is the incidence relation. The elements of I are called *flags*. The bipartite incidence graph $G(\mathcal{C})$ with black vertices P, white vertices L and edges I is known as the *Levi graph* of $\mathcal{C}$. We form the *Menger graph* of the configuration $\mathcal{C}$; its vertices are the points and two vertices are joined by en edge when the points belong to the same line of $\mathcal{C}$ [8]. A (v_r, b_k) configuration is an incidence structure $\mathcal{C} = (P, L, I)$ in which two lines meet in at most one point such that $v = |P|, b = |L|$, there are r lines through a point and there are k points on a line. If follows easily that $vr = |I| = bk$. Note: the Levi graph of a (v_r, b_k) configuration is semiregular of girth ≥ 6. A (v_r, b_k) configuration is *symmetric* if $v = b$ (which is equivalent to saying that $r = k$). We will call a (v_k, b_k) configuration a (v_k) configuration.

§ 3. Duality. For each incidence structure $\mathcal{C} = (P, L, I)$ the *dual structure* is $\mathcal{C}^d = (P^d, L^d, I^d)$ where $P^d = L, L^d = P, I^d = I)$. Both $\mathcal{C}$ and $\mathcal{C}^d$ share the same Levi graph except that the black-white coloring of vertices is reversed. The Menger graph of $\mathcal{C}^d$ is known as the *dual Menger graph* of $\mathcal{C}$. For example, Figure 1 shows the *Pasch configuration* $(6_2, 4_3)$ which is also known as *complete quadrilateral* and its dual the *complete quadrangle* $(4_3, 6_2)$, their shared Levi graph and their Menger graphs.

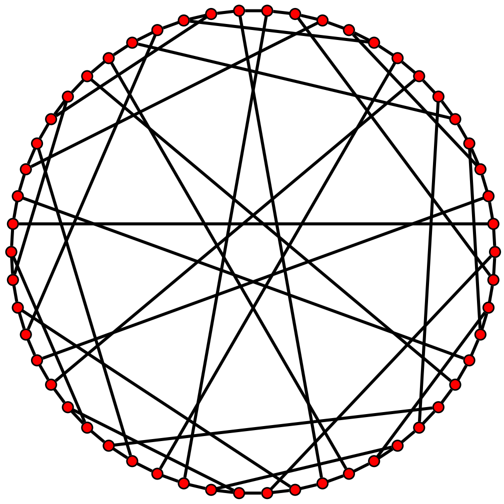

Figure 2: The Gray graph as drawn by Milan Randić.

§ 4. Automorphisms and Anti-Automorphisms. If $\mathcal{C}$ is isomorphic to its dual $\mathcal{C}^d$, we say that it is self-dual and the isomorphism is called a duality. A duality of order 2 is called a polarity. An isomorphism of $\mathcal{C}$ to itself is called an automorphism or colinearity. Automorphisms of $\mathcal{C}$ form a group denoted by $\mathrm{Aut}_0\mathcal{C}$. We may consider automorphisms and dualities (anti-automorphisms) as acting on the disjoint union $P \cup L$. They together form the extended group of automorphisms $\mathrm{Aut}\mathcal{C}$. The Levi graph L of a configuration $\mathcal{C}$ is bipartite and caries complete information about configuration. The automorphism group $\mathrm{Aut}\mathcal{C}$ coincides with the automorphism group of L while $\mathrm{Aut}_0\mathcal{C}$ is the subgroup which fixes the two bipartition sets setwise.

§ 5. The Gray Graph and the Gray Configuration. The smallest known cubic edge- but not vertex-transitive graph has 54 vertices and is known as the Gray graph, which we will call G in this paper [3, 4]. It is shown in Figure 2. Since its girth is 8, it is the Levi graph of two dual, triangle-free, point-, line- and flag-transitive, non-self-dual (27_3)-configurations [17], and this pair is the smallest such. Cyclic drawings of these two

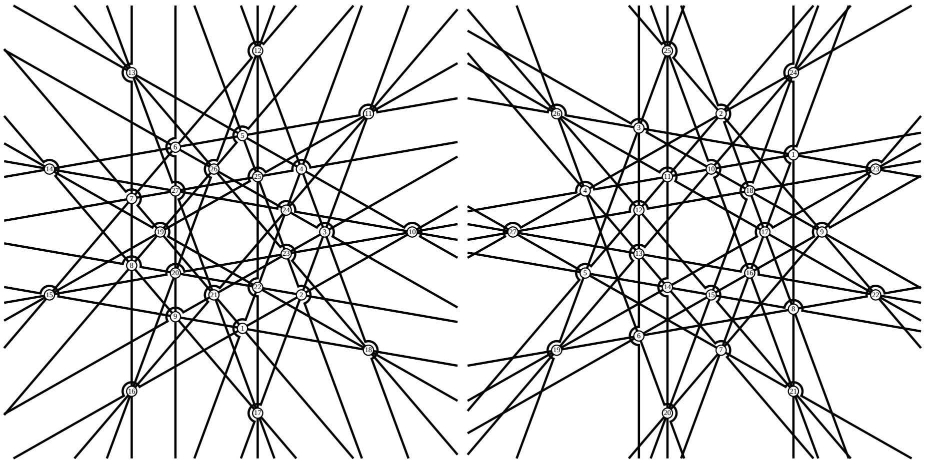

Figure 3: Dual configurations arising from the Gray graph. Both drawings contain false incidences that are clearly marked. However, if we admit all incidences, we may view these figures as a pair of dual (27_4) configurations.

configurations are shown in Figure 3; see [2] for the theoretical background. These drawings illustrate a problem in straight-line realizations of configurations, that sometimes drawings contain false incidences. Following [5] we realize in Figure 4 the first of these two as a cube-shaped configuration of 27 points in 27 lines in $\mathbb{R}^3$ and we will refer to this configuration as the Gray configuration. Using this drawing it becomes clear that the Menger graph M of the Gray configuration, shown in Figure 5(a), is isomorphic to $K_3 \square K_3 \square K_3$. (Here we use the notation $\square$ of [12] to represent the Cartesian product of graphs.)

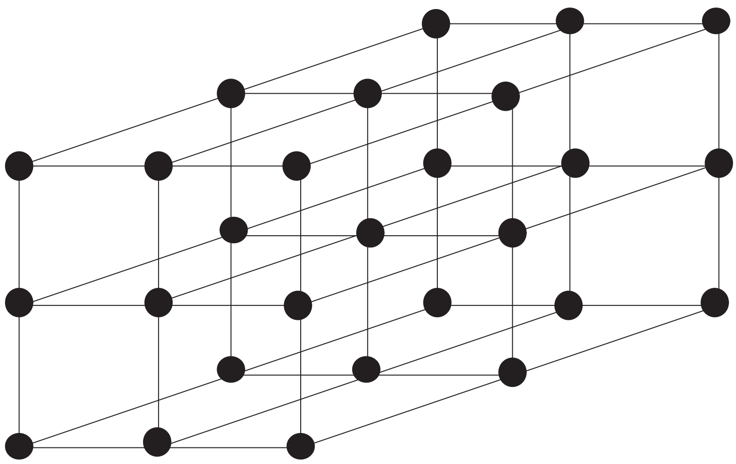

Figure 4: Spatial version of the Gray configuration.

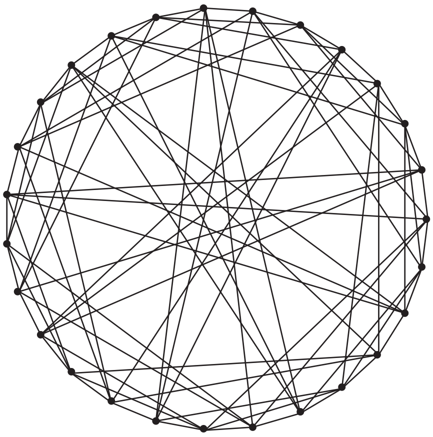

Figure 5: Menger graph M and dual Menger graph D of the Gray configuration.

§ 6. The Genus Embedding. Let $\gamma(G)$ denote the genus of the graph G. This parameter denotes the least integer k, such that G admits an embedding into an orientable surface of genus k. Several years ago it was shown that $\gamma(K_3 \square K_3 \square K_3) = 7$. The genus embedding was constructed by Mohar, Pisanski, Škoviera and White [19]. One nice feature of the genus embedding is that its dual is bipartite; i.e., its faces can be colored in two coulors so that each edge separates faces of different colors. All of the faces of one color are triangles. These 27 triangles are the only triangles in the graph and correspond to lines in the configuration. The Gray graph thus admits an embedding into the surface of genus 7. If we keep the original vertices and introduce the centers of triangles as new vertices with an old vertex v adjacent to a new vertex t if and only if v lies on the boundary of the triangle t, the resulting graph is the Gray graph. Hence the Gray graph fits onto the same surface!

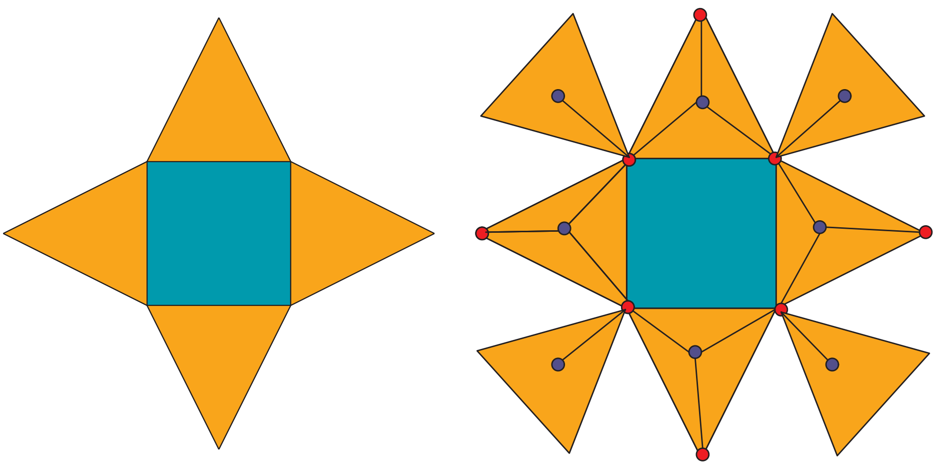

Figure 6: The Gray graph fits into the same surface.

§ 7. The lower bound. This shows that the upper bound for the genus is 7. The lower bound 7 follows from the following:

Proposition 1 *Let L be the Levi graph and let M be the Menger graph of some (v_3) configuration $\mathcal{C}$, then $\gamma(M) \leq \gamma(L)$.*

PROOF. Start with the genus embedding of cubic bipartite graph L with vertices colored, say black and white. By the reverse process depicted in Figure 6 one can obtain the embedding of M in the same surface. For each white vertex w of L having three adjacent black vertices, say a, b, c, introduce three new edges, forming a triangle that joins the three black vertices a, b, c. Remove all original edges and all white vertices.

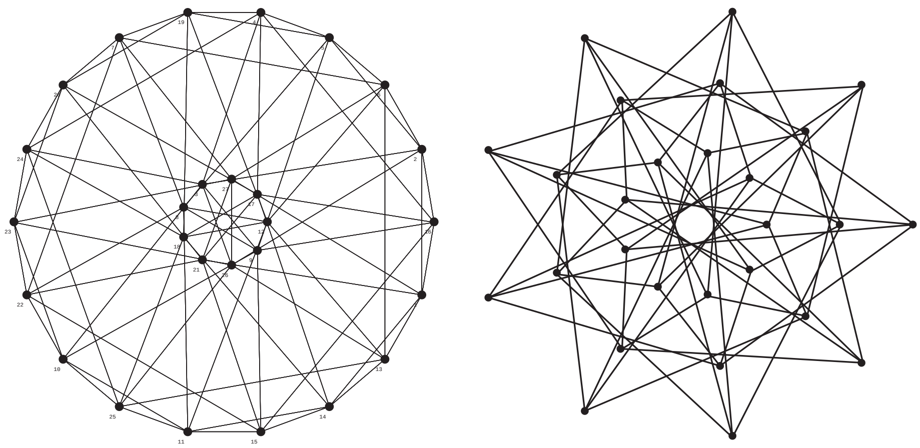

Figure 7: The dual Menger graph D and the Holt graph H, its spanning subgraph.

§ 8. The dual Menger graph D and the Holt graph. There is just one unfinished case to consider. Namely, the dual Menger graph D can also be embedded into the surface of genus 7. It turns out that this graph is quite interesting. It is the Cayley graph $\mathrm{Cay}(G, S)$ where G is the semidirect product $G = \mathbb{Z}_9 \rtimes \mathbb{Z}_3 = \langle a, b | a^9 = b^3 = 1, b^{-1}ab = a^4 \rangle$ and $S = \{a, a^{-1}, ab, (ab)^{-1}, (ab^2), (ab^2)^{-1}\}$. Therefore D can be described as a Z_9-covering graph over the base graph in Figure 8. The three gray vertices coincide with the three orbits of the group element a. Letting $x = a, y = ab, z = ab^2$, it may be verified that the group G admits the presentation:

$$D = \langle x, y, z | x^9 = y^9 = z^9 = 1, y^{-1}xy = x^4, x^{-1}yx = y^7, xyz = 1 \rangle$$

Consequently we also have

$$\{x^{-1}yx = y^7, x^{-1}zx = z^4, z^{-1}xz = x^7, z^{-1}yz = y^4\}$$

Note the cyclic symmetry of x, y, z in the presentation of G [11]. We remark that a deletion of any of the three 2-factors isomorphic to $3C_9$ corresponding to x,y, or z gives rise to a graph isomorphic to the Holt graph H. The 4-valent Holt graph of girth 5 is the smallest 1/2-arc transitive graph, that is, vertex- and edge- but not arc-transitive; see Figure 7. Figure 8 shows two essentially different ways to describe H as a $\mathbb{Z}_9$ cover graph of a graph on three vertices. In one case the base graph is a doubled triangle, in the other, it is a triangle with loops. In each case, the voltage assignment from $\mathbb{Z}_9$ is shown.

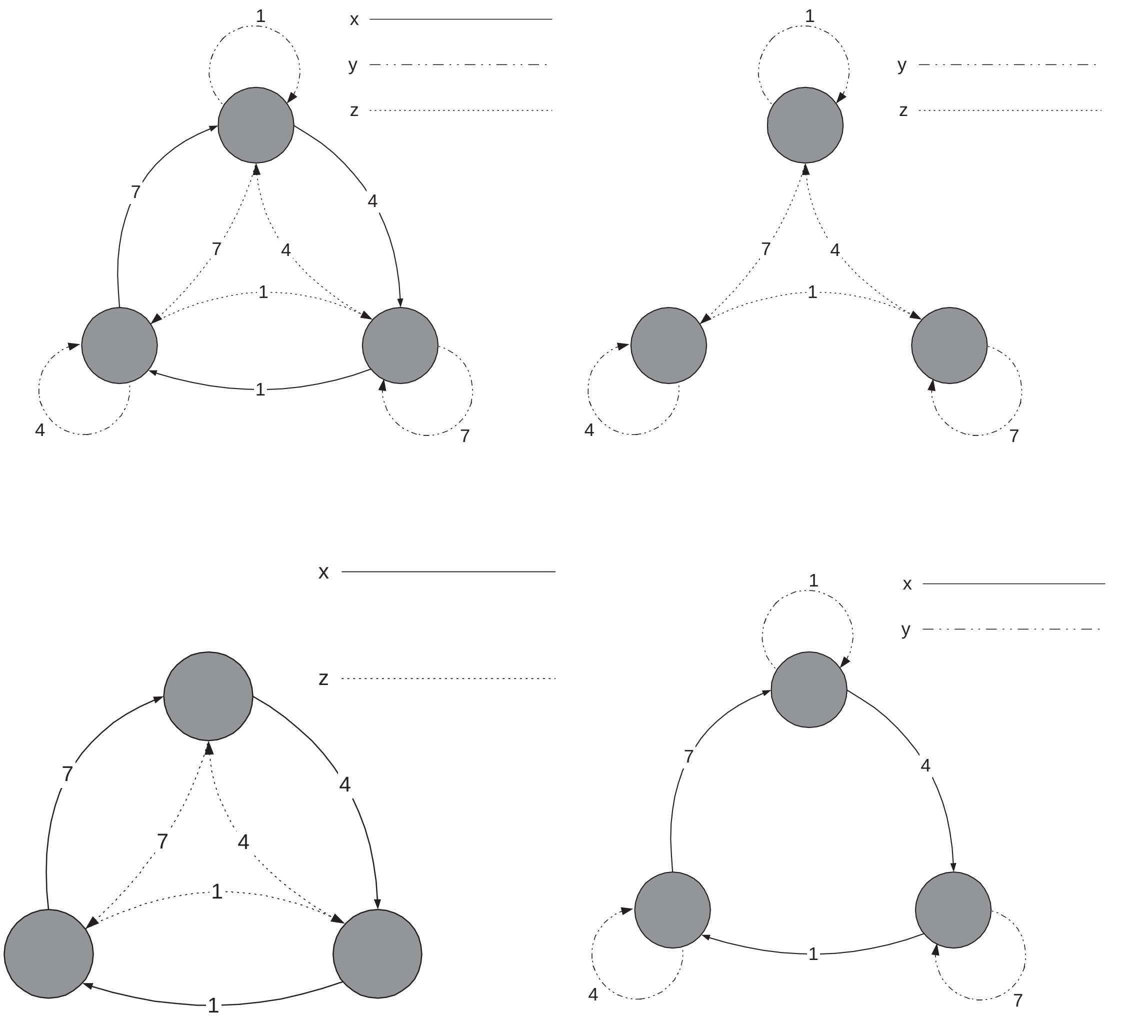

Figure 8: The $\mathbb{Z}_9$ voltage graph for the dual Menger graph and the three voltage subgraphs for the Holt graph.

§ 9. The Final Problems. What is the genus of D? What is the genus of H? The genus of $\mathbb{Z}_9 \rtimes \mathbb{Z}_3$ is known (Brin, Rauschenberg, Squier, [7]). Namely, $\gamma(\mathbb{Z}_9 \rtimes \mathbb{Z}_3) = 4$. On the other hand we proved that D admits an embedding into the surface of genus 7. Since H is a subgraph of D it follows that $4 \leq g(H) \leq g(D) \leq 7$. The Gray graph is the smallest semisymmetric cubic graph. There are others: The next largest one has 110 vertices [13] and is shown in Figure 9. Since it is bipartite, of girth 10 the corresponding dual configurations may also be the clue to its genus. The genus question can also be asked for the third graph on 112 vertices [5].

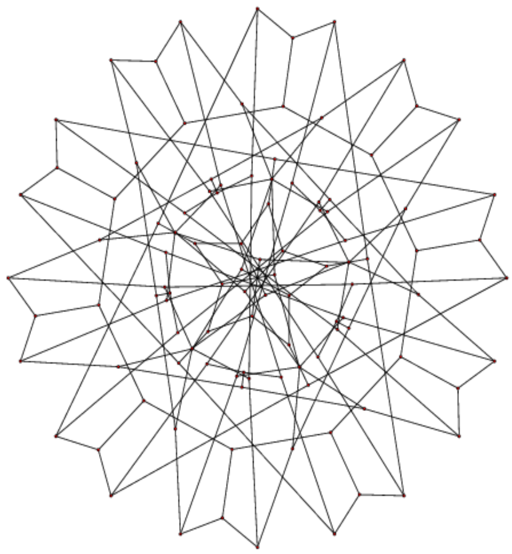

Figure 9: The second semisymmetric cubic graph on 110 vertices is bipartite and has girth 10.

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