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Abstract

We will show that every Q-polynomial distance-regular graph with diameter $d \geq 3$ and intersection number $a_1 = 0$ is 1-homogeneous in the sense of Nomura.

Key words: distance-regular graphs, Q-polynomial distance-regular graphs, 1-homogeneous distance-regular graphs.

1 Introduction

Let Γ denote a Q-polynomial distance-regular graph with diameter $d \geq 3$ and intersection numbers c_i, a_i, b_i. We will show that if $a_1 = 0$ then Γ is 1-homogeneous in the sense of Nomura [7]. We will also give explicit formulae for the parameters of the equitable partition of the vertices of Γ, corresponding to the distance from a pair of adjacent vertices x and y. To obtain our results, we use Terwilliger's "balanced set" characterization of the Q-polynomial property [8].

The Hermitean forms graphs with $r = 2$ (see Brouwer et al. [2, p. 285]) provide examples of Q-polynomial distance-regular graphs with $a_1 = 0$.

After some preliminaries in the next section, we will discuss the 1homogeneous and the Q-polynomial property in Section 3 and Section 4. We will prove the main theorem and some of its consequences in Section 5 and Section 6. In Section 7 we will discuss the Hermitean forms graphs.

2 Preliminaries

In this Section, we review some definitions and basic concepts. See the book of Brouwer et al. [2] for more background information.

Throughout this paper, Γ will denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set $V\Gamma$, edge set $E\Gamma$, path length distance function ∂, and diameter $d := \max\{\partial(x,y) | x, y \in V\Gamma\}$. For a vertex $x \in V\Gamma$ define $\Gamma_i(x)$ to be the set of vertices at distance i from x. We abbreviate $\Gamma(x) := \Gamma_1(x)$. The graph Γ is said to be *distance-regular* whenever for all integers $h, i, j (0 \leq h, i, j \leq d)$, and all $x, y \in V\Gamma$ with $\partial(x,y) = h$, the number

$$p^h_{ij} := |\{z \,|\, z \in V\Gamma,\, \partial(x,z) = i,\, \partial(y,z) = j\}| \tag{1}$$

is independent of x, y. The constants $p^h_{ij}\,(0 \leq h, i, j \leq d)$ are known as the *intersection numbers* of Γ. For convenience, set $c_i := p^i_{1i-1}\,(1 \leq i \leq d)$, $a_i := p^i_{1i}\,(0 \leq i \leq d)$, $b_i := p^i_{1i+1}\,(0 \leq i \leq d-1), k_i := p^0_{ii}\,(0 \leq i \leq d)$, and $c_0 = b_d = 0$. We observe $a_0 = 0,\ c_1 = 1$. Moreover,

$$c_i + a_i + b_i = k \ (0 \leq i \leq d), \tag{2}$$

where $k := k_1$. From now on we assume Γ is distance-regular with diameter $d \geq 3$.

In the following two lemmas we cite some well known facts about the intersection numbers; see for example Brouwer et al. [2, p. 127, 134]

Lemma 2.1 *Let Γ denote a distance-regular graph with diameter $d \geq 3$. Then for all integers h, i, j $(0 \leq h, i, j \leq d)$ the following (i), (ii) hold.*

(i) If one of h, i, j is greater then the sum of the other two, then $p^h_{ij} = 0$.

(ii) If one of h, i, j is equal to the sum of the other two, then $p^h_{ij} \neq 0$. ∎

Lemma 2.2 *Let Γ denote a distance-regular graph with diameter $d \geq 3$. Then the following (i)-(iii) hold.*

(i) $k_i = \dfrac{b_0 b_1 \cdots b_{i-1}}{c_1 c_2 \cdots c_i} \quad (0 \leq i \leq d)$,

(ii) $p^1_{i,i-1} = \dfrac{c_i k_i}{k} \quad (1 \leq i \leq d)$,

(iii) $p^1_{ii} = \dfrac{a_i k_i}{k} \quad (0 \leq i \leq d)$.

∎

Let Γ denote a distance-regular graph with diameter $d \geq 3$. For each integer $i\,(0 \leq i \leq d)$, the ith *distance matrix* A_i has rows and columns indexed by $V\Gamma$, and x, y entry

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } \partial(x,y) = i, \\ \\ 0 & \text{if } \partial(x,y) \neq i \end{cases} \quad (x, y \in V\Gamma). \tag{3}$$

Then

$$A_0 = I, \tag{4}$$

$$A_0 + A_1 + \cdots + A_d = J \ (J = \text{ all 1's matrix}), \tag{5}$$

$$A_i^t = A_i \ (0 \leq i \leq d), \tag{6}$$

and

$$A_i A_j = \sum_{h=0}^{d} p^h_{ij} A_h \ (0 \leq i, j \leq d). \tag{7}$$

By (4), (6) and (7), the matrices $A_0, A_1, \ldots, A_d$ form a basis for a commutative semi-simple $\mathbb{R}$-algebra M, known as the *Bose-Mesner algebra.* By Godsil [4, Theorem 12.2.1], the algebra M has a second basis $E_0, E_1, \ldots, E_d$ such that

$$E_0 = |V\Gamma|^{-1} J, \tag{8}$$

$$E_i E_j = \delta_{ij} E_i \ (0 \leq i, j \leq d), \tag{9}$$

$$E_0 + E_1 + \cdots + E_d = I, \tag{10}$$

$$E_i^t = E_i \ (0 \leq i \leq d). \tag{11}$$

The $E_0, E_1, \ldots, E_d$ are known as the *primitive idempotents* of Γ, and E_0 is the *trivial* idempotent.

Set $A := A_1$, and define the real numbers θ_i $(0 \leq i \leq d)$ by

$$A = \sum_{i=0}^{d} \theta_i E_i. \tag{12}$$

Then $AE_i = E_i A = \theta_i E_i$ $(0 \leq i \leq d)$, and $\theta_0 = k$. The scalars $\theta_0, \theta_1, \ldots, \theta_d$ are distinct, since A generates M [1, p. 197]. The $\theta_0, \theta_1, \ldots, \theta_d$ are known as the *eigenvalues* of Γ.

For notational convenience, we identify $V\Gamma$ with the standard orthonormal basis in the Euclidean space V, $\langle\, , \rangle$, where $V = \mathbb{R}^{|V\Gamma|}$ (column vectors), and where $\langle\, , \rangle$ is the dot product

$$\langle u, v \rangle = u^t v \ (u, v \in V\Gamma).$$

Observe M acts on V by left multiplication. The Euclidean space V, $\langle\, , \rangle$ is known as the *standard module* of Γ.

In the following lemma, we cite some well known results about primitive idempotents.

Lemma 2.3 (Terwilliger [8, Lemma 1.1]) *Let Γ denote a distance-regular graph with diameter $d \geq 3$. Pick any $\theta, \theta_0^*, \theta_1^*, \ldots, \theta_d^* \in \mathbb{R}$, and set*

$$E := |V\Gamma|^{-1} \sum_{i=0}^{d} \theta_i^* A_i. \tag{13}$$

Then the following (i)-(iii) are equivalent:

(i) θ is an eigenvalue of Γ, and E is the associated primitive idempotent.

(ii) For all $x, y \in V\Gamma$,

$$\langle Ex, Ey \rangle = |V\Gamma|^{-1} \theta_i^* \quad \textit{whenever } \partial(x, y) = i,$$

and

$$\sum_{\substack{z \in V\Gamma \\ \partial(x,z)=1}} Ez = \theta Ex.$$

(iii) The intersection numbers of Γ *satisfy*

$$c_i\theta^*_{i-1} + a_i\theta^*_i + b_i\theta^*_{i+1} = \theta\theta^*_i \ (0 \leq i \leq d),$$

and $\theta^*_0 =$ rank E. ∎

If (i)-(iii) hold, we call the sequence $\theta^*_0, \theta^*_1, \ldots, \theta^*_d$ the *dual eigenvalue sequence* associated with θ, E. The sequence is trivial whenever $E = E_0$ (in which case $\theta^*_0 = \theta^*_1 = \cdots = \theta^*_d = 1$).

3 The 1-homogeneous property

In this section, we recall the 1-homogeneous property.

Definition 3.1 *Let* Γ *denote a distance-regular graph with diameter* $d \geq 3$ *and let* x, y *denote adjacent vertices in* $V\Gamma$. *For all integers* i *and* j *we define* $D^i_j = D^i_j(x, y)$ *by*

$$D^i_j = \Gamma_i(x) \cap \Gamma_j(y).$$

We observe $D^i_j = \emptyset$ unless $0 \leq i, j \leq d$. Moreover if $0 \leq i, j \leq d$ then $|D^i_j| = p^1_{ij}$.

Lemma 3.2 *Let* Γ *denote a distance-regular graph with diameter* $d \geq 3$ *and let* x, y *denote adjacent vertices in* $V\Gamma$. *Then the following (i), (ii) hold.*

(i) For $0 \leq i, j \leq d$, *if* $|i - j| > 1$ *then* $D^i_j = \emptyset$. *If* $|i - j| = 1$ *then* $D^i_j \neq \emptyset$.

(ii) For $0 \leq i \leq d$ *we have* $D^i_i = \emptyset$ *if and only if* $a_i = 0$.

Proof. Immediate from Lemma 2.1 and Lemma 2.2. ∎

We visualize the D^i_j as follows in Figure 1.

Lemma 3.3 (Jurišić et al. [5, Lemma 2.11]) *Let* Γ *denote a distance-regular graph with diameter* $d \geq 3$. *Fix adjacent vertices* $x, y \in V\Gamma$, *and pick an integer* i $(1 \leq i \leq d)$. *Then with reference to Definition 3.1, the following (i), (ii) hold.*

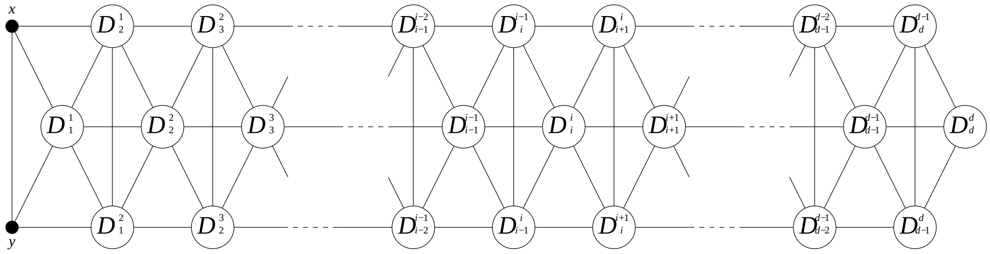

Figure 1: The distance partition corresponding to a pair of adjacent vertices x and y. Observe that $D^i_{i-1} \cup D^i_i \cup D^i_{i+1} = \Gamma_i(x)$ and $D^{i-1}_i \cup D^i_i \cup D^{i+1}_i = \Gamma_i(y)$.

(i) Each $z \in D^i_{i-1}$ *(resp.* D^{i-1}_i*) is adjacent to*

- *(a) precisely* c_{i-1} *vertices in* D^{i-1}_{i-2} *(resp.* D^{i-2}_{i-1}*),*
- *(b) precisely* $c_i - c_{i-1} - |\Gamma(z) \cap D^{i-1}_{i-1}|$ *vertices in* D^{i-1}_i *(resp.* D^i_{i-1}*),*
- *(c) precisely* $a_{i-1} - |\Gamma(z) \cap D^{i-1}_{i-1}|$ *vertices in* D^i_{i-1} *(resp.* D^{i-1}_i*),*
- *(d) precisely* b_i *vertices in* D^{i+1}_i *(resp.* D^i_{i+1}*),*
- *(e) precisely* $a_i - a_{i-1} + |\Gamma(z) \cap D^{i-1}_{i-1}|$ *vertices in* D^i_i*.*

(ii) Each $z \in D^i_i$ *is adjacent to*

- *(a) precisely* $c_i - |\Gamma(z) \cap D^{i-1}_{i-1}|$ *vertices in* D^i_{i-1}*,*
- *(b) precisely* $c_i - |\Gamma(z) \cap D^{i-1}_{i-1}|$ *vertices in* D^{i-1}_i*,*
- *(c) precisely* $b_i - |\Gamma(z) \cap D^{i+1}_{i+1}|$ *vertices in* D^{i+1}_i*,*
- *(d) precisely* $b_i - |\Gamma(z) \cap D^{i+1}_{i+1}|$ *vertices in* D^i_{i+1}*,*
- *(e) precisely* $a_i - b_i - c_i + |\Gamma(z) \cap D^{i-1}_{i-1}| + |\Gamma(z) \cap D^{i+1}_{i+1}|$ *vertices in* D^i_i*.*

∎

An *equitable partition* of a graph is a partition $\pi = \{C_1, C_2, \ldots, C_s\}$ of its vertex set into nonempty cells, such that for all i, j ($1 \leq i, j \leq s$) the number c_{ij} of neighbours, which a vertex in the cell C_i has in the cell C_j, is independent of the choice of the vertex in C_i. We call the c_{ij} the *corresponding parameters.*

Let Γ denote a distance-regular graph with diameter $d \geq 3$. The graph Γ is said to be *1-homogeneous*, whenever for all pairs x, y of adjacent vertices, the partition of $V\Gamma$ given by $\{D^i_j(x,y) | 0 \leq i, j \leq d,\ D^i_j(x,y) \neq \emptyset\}$ is equitable, and moreover the corresponding parameters are independent of the choice of x, y.

Recall Γ is *bipartite* whenever the intersection number $a_i = 0$ for $0 \leq i \leq d$. We say Γ is *almost bipartite* whenever $a_i = 0$ for $0 \leq i \leq d-1$ and $a_d \neq 0$.

Corollary 3.4 *Let* Γ *denote a distance-regular graph with diameter* $d \geq 3$*. Assume* Γ *is bipartite or almost bipartite. Then* Γ *is 1-homogeneous.*

Proof. Immediate from of Lemma 3.3. ∎

4 The Q-polynomial property

Let Γ denote a distance-regular graph with diameter $d \geq 3$. The *Krein parameters* q_{ij}^h $(0 \leq h, i, j \leq d)$ of Γ are defined by

$$E_i \circ E_j = |V\Gamma|^{-1} \sum_{h=0}^{d} q_{ij}^h E_h \quad (0 \leq i, j \leq d), \tag{14}$$

where $\circ$ denotes entrywise multiplication. We say Γ is *Q-polynomial* (with respect to the given ordering $E_0, E_1, \ldots, E_d$ of the primitive idempotents), whenever for all distinct integers i, j $(0 \leq i, j \leq d)$,

$$q_{ij}^1 \neq 0 \text{ if and only if } |i - j| = 1.$$

Let E denote a nontrivial primitive idempotent of Γ. We say Γ is *Q-polynomial with respect to E* whenever there exists an ordering $E_0, E_1 = E, \ldots, E_d$ of the primitive idempotents of Γ, with respect to which Γ is Q-polynomial.

We have the following useful lemmas about Q-polynomial distance-regular graphs.

Lemma 4.1 (Brouwer et al. [2, Thm. 8.1.1]) *Let Γ denote a distance-regular graph with diameter $d \geq 3$. Let E denote a nontrivial primitive idempotent of Γ and let $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$ denote the corresponding dual eigenvalue sequence. Suppose Γ is Q-polynomial with respect to E. Then $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$ are mutually distinct.* ∎

Lemma 4.2 (Caughman [3, Lemma 8.2]) *Let Γ denote a distance-regular graph with diameter $d \geq 3$. Let E denote a nontrivial primitive idempotent of Γ and let $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$ denote the corresponding dual eigenvalue sequence. Assume Γ is Q-polynomial with respect to E. Then*

$$(\theta_2^* - \theta_i^*)(\theta_0^* - \theta_i^*) = (\theta_1^* - \theta_{i+1}^*)(\theta_1^* - \theta_{i-1}^*) \tag{15}$$

for $1 \leq i \leq d-1$. ∎

Lemma 4.3 (Terwilliger [8, Thm. 3.3]) *Let Γ denote a distance-regular graph with diameter $d \geq 3$. Let E denote a nontrivial primitive idempotent of Γ and let $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$ denote the corresponding dual eigenvalue sequence. Then the following (i), (ii) are equivalent:*

(i) Γ *is Q-polynomial with respect to E.*

(ii) $\theta_0^* \neq \theta_i^*$ $(1 \leq i \leq d)$, *and for all integers h, i, j $(1 \leq h \leq d)$, $(0 \leq i, j \leq d)$ and for all vertices $x, y \in V\Gamma$ with $\partial(x, y) = h$ the following hold:*

$$\sum_{\substack{z \in V\Gamma \\ \partial(x,z)=i \\ \partial(y,z)=j}} Ez - \sum_{\substack{z \in V\Gamma \\ \partial(x,z)=j \\ \partial(y,z)=i}} Ez \in span\{Ex - Ey\}.$$

Suppose (i), (ii) hold. Then for all integers h, i, j $(1 \leq h \leq d)$, $(0 \leq i, j \leq d)$ and for all $x, y \in V\Gamma$ such that $\partial(x, y) = h$,

$$\sum_{\substack{z \in V\Gamma \\ \partial(x,z)=i \\ \partial(y,z)=j}} Ez - \sum_{\substack{z \in V\Gamma \\ \partial(x,z)=j \\ \partial(y,z)=i}} Ez = p_{ij}^h \frac{\theta_i^* - \theta_j^*}{\theta_0^* - \theta_h^*}(Ex - Ey). \tag{16}$$

∎

5 The main theorem

In this section we prove that if Γ is Q-polynomial with $a_1 = 0$, then Γ is 1-homogeneous.

Theorem 5.1 *Let Γ denote a distance-regular graph with diameter $d \geq 3$ and intersection number $a_1 = 0$. Let E denote a nontrivial primitive idempotent of Γ and let $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$ denote the corresponding dual eigenvalue sequence. Assume Γ is Q-polynomial with respect to E. Then with reference to Definition 3.1 the following (i)-(iii) hold.*

(i) For all integers i $(2 \leq i \leq d)$, and for all $z \in D_i^{i-1} \cup D_{i-1}^i$,

$$|\Gamma(z) \cap D_{i-1}^{i-1}| = a_{i-1} \frac{(\theta_1^* - \theta_i^*)(\theta_{i-1}^* - \theta_1^*) + (\theta_2^* - \theta_i^*)(\theta_0^* - \theta_{i-1}^*)}{(\theta_0^* - \theta_{i-1}^*)(\theta_{i-1}^* - \theta_i^*)}.$$

(ii) For all integers i $(2 \leq i \leq d)$ such that $a_i \neq 0$, and for all $z \in D_i^i$,

$$|\Gamma(z) \cap D_{i-1}^{i-1}| = c_i \frac{(\theta_1^* - \theta_i^*)(\theta_{i-1}^* - \theta_1^*) + (\theta_2^* - \theta_i^*)(\theta_0^* - \theta_i^*)}{(\theta_0^* - \theta_i^*)(\theta_{i-1}^* - \theta_i^*)}.$$

(iii) For all integers i $(2 \leq i \leq d-1)$ such that $a_i \neq 0$, and for all $z \in D_i^i$,

$$|\Gamma(z) \cap D_{i+1}^{i+1}| = b_i \frac{(\theta_1^* - \theta_i^*)(\theta_{i+1}^* - \theta_1^*) + (\theta_2^* - \theta_i^*)(\theta_0^* - \theta_i^*)}{(\theta_0^* - \theta_i^*)(\theta_{i+1}^* - \theta_i^*)}.$$

We remark the denominators in the above quotients are nonzero by Lemma 4.1.

PROOF: We split the proof into several cases.

Case 1. Let z be a vertex in $D_2^1 \cup D_1^2$. Then z has no neighbours in D_1^1 since $D_1^1 = \emptyset$.

Case 2. Let z be a vertex in D_2^2. Then z has no neighbours in D_1^1 since $D_1^1 = \emptyset$. We abbreviate $\tau = |\Gamma(z) \cap D_3^3|$ and $\eta = |\Gamma(z) \cap D_2^3|$. We observe $\tau + \eta = b_2$. By Lemma 4.3 we have

$$\sum_{\substack{w \in V\Gamma \\ \partial(x,w)=3 \\ \partial(z,w)=1}} Ew \;-\; \sum_{\substack{w \in V\Gamma \\ \partial(x,w)=1 \\ \partial(z,w)=3}} Ew = b_2 \frac{\theta_3^* - \theta_1^*}{\theta_0^* - \theta_2^*}(Ex - Ez). \tag{17}$$

Observe that $\{w \in V\Gamma \,|\, \partial(x,w) = 1,\, \partial(z,w) = 3\} \subseteq D_2^1$. Taking the inner product of (17) with Ey using Lemma 2.3 (ii), we get (after multiplying by $|V\Gamma|$)

$$\eta\theta_2^* + \tau\theta_3^* - b_2\theta_2^* = b_2 \frac{\theta_3^* - \theta_1^*}{\theta_0^* - \theta_2^*}(\theta_1^* - \theta_2^*).$$

Evaluating the above line using $\eta = b_2 - \tau$, we obtain

$$\tau = b_2 \frac{(\theta_1^* - \theta_2^*)(\theta_3^* - \theta_1^*)}{(\theta_0^* - \theta_2^*)(\theta_3^* - \theta_2^*)}.$$

Case 3. Let $z \in D_i^{i-1}$ $(3 \leq i \leq d)$. We abbreviate $\rho = |\Gamma(z) \cap D_{i-1}^{i-1}|$ and $\zeta = |\Gamma(z) \cap D_i^{i-1}|$. We observe $\rho + \zeta = a_{i-1}$. By Lemma 4.3 we have

$$\sum_{\substack{w \in V\Gamma \\ \partial(x,w)=i-1 \\ \partial(z,w)=1}} Ew \;-\; \sum_{\substack{w \in V\Gamma \\ \partial(x,w)=1 \\ \partial(z,w)=i-1}} Ew = a_{i-1} \frac{\theta_{i-1}^* - \theta_1^*}{\theta_0^* - \theta_{i-1}^*}(Ex - Ez). \tag{18}$$

Taking the inner product of (18) with Ey using Lemma 2.3, we get (after multiplying by $|V\Gamma|$)

$$\rho\theta^*_{i-1} + \zeta\theta^*_i - a_{i-1}\theta^*_2 = a_{i-1}\frac{\theta^*_{i-1} - \theta^*_1}{\theta^*_0 - \theta^*_{i-1}}(\theta^*_1 - \theta^*_i).$$

Evaluating the above line using $\zeta = a_{i-1} - \rho$ we obtain

$$\rho = a_{i-1}\frac{(\theta^*_1 - \theta^*_i)(\theta^*_{i-1} - \theta^*_1) + (\theta^*_2 - \theta^*_i)(\theta^*_0 - \theta^*_{i-1})}{(\theta^*_0 - \theta^*_{i-1})(\theta^*_{i-1} - \theta^*_i)}.$$

The cases $z \in D^i_{i-1}$ $(3 \le i \le d)$ are treated similarly.

Case 4. Let $z \in D^i_i$ $(3 \le i \le d)$. We abbreviate $\sigma = |\Gamma(z) \cap D^{i-1}_{i-1}|$ and $\delta = |\Gamma(z) \cap D^{i-1}_i|$. We observe $\sigma + \delta = c_i$. By Lemma 4.3 we have

$$\sum_{\substack{w\in V\Gamma \\ \partial(x,w)=i-1 \\ \partial(z,w)=1}} Ew - \sum_{\substack{w\in V\Gamma \\ \partial(x,w)=1 \\ \partial(z,w)=i-1}} Ew = c_i\frac{\theta^*_{i-1} - \theta^*_1}{\theta^*_0 - \theta^*_i}(Ex - Ez). \tag{19}$$

Again, taking the inner product of (19) with Ey using Lemma 2.3 we get

$$\delta\theta^*_i + \sigma\theta^*_{i-1} - c_i\theta^*_2 = c_i\frac{\theta^*_{i-1} - \theta^*_1}{\theta^*_0 - \theta^*_i}(\theta^*_1 - \theta^*_i).$$

Evaluating the above line using $\delta = c_i - \sigma$ we obtain

$$\sigma = c_i\frac{(\theta^*_1 - \theta^*_i)(\theta^*_{i-1} - \theta^*_1) + (\theta^*_2 - \theta^*_i)(\theta^*_0 - \theta^*_i)}{(\theta^*_0 - \theta^*_i)(\theta^*_{i-1} - \theta^*_i)}.$$

Case 5. Let $z \in D^i_i$ $(3 \le i \le d-1)$. We abbreviate $\tau = |\Gamma(z) \cap D^{i+1}_{i+1}|$ and $\gamma = |\Gamma(z) \cap D^{i+1}_i|$. We observe $\tau + \gamma = b_i$. By Lemma 4.3 we have

$$\sum_{\substack{w\in V\Gamma \\ \partial(x,w)=i+1 \\ \partial(z,w)=1}} Ew - \sum_{\substack{w\in V\Gamma \\ \partial(x,w)=1 \\ \partial(z,w)=i+1}} Ew = b_i\frac{\theta^*_{i+1} - \theta^*_1}{\theta^*_0 - \theta^*_i}(Ex - Ez). \tag{20}$$

Again, taking the inner product of (20) with Ey using Lemma 2.3 we get

$$\gamma\theta^*_i + \tau\theta^*_{i+1} - b_i\theta^*_2 = b_i\frac{\theta^*_{i+1} - \theta^*_1}{\theta^*_0 - \theta^*_i}(\theta^*_1 - \theta^*_i).$$

Evaluating the above line using $\gamma = b_i - \tau$ we obtain

$$\tau = b_i \frac{(\theta^*_1 - \theta^*_i)(\theta^*_{i+1} - \theta^*_1) + (\theta^*_2 - \theta^*_i)(\theta^*_0 - \theta^*_i)}{(\theta^*_0 - \theta^*_i)(\theta^*_{i+1} - \theta^*_i)}.$$

∎

Lemma 5.2 *Let Γ denote a distance-regular graph with diameter $d \geq 3$ and intersection number $a_1 = 0$. Let E denote a nontrivial primitive idempotent of Γ and let $\theta^*_0, \theta^*_1, \ldots, \theta^*_d$ denote the corresponding dual eigenvalue sequence. Assume Γ is Q-polynomial with respect to E. Then with reference to Definition 3.1 the following (i), (ii) hold.*

(i) For all integers i $(2 \leq i \leq d-1)$ such that $a_i \neq 0$, and for all $z \in D^i_i$,

$$|\Gamma(z) \cap D^{i-1}_{i-1}| = c_i \frac{(\theta^*_1 - \theta^*_{i-1})(\theta^*_i - \theta^*_{i+1})}{(\theta^*_0 - \theta^*_i)(\theta^*_{i-1} - \theta^*_i)}.$$

(ii) For all integers i $(2 \leq i \leq d-1)$ such that $a_i \neq 0$, and for all $z \in D^i_i$,

$$|\Gamma(z) \cap D^{i+1}_{i+1}| = b_i \frac{(\theta^*_1 - \theta^*_{i+1})(\theta^*_{i-1} - \theta^*_i)}{(\theta^*_0 - \theta^*_i)(\theta^*_i - \theta^*_{i+1})}.$$

PROOF. Simplify the formulae in Theorem 5.1 (ii), (iii) using Lemma 4.2. ∎

Corollary 5.3 *Let Γ denote a distance-regular graph with diameter $d \geq 3$ and intersection number $a_1 = 0$. If Γ is Q-polynomial then Γ is 1-homogeneous.*

PROOF. Immediate from Theorem 5.1 and Lemma 3.3. ∎

Remark: The converse of the above corollary is not true. Indeed the Coxeter graph is a 1-homogeneous distance-regular graph with $d = 4$ and $a_1 = 0$, but it is not Q-polynomial, see Brouwer et al. [2, Section 6.10].

6 Comments on the intersection numbers

Let Γ denote a Q-polynomial distance-regular graph with diameter $d \geq 3$ and intersection number $a_1 = 0$. In this section we show that one of the following hold: (i) Γ is bipartite; (ii) Γ is almost bipartite; or (iii) $a_i \neq 0$ for $2 \leq i \leq d$. We will use the following two lemmas.

Lemma 6.1 *Let* Γ *denote a* Q*-polynomial distance-regular graph with diameter* $d \geq 3$ *and intersection number* $a_1 = 0$*. Suppose there exists an integer* i $(2 \leq i \leq d-1)$ *such that* $a_i \neq 0$*. Then with reference to Definition 3.1 the following (i)-(iii) hold.*

(i) For all $z \in D_i^i$ *we have* $|\Gamma(z) \cap D_{i+1}^{i+1}| \neq 0$*.*

(ii) $D_{i+1}^{i+1} \neq \emptyset$*.*

(iii) $a_{i+1} \neq 0$*.*

PROOF. (i) Suppose $|\Gamma(z) \cap D_{i+1}^{i+1}| = 0$. By Lemma 5.2 (ii) we have either $\theta_1^* = \theta_{i+1}^*$ or $\theta_{i-1}^* = \theta_i^*$. But this is in contradiction with Lemma 4.1.

(ii), (iii) Immediate from (i) above. ∎

Lemma 6.2 *Let* Γ *denote a* Q*-polynomial distance-regular graph with diameter* $d \geq 3$ *and intersection number* $a_1 = 0$*. Suppose there exists an integer* i $(3 \leq i \leq d-1)$ *such that* $a_i \neq 0$*. Then with reference to Definition 3.1 the following (i)-(iii) hold.*

(i) For all $z \in D_i^i$ *we have* $|\Gamma(z) \cap D_{i-1}^{i-1}| \neq 0$*.*

(ii) $D_{i-1}^{i-1} \neq \emptyset$*.*

(iii) $a_{i-1} \neq 0$*.*

PROOF. Similar to the proof of Lemma 6.1. ∎

Corollary 6.3 *Let* Γ *denote a* Q*-polynomial distance-regular graph with diameter* $d \geq 3$ *and intersection number* $a_1 = 0$*. Then exactly one of the following (i)-(iii) hold.*

(i) Γ *is bipartite,*

(ii) Γ *is almost bipartite,*

(iii) $a_i \neq 0$ *for* $2 \leq i \leq d$*.*

PROOF. Immediate from Lemma 6.1 and Lemma 6.2. ∎

7 An example

In this section, we recall the Hermitean forms graph.

Let Γ be a Hermitean forms graph for a prime power r and let d denote the diameter of Γ. Then the intersection numbers of Γ are given by

$$b_i = \frac{r^{2d} - r^{2i}}{r+1} \quad 0 \leq i \leq d-1,$$

$$c_i = r^{i-1}\frac{r^i - (-1)^i}{r+1} \quad 1 \leq i \leq d,$$

see Brouwer et al. [2, Thm. 9.5.7]. Therefore we have

$$a_i = \frac{r^{2i} - r^{2i-1} - (-r)^{i-1} - 1}{r+1} \quad 0 \leq i \leq d.$$

Let θ denote the minimal eigenvalue of Γ. By Brouwer et al. [2, Cor. 8.4.2] θ is given by

$$\theta = -\frac{r^{2d-1}+1}{r+1}.$$

Let E denote the primitive idempotent of Γ corresponding to θ and let $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$ denote the corresponding dual eigenvalue sequence. By Brouwer et al. [2, Cor. 8.4.2], Γ is Q-polynomial with respect to E. Moreover, by Brouwer et al. [2, Cor. 8.4.2] and since the Q-polynomial structure is self-dual ([2, Cor. 8.4.4]) we have

$$\theta_i^* = \frac{b_i}{(-r)^i} + \frac{(-r)^i - 1}{r+1} = \frac{(-r)^{2d-i} - 1}{r+1} \quad (0 \leq i \leq d).$$

For the rest of this section assume $r = 2$. In this case we have $a_1 = 0$; therefore Γ is 1-homogeneous by Corollary 5.3. The parameters of the corresponding equitable partition are given by the following theorem. We remark $a_2 = 3$. Since $a_2 \neq 0$ we find $a_i \neq 0$ for $2 \leq i \leq d$ in view of Corollary 6.3 (iii).

Theorem 7.1 *Let Γ denote a Hermitean forms graph with $r = 2$ and diameter $d \geq 3$. Then with reference to Definition 3.1, the following (i), (ii) hold.*

(i) *For* $1 \leq i \leq d$, *each* $z \in D^i_{i-1}$ *(resp.* D^{i-1}_i*) is adjacent to*

(a) precisely	0	*vertices in* D^{i-1}_{i-1},
(b) precisely	$2^{i-2}(2^{i-1}-(-1)^{i-1})/3$	*vertices in* D^{i-1}_{i-2} *(resp.* D^{i-2}_{i-1}*),*
(c) precisely	$2^{i-2}(2^{i-1}+(-1)^{i-1})$	*vertices in* D^{i-1}_i *(resp.* D^i_{i-1}*),*
(d) precisely	$(2^{2i-3}-(-2)^{i-2}-1)/3$	*vertices in* D^i_{i-1} *(resp.* D^{i-1}_i*),*
(e) precisely	$(2^{2d}-2^{2i})/3$	*vertices in* D^{i+1}_i *(resp.* D^i_{i+1}*),*
(f) precisely	$2^{i-2}(2^{i-1}-(-1)^{i-1})$	*vertices in* D^i_i.

(ii) *For* $2 \leq i \leq d$, *each* $z \in D^i_i$ *is adjacent to*

(a) precisely	$(-1)^i 2^{i-1}((-2)^{i-2}-1)/3$	*vertices in* D^{i-1}_{i-1},
(b) precisely	$(2^{2d}-2^{2i})/3$	*vertices in* D^{i+1}_{i+1},
(c) precisely	2^{2i-3}	*vertices in* D^i_{i-1},
(d) precisely	2^{2i-3}	*vertices in* D^{i-1}_i,
(e) precisely	0	*vertices in* D^{i+1}_i,
(f) precisely	0	*vertices in* D^i_{i+1},
(g) precisely	$(2^{2i-3}-(-2)^{i-1}-1)/3$	*vertices in* D^i_i.

Proof. From Theorem 5.1 we get the following formulae.

(i) For all integers i $(2 \leq i \leq d)$ and for all $z \in D^{i-1}_i \cup D^i_{i-1}$, we have $|\Gamma(z) \cap D^{i-1}_{i-1}| = 0$.

(ii) For all integers i $(2 \leq i \leq d)$ and for all $z \in D^i_i$, we have $|\Gamma(z) \cap D^{i-1}_{i-1}| = c_i((-2)^{i-2}-1)/((-2)^i-1)$.

(iii) For all integers i $(2 \leq i \leq d-1)$ and for all $z \in D^i_i$, we have $|\Gamma(z) \cap D^{i+1}_{i+1}| = b_i$.

The present theorem now follows by Lemma 3.3 and from the formulae of the intersection numbers b_i and c_i ∎

We conclude the paper with some conjectures.

Conjecture 7.1 *Let* Γ *denote a distance-regular graph with diameter* $d \geq 3$ *and intersection numbers* $a_1 = 0$, $a_2 \neq 0$. *Let* θ *denote a nontrivial eigenvalue of* Γ *and let* E *denote the corresponding primitive idempotent. Assume* Γ *is Q-polynomial with respect to* E. *Then* θ *is the minimal eigenvalue of* Γ.

Conjecture 7.2 *Let* Γ *denote a distance-regular graph with diameter* $d \geq 3$ *and intersection numbers* $a_1 = 0$, $a_2 \neq 0$. *Assume* Γ *has classical parameters* (d, b, α, β), *see Brouwer et al. [2, Sec. 6.1]. Then* $b = -2$.

Conjecture 7.3 *Let Γ denote a distance-regular graph with diameter $d \geq 3$ and intersection numbers $a_1 = 0$, $a_2 \neq 0$. Assume Γ has classical parameters. Let i denote an integer $(2 \leq i \leq d)$ and let x, y denote vertices of Γ at distance $\partial(x, y) = i$. Then x, y are contained in a weak-geodetically closed subgraph of Γ which has diameter i. See [6], [9], [10], [11], [12] for more information on this topic.*

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