University of Ljubljana
Institute of Mathematics, Physics and Mechanics
Department of Mathematics
Jadranska 19, 1000 Ljubljana, Slovenia

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EXTENSION OF MAPS TO NILPOTENT SPACES III

M. Cencelj A. N. Dranishnikov

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Extension of Maps to Nilpotent Spaces III

M. Cencelj [1] and A. N. Dranishnikov [2]

Abstract: Let M be a nilpotent CW-complex. We give necessary and sufficient cohomological dimension theory conditions for a finite-dimensional metric compactum X so that every map $A \to M$, where A is a closed subset of X can be extended to a map $X \to M$.

This is a generalization of a result by A. N. Dranishnikov (Matem. Sbornik, 1991) where such conditions were found for simply-connected CW-complexes M, and M. Cencelj and A. N. Dranishnikov (Can. Bull. Math., 2001 and Topol. Appl.2002) where a condition of finitely generatedness was imposed on the nilpotent CW-complex M.

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We generalize the main theorem of [3] and Theorem 7 of [1] to obtain the following theorem.

Theorem 1. *For any nilpotent CW-complex M and finite-dimensional metric compactum X, the following are equivalent:*

1. $X\tau M$*;*

2. $X\tau SP^{\infty}M$*;*

3. $\dim_{H_i(M)}X \leq i$ *for every* $i > 0$*;*

4. $\dim_{\pi_i(M)}X \leq i$ *for every* $i > 0$*.*

We use the Kuratowski notation $X\tau M$ for the case every map from a closed subset of X to M can be extended over all of X. We recall that the cohomological dimension of a space X can be defined in this notation as follows: $\dim_G X \leq n$ iff and only if $X\tau K(G,n)$, where $K(G,n)$ is an Eilenberg-MacLane complex. We recall that a group G is called *nilpotent* if its lower central series $G = \Gamma^1 G \supset \Gamma^2 G \supset \dots \Gamma^k G \subset \Gamma^{k+1} G = 1$ has a finite length k called the nilpotency class of G. Here $\Gamma^2 G = [G,G]$ and $\Gamma^i G = [G, \Gamma^{i-1} G]$. The main examples of nilpotent groups are upper triangular matrix groups. The action of an upper triangular matrix group on a corresponding vector space suggest a definition of a *nilpotent action.* An action $\alpha : G \to \mathrm{Aut}H$ is called nilpotent if there is a G-invariant normal stratification $H = H_1 \supset \dots \supset H_i \supset \dots \supset H_n = *$ such that H_i/H_{i+1} is abelian and the induced action on H_i/H_{i+1} is trivial for all i. A topological space is called *nilpotent* if $\pi_1(X)$ is a nilpotent group and the action of $\pi_1(X)$ on every higher dimensional homotopy group is nilpotent.

We refer the reader to [2] and [5] for the facts and terminology from the cohomological dimension theory which we are using in this paper.

As opposed to Theorem 7 of [1] and Theorem 1 of [2] we do not impose any additional condition on M except nilpotency. The proof of the main theorem, however, is not dimension-wise in all cases, in some cases dimension over $H_1(M)$ does not provide as much information as dimension over $\pi_1(M)$, however, dimension over $H_1(M)$ and over $H_2(M)$ together suffice.

Theorem 2. *For a nilpotent group N and every metric compactum X the following equivalence holds*

$$\dim_N X \le 1 \Longleftrightarrow \dim_{\mathrm{Ab}N} X \le 1 \,.$$

provided N has one of the following properties:

(1) N is a torsion group;

(2) for every prime p for which $p - \mathrm{tor} N \neq 1$

(a) N is not p-divisible, or

(b) $p - \mathrm{tor}(\mathrm{Ab}N) \neq 0$.

Proof: We need to prove

$$\dim_{\mathrm{Ab}N} X \le 1 \Longrightarrow \dim_N X \le 1$$

for every metric compactum X, the reverse implication holds for every group.

Recall some general results which we will use in the sequel. A nilpotent group is the result of finitely many central extensions

$$0 \to A_i \to N_{i+1} \to N_i \to 1$$

of an Abelian group N_1 (note that in this case $\dim_{A_i} X \le 1$ and $\dim_{N_i} X \le 1$ implies $\dim_{N_{i+1}} X \le 1$).

At such an extension we have a natural epimorphism

$$\mathrm{Ab}N_i \longrightarrow \mathrm{Ab}N_{i+1} \,. \tag{1}$$

For every group G we also have an epimorphism

$$\otimes^n \mathrm{Ab}G \to \Gamma^n(G)/\Gamma^{n+1}G \,, \quad x_1 \otimes \cdots \otimes x_n \mapsto [\ldots[x_1, x_2], x_3], \ldots, x_n] \,.$$

If the nilpotent group N is of nilpotency class n we therefore have an epimorphism

$$\otimes^n \mathrm{Ab}N \longrightarrow \Gamma^n = A_n \,. \tag{2}$$

(1) If N is a torsion nilpotent group it is the direct sum of p-torsion nilpotent groups, p prime. Obvously every p-torsion nilpotent group is the result of finitely many central extensions of a p-torsion Abelian group by p-torsion Abelian groups.

If the p-torsion nilpotent group N is also p-divisible (radicable) we see from the epimorphisms (1) and (2) that every group in the central extensions has to be p-divisible. The same holds for AbN. Therefore $\{\mathbb{Z}_{p^\infty}\} = \sigma(\mathrm{Ab}N)$ (and similarly for all groups in the central extensions in the construction of N), from $\dim_{\mathrm{Ab}N} X \leq 1$ we obtain in finitely many steps $\dim_N X \leq 1$.

If the p-torsion nilpotent group N is not p-divisible the same holds for AbN and therefore $\{\mathbb{Z}_p\} = \sigma(\mathrm{Ab}N)$. All groups in the central extensions in the construction of N have either $\mathbb{Z}_p$ or $\mathbb{Z}_{p^\infty}$ in their Bockstein family. Since $\dim_{\mathbb{Z}_p} X \leq 1$ implies also $\dim_{\mathbb{Z}_{p^\infty}} X \leq 1$ we obtain from $\dim_{\mathrm{Ab}N} X \leq 1$ also in this case $\dim_N X \leq 1$ in finitely many steps.

(2) If N is not torsion the same holds for AbN and therefore $\dim_{\mathrm{Ab}N} X \leq 1$ implies $\dim_{\mathbb{Q}} X \leq 1$.

With the prime numbers p for which $\mathbb{Z}_{(p)} \in \sigma(\mathrm{Ab}N)$ we have no problems (i.e. any of $\mathbb{Z}_{(p)}$, $\mathbb{Z}_p$ or $\mathbb{Z}_{p^\infty}$ can appear in the Bockstein family σ of any Abelian group in the central extensions of N).

Consider those prime p for which $\mathbb{Z}_{(p)} \notin \sigma(\mathrm{Ab}N)$, $\mathbb{Z}_p \in \sigma(\mathrm{Ab}N)$. Let N be the central extension

$$0 \to \Gamma^n \longrightarrow N \longrightarrow M \to 1\,.$$

The surjection $N \to M$ implies the surjection $\mathrm{Ab}N \to \mathrm{Ab}M$, therefore also the surjection $\mathrm{Ab}N/Tor \to \mathrm{Ab}M/Tor$. Therefore $\mathbb{Z}_{(p)} \notin \sigma(\mathrm{Ab}M)$.

The surjection

$$\otimes^n \mathrm{Ab}N \longrightarrow \Gamma^n$$

and the surjection $\otimes^n \mathrm{Ab}(N/Tor) \cong \otimes^n \mathrm{Ab}N/Tor \to \Gamma^n$ imply $\mathbb{Z}_{(p)} \notin \sigma(\Gamma^n)$. By induction we see that for such a prime p and every Abelian group A appearing in the central extensions in the construction of N we have $\mathbb{Z}_{(p)} \notin \sigma(A)$.

Consider the prime numbers p for which we only have $\mathbb{Z}_{p^\infty} \in \sigma(\mathrm{Ab}N)$. This means that Ab$N$ as well as N are p-divisible groups with p-torsion (this must hold also for N; if N is without p-torsion it is $\bar{p}$-local, $\bar{p}$ denotes the complement of $\{p\}$ in the set of all primes, implying that also H_1 is $\bar{p}$-local and in particular without p-torsion). In this case we see that no Abelian group A in the central extensions of N has neither $\mathbb{Z}_{(p)}$ nor $\mathbb{Z}_p$ in its Bockstein family.

If the prime p does not appear in $\sigma(\text{Ab}N)$ this group is p-divisible and without p-torsion. Then also N is p-divisible and by assumption it has no p-torsion. By induction we see that also all groups A in the central extensions of N are p-divisible and without p-torsion.

□

Proof of Theorem 1: To start the proof (i.e. in dimension 1) we need only check those nilpotent groups $G = \pi_1(M)$ which are not torsion and for some prime p the abelianization AbG has no p-torsion and is p-divisible while G has p-torsion (and is also p-divisible), e.g. [7], Example 5.2. For all other prime numbers p the proof follows the proof of Theorem 2.

Since G_1 is the quotient of AbG also G_1 must be p-divisible. If G_1 had p-torsion one of the A_i would not be divisible by p, but this is impossible since the p-divisibility of G implies the p-divisibility of all G_i and AbG_i. Therefore G_1 has no p-torsion and is therefore $\bar{p}$-local. The same holds for its homology $H_*(G_1)$. We can construct G from G_1 only with central extensions by p-divisible Abelian groups due to the epimorphisms (2). If we extend G_1 (or any other $\bar{p}$-local G_i) by an Abelian group without p-torsion the resulting group is $\bar{p}$-local. Therefore in this case we extend the $\bar{p}$-local group G_i at least once by an Abelian group A which has p-divisible p-torsion. Since the p-torsion does not appear in the abelianization of the extension it has to be eliminated by the boundary homomorphism

$$\partial : E^2_{2,0} = H_2(G_i) \longrightarrow A = E^2_{0,1}$$

of the Lyndon-Hochschild-Serre spectral sequence(e.g. [6]) the kernel of which is the quotient of $H_2(G_{i+1})$. Since $H_2(G_i)$ is $\bar{p}$-local group the p-torsion of A can be eliminated only if the kernel of ∂ contains an element of infinite order which is not

p-divisible. Therefore $\mathbb{Z}_{(p)} \in \sigma H_2(G_{i+1})$ and thus $\mathbb{Z}_{(p)} \in \sigma H_2(G)$. Since there is an epimorphism

$$H_2(M) \longrightarrow H_2(K(\pi_1(M),1)) = H_2(\pi_1(M))\,,$$

we have also $\mathbb{Z}_{(p)} \in \sigma H_2(M)$.

Since by assumption $\mathrm{Ab}(\pi_1(M)) = H_1(M)$ is not torsion $\dim_{H_1(M)} X \leq 1$ implies $\dim_{\mathbb{Q}} X \leq 1$. Therefore if X is p-regular we have

$$\dim_{\mathbb{Q}} X = \dim_{\mathbb{Z}_{p^\infty}} X = \dim_{\mathbb{Z}_p} X = \dim_{\mathbb{Z}_{(p)}} X \leq 1\,,$$

if, however, X is p-singular $\dim_{H_1(M)} X \leq 1$ and $\dim_{H_2(M)} X \leq 1$ imply $\dim_{\mathbb{Z}_{p^\infty}} X \leq 1$, this on the other hand implies (since this holds for all such p) that in finitely many steps we obtain $\dim_{\pi_1(M)} X \leq 1$.

The rest of the proof is essentially the same as for $\pi_1(M)$ finitely generated [2].

□

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IMFM, University of Ljubljana
P. O. B. 2964
SI-1001 Ljubljana
Slovenia

University of Florida
Department of Mathematics
P. O. Box 118105
358 Little Hall
Gainesville, FL 32611-8105
USA

E-mail addresses:
`matija.cencelj@uni-lj.si`
`dranish@math.ufl.edu`