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Abstract. If G is a toral group, i.e. an extension of a torus by a finite group, and X is a G-CW complex we prove that there exists a G-homotopy equivalent CW complex Y with the property that the action map $\rho\colon G \times Y \to Y$ is a cellular map.

1. Formulation of the result

Let G be a compact Lie group. A *G-cell* of dimension n is a space of the form $G/H \times D^n$, where H is a closed subgroup of G and D^n is an n-cell. A *G-CW complex* X (or an *equivariant CW complex* in the terminology of [9]) is constructed by iterated attaching of G-cells. It is the union of G-spaces $X^{(n)}$ such that $X^{(0)}$ is a disjoint union of G-cells of dimension 0, i.e. orbits G/H, and $X^{(n+1)}$ is obtained from $X^{(n)}$ by attaching G-cells of dimension $n+1$ along equivariant attaching maps $G/H \times \partial D^{n+1} \to X^{(n)}$. The space $X^{(n)}$, which is called the *n-skeleton* of X, is thus the union of all G-cells of dimension at most n (the topological dimension of the $X^{(n)}$ is in general greater than n). For

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basic facts about G-complexes see the original papers of Matumoto [6] and Illman [4] or the exposition in [9].

For discrete groups G it is well known that every G-CW complex is also a CW complex with a cellular action of G (this follows for example from [9, Propositon 1.16, p. 102]). For non-discrete groups, Illman [5] gave an example showing that a G-CW complex X does not always admit a CW decomposition, compatible with the given G-CW decomposition, and proved that there always exists a homotopy equivalent CW complex Y which is finite if X is a finite G-complex.

In this paper we consider the following problem: given a G-CW complex X, does there exist a G-space Y, G-homotopy equivalent to X, with a CW decomposition such that the action $\rho\colon G \times Y \to Y$ is a cellular map with respect to some decomposition of G. The existence of such a Y is interesting from the point of view of equivariant homology and cohomology. For example, Greenlees and May showed that for some groups G the generalized Tate cohomology defined in [3] can be calculated from the CW decomposition of Y. Also, the Borel equivariant cohomology $H^*_G(X) = H^*(EG \times_G X)$ of a G-CW complex X can be computed using the cellular cohomology of the CW complex Y which is G-homotopy equivalent to $EG \times_G X$.

For a general compact Lie group G it is not known if every G-CW complex is G-homotopy equivalent to a CW complex Y with the required properties. Greenlees and May [3, Lemma 14.1] gave a construction of Y for any $SO(2)$-CW complex X. For non-abelian groups, the construction of Y is more difficult, since the fixed point sets $(G/H)^K$ of actions of subgroups $K < G$ on the orbits G/H can be nontrivial.

In [7] the original proof for $G = SO(2)$ was generalized to the two non-abelian 1-dimensional compact Lie groups, the orthogonal group $O(2)$ and the continuous quaternionic group $N_{SU(2)}T$. In [1] a sufficient condition for the existence of Y in the non-commutative case was identified and it was shown that the group $SU(2)$ satisfies this condition. Here we consider general toral groups, i.e. groups G which are extensions

$$T \rightarrowtail G \twoheadrightarrow F$$

of a torus T over a finite group F. The two groups in [7] are both toral groups, but there the construction of Y rests on a property of these two groups which is satisfied only for a few particular groups G. It is not satisfied for any group G containing a copy of S_7, the symmetric group on 7 letters [1], and in particular for a general toral group, since a toral group may well contain a copy of S_7. We prove

Theorem 1. *For any toral group G and any G-CW complex X, there exists a G-homotopy equivalent CW complex Y with a cellular action of G.*

The construction of the complex Y is similar to the construction of Greenlees and May for $G = SO(2)$, generalized to the non-abelian case $G = SU(2)$ in [1]. It requires the existence of a CW decomposition of every orbit G/H such that first, the action $\rho\colon G \times G/H \to G/H$ is cellular with respect to some given decomposition of G, and second, the fixed point set $(G/H)^K$ of the natural action of K on G/H is a subcomplex for every $K < G$. More precisely, since the orbit type of a cell is determined only up to the conjugacy type of the group H, it suffices to show that there exists a family of subgroups $\mathcal{K}$, containing at least one representative from every conjugacy class of subgroups of G, and a CW decomposition of every G/H, $H \in \mathcal{K}$, such that the action $G \times G/H \to G/H$ is a cellular map and every fixed point set $(G/H)^K$, $K \in \mathcal{K}$, is a subcomplex of G/H. In the terminology of [1], such a $\mathcal{K}$ is a good representative family of subgroups. In section 2 we consider the case where G is a torus. In this case the situation is simpler, since conjugation in an abelian group is trivial. This implies first, that the only good representative family $\mathcal{K}$ is the family of all subgroups of G, and second, that the fixed point sets $(G/H)^K = \{gH \mid g^{-1}Kg \subset H\}$ are either the whole space G/H (if $K < H$) or empty, and therefore automatically subcomplexes of G/H. Therefore it suffices to give an explicit description of decompositions of orbits G/H, $H < G$, such that the natural action of G with the standard decomposition is cellular. In section 3 we find a good representative family $\mathcal{K}$ of subgroups in a general toral group G and extend the decompositions of tori from section 2 to decompositions of G and of G/H, $H \in \mathcal{K}$ with the required properties. The proof of the theorem now follows from [1, Proposition 1]. Nevertheless, to complete the arguments in the context of this paper, we give a proof of Theorem 1 in section 4.

2. Decompositions of tori

In this section, G is a compact connected abelian group, i.e. a torus $T = (SO(2))^s$. We can view T as $\mathbb{R}^s/\mathbb{Z}^s$, where $\mathbb{R}^s$ is identified with the tangent space of T at the identity, or equivalently, as the cube $I^s \subset \mathbb{R}^s$, (where $I = [0,1]$), with identified parallel sides. Let $\{a_1, \dots, a_s\}$ be the standard basis of $\mathbb{R}^s$ and $\pi : \mathbb{R}^s \to T^s$ the projection. The standard (product) CW decomposition of T has one 0-cell $e = \pi(0) = e^0$ (the unit of T), s closed 1-cells $e^1_i = \pi(L^1_i)$, where L^1_i is the 1-dimensional subspace of $\mathbb{R}^s$ spanned by a_i, and $\binom{s}{j}$ closed j-cells $e^j_J = \pi(L^j_J)$ for every $j \leq s$, where $J = \{i_1, \ldots, i_j\} \subset \{1, \ldots, s\}$ and L_J^j is the j-dimensional linear subspace spanned by $\{a_i, i \in J\}$. Clearly, every cell of this decomposition is a closed subgroup of T, and a j-cell e_J^j is the product $e_J^j = e_{i_1}^1 \cdots e_{i_j}^1$. Let $\mathbf{T}$ denote the torus T with this standard decomposition.

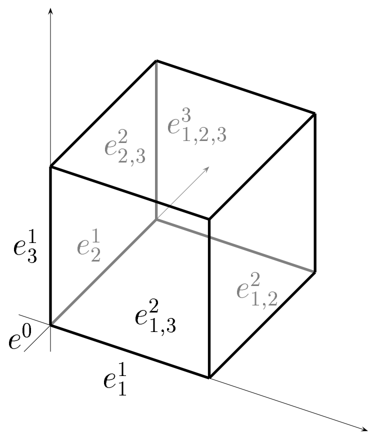

FIGURE 1. The standard decomposition $\mathbf{T}^3$.

We call a CW decomposition of T *linear* if every j-cell, $j = 0, \ldots, s$, lies on $\pi(L)$, where L is a j-dimensional linear subspace of $\mathbb{R}^s$. Clearly, the standard decomposition $\mathbf{T}$ is linear.

Theorem 2. *For any closed subgroup $H \leq T$, there exists a linear CW decomposition of T, inducing a CW decomposition on the quotient T/H, such that the actions $\rho\colon \mathbf{T} \times T \to T$ and $\rho\colon \mathbf{T} \times T/H \to T/H$ are cellular maps.*

Proof. Since the quotient map $q\colon T \to T/H$ is a homomorphism of groups, every orbit T/H ($H \leq T$ closed) is a compact connected abelian group, i.e. a torus.

A closed subgroup $H < T$ is a product $H = H_0 \times D$, where $H_0 \cong T^r$ is a torus of dimension $r \leq s$, and $D \cong \mathbb{Z}/n_1 \times \cdots \times \mathbb{Z}/n_k$ is a discrete torus. We first consider the case where $H = H_0 \cong T^r$. The tangent space of H at the identity is a subspace $L \subset \mathbb{R}^s$ spanned by vectors $b_i = \alpha_{1i}a_1 + \cdots + \alpha_{si}a_s, \alpha_{ji} \in \mathbb{Z}$, $i = 1, \ldots, r$. If we imagine the torus as I^s with identified parallel sides, then $H = \pi(L)$ consists of finitely many parallel r-dimensional planes inside the cube I^s. We cut the cube I^s along all possible $(s-1)$-planes which are spanned by one of these planes and any $s-r-1$ basis vectors $a_{i_1}, \ldots, a_{i_{s-r-1}}$. Since there are finitely many such hyperplanes this gives a subdivision of I^s into convex polyhedra, and since the cuts along parallel sides coincide, this subdivision determines a CW decomposition $\tilde{T}$ of T which is linear and has H as a subcomplex. For every $k \geq 0$, the $(k+r)$-skeleton of $\tilde{T}$ consists of all $(k+r)$-planes in I^s, parallel to some $(k+r)$-subspace of $\mathbb{R}^s$ spanned by L and by k vectors $\{a_{i_1}, \ldots, a_{i_k}\}$.

Figure 2 shows the decomposition $\tilde{T}^2$ with respect to the subgroup $H_{3,1} < T^2$ generated by the vector $b = 3a_1 + a_2 \in \mathbb{R}^2$ (in this case $r = 1$, so $s - r - 1 = 0$).

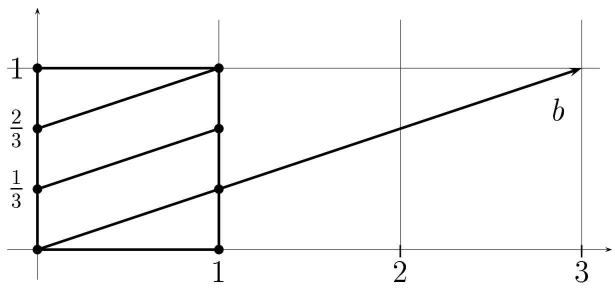

FIGURE 2. The decomposition $\tilde{T}^2$ with respect to $H_{3,1}$

The quotient map $T \to T/H$ is covered by the projection $\mathbb{R}^s \to M$ in the direction of L onto any linear subspace M of $\mathbb{R}^s$ spanned by a subset $a_{i_1}, \ldots, a_{i_{s-r}}$ of basis vectors such that $M \oplus L = \mathbb{R}^s$. This projection maps the subdivision of I^s into polyhedra to a subdivision of the unit cube I^{s-r} in M which determines a linear CW decomposition of T/H.

For example, let $H_{1,1,1}$ be the subgroup of T^3 generated by the vector $b = a_1 + a_2 + a_3 \in \mathbb{R}^3$ (in this case $s - r - 1 = 1$). Figure 3 shows the decomposition $\tilde{T}^3$ with respect to $H_{1,1,1}$. If M is the complementary

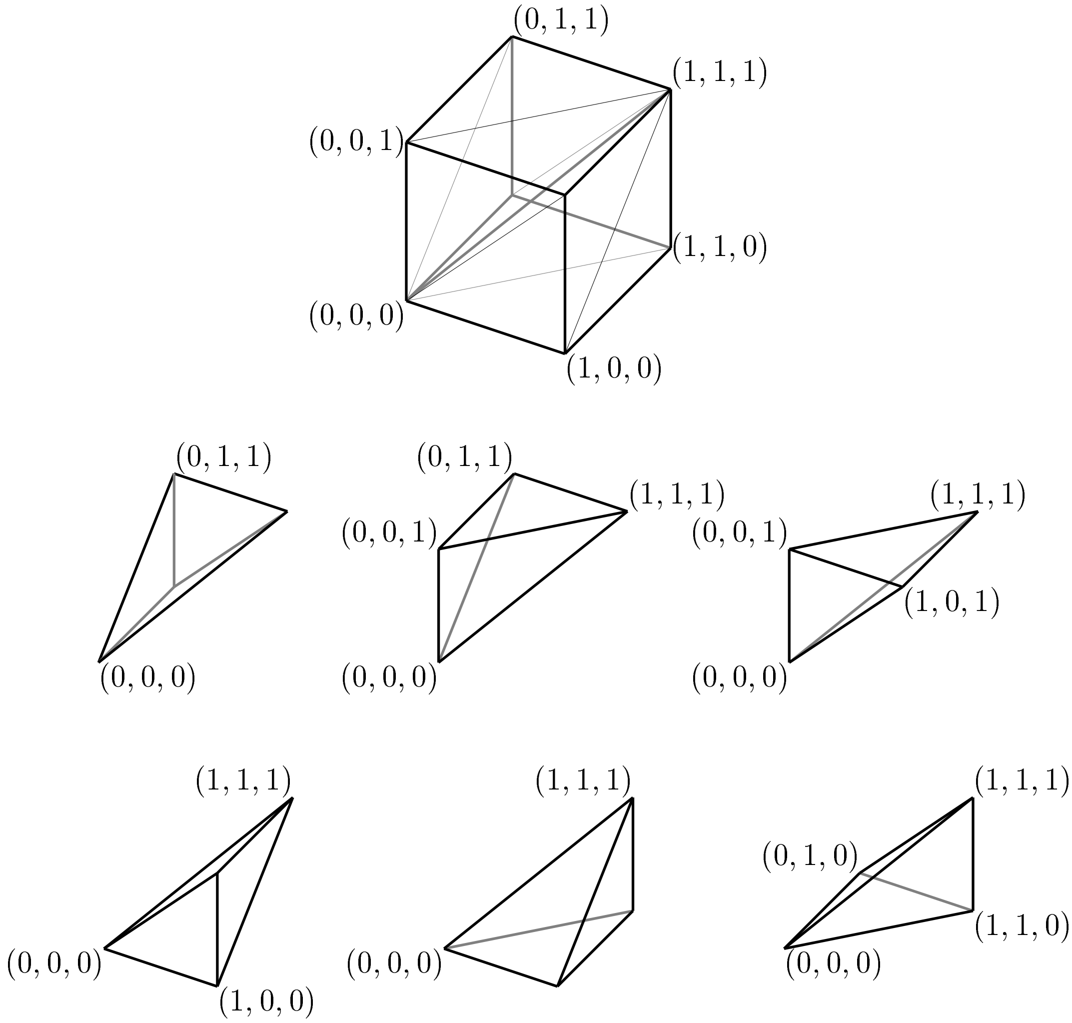

FIGURE 3. The decomposition of $\tilde{T}^3$ with respect to $H_{1,1,1}$

subspace to L in $\mathbb{R}^3$ spanned by a_1 and a_2, then the projection $\mathbb{R}^3 \to M$ in the direction of L maps the cube I^3 onto the hexagon shown in Figure

4. We can imagine T^2 as the shaded square I^2 with identified parallel sides. Thus, the induced decomposition $\tilde{T}^2$ has two 2-simplices and projection $\tilde{T}^3 \to \tilde{T}^3/H_{1,1,1} = \tilde{T}^2$ maps three 3-simplices onto any one of these two 2-simplices.

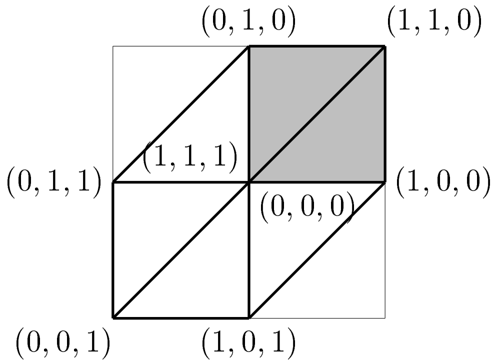

FIGURE 4. The induced decomposition of the quotient $\tilde{T}^3/H_{1,1,1}$

Let us prove that the actions of $\mathbf{T}$ on T and on T/H with these decompositions is cellular. In $T = \mathbb{R}^s/\mathbb{Z}^s$, the product of two points $\pi(x), \pi(y)$, $x, y \in \mathbb{R}^s$, equals $\pi(z)$ where $z = x + y$. For any $(k+r)$-cell τ^{k+r} of $\tilde{T}$ and any cell e^j_J of $\mathbf{T}$, multiplication in T maps the product $e^j_J \times \tau^{k+r}$ into the plane in I^s spanned by τ^{k+r} and $\{a_i, i \in J\}$, and this is contained in the $(k+r+j)$-skeleton of $\tilde{T}$. Passing to the quotient, this implies that the product of a j-cell of $\mathbf{T}$ and a k-cell of T/H is in the $(j+k)$-skeleton of T/H, so the action

$$\rho\colon \mathbf{T} \times T/H \to T/H$$

is a cellular map.

If $H = H_0 \times D$, where $H_0 \cong T^r$ is a torus and $D = \mathbb{Z}/n_1 \times \cdots \times \mathbb{Z}/n_k$, is a discrete torus, the proof of the proposition follows directly from the following simple lemma, applied to the torus T with the decomposition $\tilde{T}$ and to the torus $T' = T^{s-r}$ with the induced decomposition. $\square$

Lemma 1. *Let $T' = T/H$ be a torus with a given linear CW decomposition, such that the action of* $\mathbf{T}$ *is cellular. For every closed discrete subgroup $D < T'$ there exists a CW decomposition of T'/D such that the induced action of* $\mathbf{T}$ *on T'/D is cellular.*

Proof. The projection $T' \to T'/D$ can be decomposed into

$$T' \to T'/D_1 = T_1 \to \cdots \to T_{k-1}/D_{k-1} = T'/D,$$

where every group D_i is isomorphic to a cyclic group $\mathbb{Z}/n_i$. Let β be a generator of D_1 and $b = (b_1, \dots, b_{s-r})$ a generator of $\pi^{-1}D_1$ in $\mathbb{R}^{s-r}$, where $r = \dim H$. Every component b_i is of the form p_i/q_i, where q_i divides the order n_1 of D_1. Let $h\colon \mathbb{R}^{r-s} \to \mathbb{R}^{r-s}$ be the linear isomorphism given by

$$h\colon (x_1 \dots, x_{r-s}) \mapsto (q_1x_1, \dots, q_{r-s}x_{r-s}).$$

The map h^{-1} induces a subdivision of the unit cube I^{s-r} into ν copies $I_1, \dots I_\nu$, where ν is a multiple of n_1. If the original decomposition of I^{s-r} into convex polyhedra arising from the given CW decomposition of T' is repeated in each one of these copies, a linear subdivision of I^{s-r} is obtained which induces a D_1-invariant CW decomposition of T'. Figure 5 illustrates this decomposition of T' in the case where $T' = T^3/H_{1,1,1} \cong T^2$ from Figure 4 and the discrete subgroup $D < T'$ is generated by $b = (1/3, 1/6) \in \mathbb{R}^2$.

The induced CW decomposition of T'/D_1 obtained in this way is linear and clearly has the property that the action of $\mathbf{T}$ is cellular.

In the same way we construct a map $h_i\colon T_{i-1} \to T_{i-1}/D_{i-1} = T_i$ for each $i = 1, \dots, k$. The CW decomposition of T'/D induced by $h = h_k \circ \cdots \circ h_1$ has the required property. □

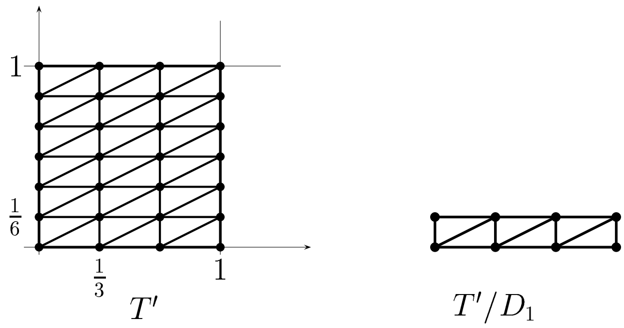

FIGURE 5. The decompositions of T' and of T'/D in the case $D \cong \mathbb{Z}/6$ is generated by $b = (1/3, 1/6) \in \mathbb{R}^2$.

3. Toral groups

In this section G is a toral group, i.e. an extension

$$T \rightarrowtail G \overset{p}{\twoheadrightarrow} F,$$

of a torus T over a finite group F. Our aim is to construct suitable CW decompositions of G and of every orbit G/H where H is a member of a good representative family $\mathcal{K}$ of subgroups of G. In order to do this, we first prove

Proposition 1. *For every toral group G, where $T \to G \to F$, there exists a finite subgroup $F' \subset G$ such that $p\colon F' \to F$ is surjective.*

Proof. Let $p_1, \dots, p_r$ be all primes that divide $|F|$ and let

$$A = (\mathbb{Z}[\frac{1}{p_1}, \dots, \frac{1}{p_r}]/\mathbb{Z})^s.$$

The group A is a subgroup of T^s. Since $H^2(F, A) \cong H^2(F, T^s)$ (Lemma 2) there exists a subgroup B of G which is an extension of A by F. We can write A as the union

$$A = \cup_{n=1}^{\infty} A_n, \quad A_n = \{x \in A \mid (p_1 \cdots p_r)^n x = 0\} \subset A.$$

Let $[\phi] \in H^2(F, A)$ be an element representing the extension B. Because $\phi\colon F \times F \to A$ is a map from a finite set, there exists an n such that $\mathrm{Im}\phi \subset A_n$. So $[\phi] \in H^2(F, A_n)$ which means that there exists a finite subgroup F' of B which is an extension of F by A_n. □

Lemma 2. *$H^2(F, A) \cong H^2(F, T^s)$.*

Proof. An exact sequence of groups $A \to T^s \to T^s/A$ induces a long exact sequence

$$\cdots \to H^{n-1}(F, T^s/A) \to H^n(F, A) \to H^n(F, T^s) \to H^n(F, T^s/A) \to \cdots$$

Let $[\phi] \in H^n(F, T^s/A)$. By [2, Corollary 4.2.3], $|F| \cdot [\phi] = 0$. There exists a cochain $\psi \in C^{n-1}(F, T^s/A)$ such that $|F| \cdot \phi = \delta(\psi)$. Therefore $\phi = \frac{1}{|F|}\delta(\psi) = \delta(\frac{1}{|F|}\psi)$, hence $[\phi] = 0$ and $H^n(F, T^s/A) = 0$. □

Next, we generalize the standard decomposition $\mathbf{T}$ to a suitable CW decomposition $\mathbf{G}$ of G with the property that multiplication $\mathbf{G} \times \mathbf{G} \to \mathbf{G}$ is a cellular map. The basic property of $\mathbf{G}$ is that every product $(f_1' e_1^1 f_1) \cdots (f_j' e_j^1 f_j)$, where $f_i, f_i' \in F'$ and e_i^1 is a 1-cell of $\mathbf{T}$, $i = 1, \ldots, j$, is in the j-skeleton $\mathbf{G}^{(j)}$. As a result, the restriction $\mathbf{G}|_T$ is a subdivision of $\mathbf{T}$, since every j-cell $e_J^j \in \mathbf{T}$ is contained in the j-skeleton of $\mathbf{G}$. In addition to the (subdivided) cells of $\mathbf{T}$, the j-skeleton $(\mathbf{G}|_T)^{(j)}$ contains products of the form

$$\sigma^j = (u_1 f_1^{-1} e_1^1 f_1) \cdots (u_j f_j^{-1} e_j^1 f_j) = u(f_1^{-1} e_1^1 f_1) \cdots (f_j^{-1} e_j^1 f_j)$$

where $u_i \in F' \cap T$, $f_i \in F'$, $i = 1, \ldots j$, and $u = u_1 \cdots u_j$. Geometrically σ^j can be described as the projection $\pi(L_u^j)$ of the affine space $L_u^j = \tilde{u} + L^j$ where L^j is the tangent space of the subgroup $(f_1^{-1} e_1^1 f_1) \cdots (f_j^{-1} e_j^1 f_j)$, and $\tilde{u} \in \pi^{-1}(u)$. If we imagine T as the cube

I^s with identified parallel sides, then every σ^j consists of finitely many parallel j-dimensional planes.

The required CW decomposition $\mathbf{G}|_T$ is constructed by cutting the cube I^s along the finitely many planes $\pi(L^{s-1})$ corresponding to all possible σ^{s-1}. Since the cuts on parallel sides coincide, this decomposition of I^s into convex polyhedra determines a CW decomposition of T, such that every σ_j is contained in a union of j-faces, and thus in the j-skeleton. This decomposition is not linear, since L^{s-1} is in general an affine and not a linear subspace of $\mathbb{R}^s$. Nevertheless, multiplication is a cellular map, since the sum $a+b$ of elements $a \in L^{j_1}_{u_1}$ and $b \in L^{j_2}_{u_2}$ is in $L^j_{u_1 u_2}$, where $j \le j_1 + j_2$.

The decomposition of T obtained in this way is extended to other components of G in the following way. For every $f \in F'$ the map $u \mapsto fu$ is a homeomorphism from T to the component fT which induces a CW decomposition $\mathbf{G}|_{fT} = f(\mathbf{G}|_T)$ on fT. The j-skeleton $(\mathbf{G}|_{fT})^{(j)}$ is the union of all products $f\sigma^j$. If $f_1 \in F' \cap fT$ then $f_1 = fu$, $u \in F' \cap T$ and $f_1\sigma^j = fu\sigma^j = f(\sigma')^j$ is also in the j-skeleton $(\mathbf{G}|_{fT})^{(j)}$, so the decompositions of fT obtained from multiplication by two different elements $f, f' \in fT$ coincide. Every product $(f_1' e_1^1 f_1)\cdots(f_j' e_j^1 f_j)$ is contained in $\mathbf{G}^{(j)}$, since it can be rewritten as $f(g_1^{-1}e_1 g_1)\cdots(g_j^{-1}e_j g_j)$, $f, g_i \in F'$, $i = 1, \dots, j$, and multiplication is clearly cellular.

Example. Let $G = N_{SU(2)}T$ be the infinite quaternionic group, which is an extension

$$T^1 \rightarrowtail G \overset{p}{\twoheadrightarrow} \mathbb{Z}/2.$$

We can represent G as the subgroup of $SU(2)$ generated by rotations

$$T = \left\{ r_\varphi = \begin{bmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{bmatrix}, \alpha \in S^1 \subset \mathbb{C} \right\} < SU(2),$$

and the element

$$u = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.$$

Since u is of order 4 in $SU(2)$, it generates a copy of $\mathbb{Z}/4$, so we can take $F' =< u >\cong \mathbb{Z}/4$, and $F'' = F' \cap T = \pm I$ (where I denotes the identity matrix in $SU(2)$). We give $T < G$ the common subdivision of the standard decomposition $\mathbf{T} = \{e^0 = I, e^1 = T\}$, and of $-\mathbf{T}$, i.e. $\{e^0_1 = I, e^0_2 = -I, e^1_1 = S^1_+, e^1_2 = S^1_-\}$, and the second component uT the decomposition induced by multiplication by u, i.e. $\{e^0_3 = u, e^0_4 = -u, e^1_3 = uS^1_+, e^1_4 = uS^1_-\}$.

Let $\mathcal{K}'$ be any family of closed subgroups, containing precisely one representative of every conjugacy class in G, and define

$$\mathcal{K} = \{f^{-1}Kf, K \in \mathcal{K}', f \in F'\}.$$

The following theorem is an extension of theorem 2 to toral groups.

Theorem 3. *Let G be a toral group and $H \in \mathcal{K}$. There exist CW decompositions of the orbit space G/H such that the action $\rho\colon \mathbf{G} \times G/H \to G/H$ is cellular and for every $K \in \mathcal{K}$, the fixed point set $(G/H)^K$ of the natural action of K on G/H, is a subcomplex of G/H.*

In the proof of the theorem we will need three additional propositions.

Proposition 2. *For any two closed subgroups $H, A \leq G$ such that $H \leq A$, and for any given $u \in T$, there exists a CW decomposition of $T/(H \cap T)$ such that $(uA \cap T)/(H \cap T)$ is a subcomplex and the action of T on $T/(H \cap T)$ with the decomposition $\mathbf{G}|_T$ is cellular.*

Proof. Again we imagine the unit component T of G as the cube I^s with identified parallel sides. Then $uA \cap T$ is the union of finitely many parallel planes $\{(u + c_q) + L_A \mid q = 1, \dots, Q\}$, where L_A is the tangent space of A at the identity. Since $H < A$, the tangent space L_H of H is a linear subspace of L_A. Let $\{b_i, i = 1, \dots, r'\}$ be a basis of L_A such that the first r vectors form a basis of L_H. We cut the cube I^s along all $(s-1)$-planes $(u + c_q) + L_{u_i}^{s-1}$, where $L_{u_i}^{s-1} = u_i + L^{s-1}$, $u_i \in F' \cap T$, and L^{s-1} is spanned by any collection of linearly independent vectors containing $b_1, \dots, b_r$, and any $s - r - 1$ vectors from the union $\{b_i, i = r+1, \dots, r'\} \cup \{f^{-1} a_i f, i = 1, \dots s, f \in F'\}$, where a_i are standard basis vectors. This gives an $(H \cap T)$-invariant decomposition of T which induces a decomposition of $T/(H \cap T)$ such that $(uA \cap T)/(H \cap T)$ is a CW subcomplex, and such that the action of T with the decomposition $\mathbf{G}|_T$ is cellular. □

Let us fix a subgroup $H \in \mathcal{K}$. For any $K \in \mathcal{K}$, let A_K denote the intersection

$$A_K = q^{-1}((G/H)^K) \cap T = \{u \in T \mid u^{-1} K u < H\},$$

where $q \colon G \to G/H$ is the quotient map. Notice that, since conjugation by elements of T preserves components, the set A_K is nonempty only for those subgroups K of G, for which $p(K) < p(H)$.

Proposition 3. *If K is a subgroup of H, then the set A_K is a subgroup of T.*

Proof. Since the conjugation $\varphi_u \colon G \to G$, $\varphi_u(g) = u^{-1}gu$, by an element $u \in T$ preserves components of G, it follows that for every $u \in T$ and $g \in G$ there exists a $v \in T$ such that $u^{-1}gu = gv$. Let $u, u' \in A_K$, and $k \in K$. Then there exist $v, v' \in T$ such that $u^{-1}ku = kv \in H$ and $(u')^{-1}ku' = kv' \in H$. Then

$$(uu')^{-1}k(uu') = (u')^{-1}(kv)u' = (u')^{-1}ku'v = kv'v = (kv')k^{-1}(kv).$$

Since this is a product of three elements from H, it is in H, so $uu' \in A_K$. The fact that A_K is a subgroup follows either from Theorem 3.5 of [8] or from the following simple argument: if $u^{-1}ku = kv \in H$, also $v \in H$, so

$$k = ukvu^{-1} = uku^{-1}v \in H,$$

and therefore $uku^{-1} = kv^{-1} \in H$. Thus $u^{-1} \in A_K$ if $u \in A_K$. □

Proposition 4. *For a given H, the family $\{A_K \mid K < H\}$ contains at most finitely many different sets.*

Proof. Let $K, K' < H$ be such that $p(K) = p(K')$, i.e. K and K' have elements in the same components of G. For every $k \in K$ there exists a $k' \in K'$ such that $k' = kv$ for some $v \in T$. Then $v = k^{-1}k' \in H \cap T$. For every $u \in T$,

$$u^{-1}k'u = u^{-1}kvu = u^{-1}kuv$$

so $u^{-1}k'u \in H$ precisely when $u^{-1}ku \in H$ which means that $u \in A_K$ precisely when $u \in A_{K'}$. The set A_K thus depends only on the projection $p(K) < F$. Since F is finite, there are only finitely many possibilities for A_K. □

Proof of Theorem 3. Let $K \in \mathcal{K}$ be such that $A_K \neq \emptyset$. Pick any element $y \in A_K$ and let $\bar{K} = y^{-1}Ky < H$. Then

$$A_K = \{u \mid u^{-1}Ku = u^{-1}y\bar{K}y^{-1}u < H\} = \{yv \mid v^{-1}\bar{K}v < H\} = yA_{\bar{K}}.$$

By Proposition 3, $A_{\bar{K}}$ is a group for every K. By Proposition 2 there exists a CW decomposition of $T/(H \cap T)$ such that $A_K/(H \cap T)$ is a subcomplex of T and the action of $\mathbf{G}|_T$ is cellular. By Proposition 3 the family $\{A_{\bar{K}} \mid \bar{K} < H\}$ is finite. For every $K \in \mathcal{K}$, the number of groups $K' \in \mathcal{K}$ which are conjugate to K equals F', so also the family $\{A_K \mid K \in \mathcal{K}\}$ is finite and there exists a common CW subdivision of $T/(H \cap T)$ such that $A_K/(H \cap T)$ is a subcomplex for every $K \in \mathcal{K}$, and the action of $\mathbf{G}|_T$ on $T/(H \cap T)$ is cellular.

This decomposition is extended to the other components of G in the same way as the standard decomposition of T: for every $f \in F'$, the homeomorphism $h_f \colon T \to fT$, $h_f(t) = ft$, determines a CW decomposition of $h_f(T)$. We let every component of G have the CW decomposition which is the common subdivision of the finitely many decompositions obtained in this way. This gives the an F'-invariant CW decomposition of G such that the restriction to T is a subdivision of the decomposition defined above. By construction, the induced decomposition of G/H is F' invariant.

For a given $K < H$, the intersection of the fixed point set with the component fT of G is

$$q^{-1}((G/H)^K) \cap fT = \{fu \mid u \in T, u^{-1}f^{-1}Kfu < H\} = fA_{f^{-1}Kf}.$$

Since the representative family $\mathcal{K}$ is closed under conjugation by elements of F', this implies that $(G/H)^K$ is a subcomplex for every K.

The proof that the action of G on G/H is cellular is similar to the argument used in the proof of Proposition 2.

□

4. Proof of Theorem 1

Now that CW decompositions of the homogeneous spaces G/H, $H \in \mathcal{K}$, are given, the CW complex Y and the G-homotopy equivalence $h\colon X \to Y$ is constructed inductively by a similar process as in [3] and [1].

The 0-skeleton $X^{(0)}$ is a disjoint union of orbits G/H_i, where $H_i \in \mathcal{K}$. Let Y_0 be $X^{(0)}$ with the CW decomposition of Theorem 3 on every G-cell G/H_i. Then the action $\rho\colon G \times Y_0 \to Y_0$ is cellular. For every $K \in \mathcal{K}$ the fixed point set $(X^{(0)})^K$ is a disjoint union of fixed point sets $(G/H_i)^K$ and is a subcomplex. We define the G-homotopy equivalence on the 0-skeleton by $h_0 = \mathrm{id}\colon X^{(0)} \to Y_0$.

By induction we assume that there exists a CW complex Y_{n-1} with a cellular action of G such that for every $K \in \mathcal{K}$, the fixed point set $(Y_{n-1})^K$ is a subcomplex of Y_{n-1} and a G-homotopy equivalence

$$h_{n-1}\colon X^{(n-1)} \to Y_{n-1}.$$

For any G-cell $e^n \in X^{(n)}$, the attaching G-map $G/H \times S^{n-1} \to X^{(n-1)}$ is determined by its restriction

$$\varphi\colon S^{n-1} \to (X^{(n-1)})^H.$$

Let ψ be a non-equivariant cellular approximation of the composition

$$h_{n-1} \circ \varphi\colon S^{n-1} \to (Y_{n-1})^H.$$

Since the action of G on Y_{n-1} is cellular, the G-extension

$$\tilde{\psi}\colon G/H \times S^{n-1} \to Y_{n-1}$$

of ψ is also cellular, and the space

$$Y_n = \coprod_{e_i^n \in X^{(n)}} (G/H_i \times D^n) \cup_{\coprod \tilde{\psi}_i} Y_{n-1}$$

is a CW complex with a cellular action of G. For every $K \in \mathcal{K}$, the fixed point set $(Y_n)^K$ is the disjoint union of subcomplexes $(G/H_i)^K$ glued to the subcomplex $(Y_{n-1})^K$ along the cellular map $\tilde{\psi}$ and is a subcomplex. The G-homotopy $h_n\colon Y_n \to X_n$ is obtained by extending the map h_{n-1} over the G-cells one by one. In the direct limit we obtain the desired CW complex Y and G-homotopy equivalence h. □

Remark. The class of toral groups contains all normalizers of maximal tori NT of compact Lie groups (including both 1-dimensional groups treated in [7]), and it might be possible to use our Theorem 1 to prove a similar theorem for general compact Lie groups. Nevertheless, as the example of $SU(2)$ in [1] shows, the step from NT to G in a general compact Lie group is nontrivial. It seems more likely that there exists a compact Lie group which does not satisfy the assumptions of the Theorem.

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