University of Ljubljana
Institute of Mathematics, Physics and Mechanics
Department of Mathematics
Jadranska 19, 1000 Ljubljana, Slovenia

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THE POINTED VERSION OF LIPSCOMB'S EMBEDDING THEOREM

Ivan Ivanšić Uroš Milutinović

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The pointed version of Lipscomb's embedding theorem

Ivan Ivanšić
University of Zagreb, FER,
Unska 3, 10000 Zagreb, Croatia
ivan.ivansic@fer.hr

Uroš Milutinović
University of Maribor, PEF,
Koroška cesta 160, 2000 Maribor, Slovenia
uros.milutinovic@uni-mb.si

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Abstract

Let $\Sigma(\tau)$ be the generalized Sierpiński curve constructed in [8, 9], which is naturally identified with the Lipscomb's space $\mathcal{J}(\tau)$ [3, 4]. Then for any n-dimensional metric space of weight τ there is an embedding of X into $L_n(\tau) \subseteq \Sigma(\tau)^{n+1}$, $L_n(\tau)$ being the set of points having at least one irrational coordinate. Here we prove that this embedding may be chosen in such a way that its value at a certain point (the base point) is given in advance. In fact, we prove a stronger result that the values of the embedding may be given in advance at any finite set of points of X.

Keywords: covering dimension, generalized Sierpiński curve, universal space, Lipscomb's universal space, embedding, extension
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Let $\tau \geq \aleph_0$ be a cardinal number and let Λ be a set of cardinality τ. In his papers [3, 4] S. L. Lipscomb has defined the space $\mathcal{J}(\tau)$ as a factor-space of Baire's universal 0-dimensional space $\Lambda^{\mathbb{N}}$ and used it in his construction of a universal n-dimensional metrizable space of weight τ (which is a subspace of $\mathcal{J}(\tau)^{n+1}$). In [8, 9] it has been proved that $\mathcal{J}(\tau)$ is naturally homeomorphic to a generalized Sierpiński curve $\Sigma(\tau)$.

The subspace $L_n(\tau) \subseteq \Sigma(\tau)^{n+1}$ of all points having at least one irrational coordinate is a universal space for n-dimensional metrizable spaces of weight $\leq \tau$ [4, 9, 8]. That means that for any n-dimensional metrizable space of weight τ there is an embedding of X into $L_n(\tau) \subseteq \Sigma(\tau)^{n+1}$. Here we prove that this embedding may be chosen in such a way that its value at a certain finite set of points of X is given in advance.

Let us remark that this is an embedding theorem of the relative type, i.e. an embedding given in advance on the finite subspace is extended to an embedding of the whole space. In [7] we have proved that the embedding may be given in advance on any compact subspace X_0 of X when $n = 0$. For general n we believe that the result presented here is (almost) the best possible, i.e. the best possible if no additional information about the embedding of X_0 is assumed. We have no formal proof of this statement, since the techniques we use are applicable only for obtaining positive results, i.e. for constructing embeddings.

We shall use the notation of [1, 3, 11] (with a few slight modifications).

$|X|$ denotes the cardinal number of the set X.

For the sake of completeness we include here the descriptions of Lipscomb's space $\mathcal{J}(\tau)$, the generalized Sierpiński curve $\Sigma(\tau)$, and the natural homeomorphism $\chi : \mathcal{J}(\tau) \longrightarrow \Sigma(\tau)$.

Baire's universal 0-dimensional space of weight τ is the set $\Lambda^{\mathbb{N}}$ (where $\mathbb{N} = \{1, 2, 3, \ldots\}$) of all sequences of elements of Λ equipped with the product topology, while Λ is equipped with the discrete topology. *Lipscomb's space* $\mathcal{J}(\tau)$ is defined as the quotient space $\mathcal{J}(\tau) = \Lambda^{\mathbb{N}}/\!\sim$, where the equivalence relation $\sim$ is defined as follows:

for $\lambda = (\lambda_1, \ldots, \lambda_m, \ldots), \mu = (\mu_1, \ldots, \mu_m, \ldots)$

$\lambda \sim \mu \iff \lambda = \mu \,\text{or}\, \exists j \in \mathbb{N} \,\text{such that :}$

i) $\forall k, k < j \implies \lambda_k = \mu_k,$

ii) $\forall s \in \mathbb{N}, \lambda_j = \mu_{j+s},$

iii) $\forall s \in \mathbb{N}, \lambda_{j+s} = \mu_j.$

In the case $\mu \neq \lambda$ such a j is uniquely determined and is called the *tail index* of λ and μ. We also say that the two sequences are *interwoven.*

The equivalence class of $\lambda = (\lambda_1, \ldots, \lambda_m, \ldots)$ will be denoted by $[\lambda]$ or $[\lambda_1, \ldots, \lambda_m, \ldots]$. An equivalence class may be a singleton — in which case it is called an *irrational point* of $\mathcal{J}(\tau)$ — or a dyad — in which case it is called a *rational point* of $\mathcal{J}(\tau)$. $\mathcal{J}(\tau)$ is a one-dimensional metrizable space of weight τ [3].

The classic *triangular Sierpiński curve (Sierpiński gasket)* may be described as a subset of $\mathbf{R}^3$ as follows:

Let $e^1 = (1, 0, 0)$, $e^2 = (0, 1, 0)$, $e^3 = (0, 0, 1)$. Let $\varphi_1, \varphi_2, \varphi_3 : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the homotheties with the coefficients $1/2$ and the centers e^1, e^2, e^3, respectively. If the convex hull of these three points (i.e. the standard 2-simplex) is denoted by Σ it is obvious that the set obtained from Σ by n removals of the middle triangles may be described as

$$\Sigma_n = \bigcup_{(\lambda_1, \ldots, \lambda_n) \in \Lambda^n} \varphi_{\lambda_1} \circ \cdots \circ \varphi_{\lambda_n} \Sigma, \qquad (1)$$

where $\Lambda = \{1, 2, 3\}$. After that, the classic triangular Sierpiński curve is obtained as

$$\Sigma(3) = \bigcap_{n \in \mathbb{N}} \Sigma_n.$$

The *generalized Sierpiński curve* $\Sigma(\tau)$ is defined analogously using the Hilbert space $\ell_2(\tau) = \{(x_\lambda) \in \mathbf{R}^\Lambda : \sum_{\lambda\in\Lambda} x_\lambda^2 < \infty\}$ as the ambient space instead of $\mathbf{R}^3$. Using e^λ, $\lambda \in \Lambda$, defined by $\forall \mu \in \Lambda$, $e^\lambda_\mu = \delta_{\lambda,\mu}$ (Kronecker's symbol) we describe the "homotheties" with the centers e^λ and the coefficients $1/2$, i.e. the functions $\varphi_\lambda : \ell_2(\tau) \longrightarrow \ell_2(\tau)$ defined by

$$(\varphi_\lambda(x))_\mu = \begin{cases} (x_\lambda + 1)/2, & \mu = \lambda \\ x_\mu/2, & \mu \neq \lambda. \end{cases}$$

Let $\sigma = \{(x_\lambda) \in \ell_2(\tau) : \sum_{\lambda\in\Lambda} x_\lambda = 1 \ \&\ \forall\lambda, 0 \leq x_\lambda \leq 1\}$. Then $\Sigma = \operatorname{Cl}\sigma = \operatorname{Cl}\sigma = \{(x_\lambda) \in \ell_2(\tau) : \sum_{\lambda\in\Lambda} x_\lambda \leq 1 \ \&\ \forall\lambda, 0 \leq x_\lambda \leq 1\}$ is the closed convex hull of the set $\{e^\lambda | \lambda \in \Lambda\}$ and it may be called the *standard τ-simplex* by an analogy to the standard n-simplex. Now the generalized Sierpiński curve $\Sigma(\tau)$ may be described in the same way as previously the classic curve: subspaces Σ_n of $\ell_2(\tau)$ are defined by (1) (only Σ now has a different meaning, and Λ is of cardinality τ), and then $\Sigma(\tau)$ is defined as

$$\Sigma(\tau) = \bigcap_{n\in\mathbb{N}} \Sigma_n.$$

The points $\varphi_{\lambda_1}\circ\cdots\circ\varphi_{\lambda_n}(e^\lambda)$, $n \geq 1$, are called the *rational points* of $\Sigma(\tau)$ (more precisely for a fixed n they are called the *nth level vertices*), and all other points (including all e^λs) are *irrational points* of $\Sigma(\tau)$.

That $\chi : \mathcal{J}(\tau) \longrightarrow \Sigma(\tau)$, defined by

$$\chi([\lambda_1, \ldots, \lambda_n, \ldots]) = \bigcap_{n\in\mathbb{N}} \varphi_{\lambda_1} \circ \cdots \circ \varphi_{\lambda_n} \Sigma \tag{2}$$

is a homeomorphism sending rational points to rational points and irrational points to irrational points has been proved in [8], and we identify the spaces $\mathcal{J}(\tau)$ and $\Sigma(\tau)$ via χ freely, choosing the description that is more convenient for use in the context.

Every point of $\Sigma(\tau)$ is thus described by a unique equivalence class of indices $[\lambda] = [(\lambda_1, \ldots, \lambda_n, \ldots)]$, where the λ_ns are the indices of the homotheties from (2). Any rational point is represented by two interwoven sequences, while any irrational point is represented by a unique sequence.

S.L. Lipscomb [4] proved that the n-dimensional subspace $L_n(\tau) \subseteq \mathcal{J}(\tau)^{n+1}$, consisting of all points having at least one irrational coordinate, is a universal space for metrizable spaces of weight $\leq \tau$ and covering dimension $\leq n$. U. Milutinović [8, 10] has used $\Sigma(\tau)$ and indexing of special type of certain sequences of decompositions of a given metrizable space X of weight $\leq \tau$ and covering dimension $\leq n$, in order to obtain an embedding of X into $L_n(\tau) \subseteq \Sigma(\tau)^{n+1}$ (and thus giving a new proof of the universality of $L_n(\tau)$). I. Ivanšić and U. Milutinović have used the same approach (but more complicated decompositions and indexing) in [6] in order to prove that $\Sigma(3)$ may be used instead of $\Sigma(\tau)$ in the construction of the universal space in the separable case. In [10] U. Milutinović proved a result on approximation of maps by embeddings using another modification of decompositions and indexing. I. Ivanšić and U. Milutinović used still another modification in [7] to prove a relative embedding theorem.

In this paper we use the same general strategy, but a special trick of enlarging the space before imbedding is employed, in addition to the use of a further modification of decompositions and indexing.

This general strategy consists of constructing certain finer and finer families of decompositions and then indexing them in such a way, that the mappings defined by

$$x \longmapsto [\lambda_1, \ldots, \lambda_k, \mu, \nu, \ldots, \nu, \ldots] = [\lambda_1, \ldots, \lambda_k, \nu, \mu, \ldots, \mu, \ldots] \tag{3}$$

(in case when x belongs to the boundaries of the sets indexed by the initial segments of the sequences $\lambda_1, \ldots, \lambda_k, \mu, \nu, \ldots, \nu, \ldots$ and $\lambda_1, \ldots, \lambda_k, \nu, \mu, \ldots, \mu, \ldots$), or by

$$x \longmapsto [\lambda_1, \ldots, \lambda_k, \ldots] \tag{4}$$

(in case when x belongs to the sets indexed by the initial segments of the sequence $\lambda_1, \ldots, \lambda_k, \ldots$ and belongs to no boundary of elements of certain decompositions) will be the $n+1$ coordinate functions of an embedding into $\Sigma(\tau)^{n+1}$. That means that one may interpret the indexing of the decompositions as a sort of coordinatization of the space that mimics the coordinates of $\Sigma(\tau)^{n+1}$.

Let $\mathcal{U}$ be a family of subsets of X. The *local order* of $\mathcal{U}$ at $x \in X$ is defined as $\operatorname{lord}_x \mathcal{U} = \inf\{k : x \textit{ has a neighborhood intersecting } k \textit{ elements of } \mathcal{U}\} \in \{0, 1, 2, \ldots, \infty\}$. The *local order* of $\mathcal{U}$ is defined as $\operatorname{lord} \mathcal{U} = \sup\{\operatorname{lord}_x \mathcal{U} : x \in X\}$.

$\operatorname{Bd} \mathcal{U} = \bigcup_{U \in \mathcal{U}} \operatorname{Bd} U$, where $\operatorname{Bd} U$ denotes the boundary of the set U; $\operatorname{Cl} \mathcal{U} = \bigcup_{U \in \mathcal{U}} \operatorname{Cl} U$, where $\operatorname{Cl} U$ denotes the closure of the set U.

A *decomposition* of the space X is a pairwise disjoint locally finite family of open nonempty subsets of X whose closures cover X.

As in [6]–[10], the main tool enabling us to construct the needed decompositions was the following Lipscomb's lemma, which we use in the present paper as well, and therefore quote it for the sake of completeness (the notation is changed, to fit ours):

Lemma 1 *([4, Lemma 4, p.152])*

Let $n \in \{0, 1, 2, \ldots\}$. *Let* X *be a metric space such that* $\dim X = n$, $wX = \tau \geq \aleph_0$.

Let $X = X_1 \cup X_2 \cup \cdots \cup X_{n+1}$, *where* $X_1, \ldots, X_{n+1}$ *are pairwise disjoint* 0*-dimensional subsets of* X.

Let $\mathcal{T}$ *be an arbitrary open covering of* X. *For each* j, $1 \leq j \leq n+1$, *let* $\mathcal{V}_j$ *be a decomposition of* X *such that* $|\mathcal{V}_j| \leq \tau$ *and* $\operatorname{lord} \mathcal{V}_j \leq 2$. *Let* $\mathcal{F}_j$, $|\mathcal{F}_j| \leq \tau$, *be a discrete closed family such that*

$$\operatorname{Bd} \mathcal{V}_j = \bigcup \mathcal{F}_j, \tag{5}$$

and let for each $k \in \{1, \ldots, n+1\}$ *and distinct* $j_1, \ldots, j_k \in \{1, \ldots, n+1\}$

$$\dim(\operatorname{Bd} \mathcal{V}_{j_1} \cap \cdots \cap \operatorname{Bd} \mathcal{V}_{j_k}) \leq n - k \tag{6}$$

hold.

Let $\mathcal{O}_j = \{O_F : F \in \mathcal{F}_j\}$ *be an open family such that* $F \subseteq O_F$ *for each* $F \in \mathcal{F}_j$.

Then for each j, $1 \le j \le n+1$, *there are discrete families* $\mathcal{W}_j^S$, $\mathcal{W}_j^B$, *and* $\mathcal{W}_j^Q$ *of cardinality* $\le \tau$, *which are disjoint in pairs, such that*

$$\mathcal{W}_j = \mathcal{W}_j^S \cup \mathcal{W}_j^B \cup \mathcal{W}_j^Q$$

is a decomposition of X *satisfying (for each* j, $1 \le j \le n+1$*):*

(a) $\operatorname{lord} \mathcal{W}_j \le 2$;
(b) $\{\operatorname{Cl} W : W \in \mathcal{W}_j^S\}$ *refines* $\mathcal{T}$; $\bigcup_{j=1}^{n+1} \mathcal{W}_j^S$ *covers* X;
(c) if $x \in \operatorname{Bd} \mathcal{W}_j$ *then there are distinct elements* W_1, W_2 *in* $\mathcal{W}_j$ *such that* $x \in \operatorname{Bd} W_1 \cap \operatorname{Bd} W_2$;
(d) $\mathcal{W}_j$ *covers* X_j *(hence* $\operatorname{Bd} \mathcal{W}_j$ *misses* X_j*);*
(e) $\operatorname{Bd} \mathcal{W}_j \cap \operatorname{Bd} \mathcal{V}_j = \emptyset$;
(f) $\mathcal{W}_j^S \cup \mathcal{W}_j^Q$ *refines* $\mathcal{V}_j$;
(g) $\mathcal{W}_j^S \cup \mathcal{W}_j^B$ *is a discrete family;*
(h) $\mathcal{W}_j^B = \{W_F : F \in \mathcal{F}_j\}$ *(the indexing is faithful, i.e. injective) and* $F \subseteq W_F \subseteq \operatorname{Cl} W_F \subseteq O_F$ *for each* $F \in \mathcal{F}_j$.

Theorem 2 *Let* X *be an* n*-dimensional* ($n \ge 0$) *metrizable space of weight* $\tau \ge \aleph_0$. *Let* $\{x_1, \dots, x_m\}$ *be any ordered set of* m *different points of* X; *similarly let* $\{y_1, \dots, y_m\}$ *be any ordered set of* m *different points of* $L_n(\tau)$. *Then there is an embedding of* $f : X \longrightarrow L_n(\tau)$ *such that* $f(x_i) = y_i$ *for any* $i = 1, \dots, m$.

Proof. For $n = 0$ the claim is a special case of Theorem 2.1 of [7]. Therefore in the rest of the proof we assume that $n \ge 1$.

We plan to construct the embedding f by means of coordinate functions which have the property that they map a point x to a rational point in $\mathcal{J}(\tau)$ if and only if x belongs to $\operatorname{Bd} U \cap \operatorname{Bd} V$, where U and V are two different elements of a decomposition of X.

Since even some x_k may be isolated points, and at the same time the corresponding y_k may be rational, it follows that this property may not be obtained for the original space X (if $x_k \in \operatorname{Bd} U \cap \operatorname{Bd} V$, then $x_k \in U \cap V$, and then, by the definition of the decomposition, $U = V$).

Hence we first include X into a larger metrizable space $\tilde{X}$ of the same dimension and weight, such that there is a decomposition $\mathcal{U} = \{U_1, \dots, U_{2m+1}\}$ of $\tilde{X}$ with $\operatorname{lord} \mathcal{U} = 2$, and such that any x_k belongs to the boundary of exactly two elements U_{2k-1}, U_{2k} of $\mathcal{U}$. Note that though only isolated points of X belonging to the set $\{x_1, \dots, x_m\}$ make the introduction of a new space $\tilde{X}$ necessary, in order to avoid complications arising from distinguishing isolates from non-isolates, we shall treat all points $x_1, \dots, x_m$ in the same manner.

Fix a metric d on X. Let $\tilde{X}$ be obtained from the disjoint union of X and $[-1, 1] \times \{1, 2, \dots, m\}$, by identification of each point x_k with $(0, k)$. Define

$\tilde{\mathrm{d}} : \tilde{X} \times \tilde{X} \longrightarrow \mathbb{R}$ as follows:

$$\tilde{\mathrm{d}}(a,b) = \begin{cases} \mathrm{d}(a,b) & \text{if } a,b \in X, \\ \mathrm{d}(a,x_k) + |t| & \text{if } a \in X, b=(t,k) \\ \mathrm{d}(x_k,b) + |t| & \text{if } b \in X, a=(t,k) \\ |s-t| & \text{if } a=(s,k), b=(t,k), \\ |s| + \mathrm{d}(x_k,x_\ell) + |t| & \text{if } a=(s,k), b=(t,\ell), k \neq \ell. \end{cases}$$

It is easily checked that $\tilde{\mathrm{d}}$ is a metric on $\tilde{X}$ extending d and that the dimension and the weight of X are preserved. See Fig. 1.

Choose $r > 0$ such that the $2r$-balls in $\tilde{X}$ around the points $x_1, \dots, x_m$ are pairwise disjoint. It is of no special importance, but we may also achieve that the balls around the isolated points (if any) contain only these points.

Let $U_{2k-1} = (B_X(x_k,r) \setminus \{x_k\}) \cup ([-1,0) \times \{k\})$, $U_{2k} = (0,1] \times \{k\}$, for $k = 1, \dots, m$, where $B_X(x_k,r)$ denotes the ball in X with the center x_k and the radius r. Also, let U_{2m+1} be the complement in $\tilde{X}$ of the union of the closures of all sets $U_1, \dots, U_{2m}$. It may happen that the complement is empty; in that case we simply omit U_{2m+1} from $\mathcal{U}$.

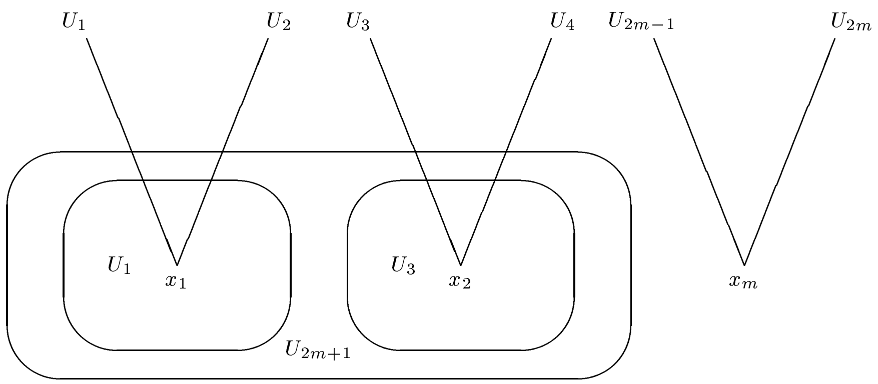

Figure 1: The space $\tilde{X}$ and the decomposition $\mathcal{U}$

It is easily seen that $\mathcal{U}$ obtained this way is a decomposition of $\tilde{X}$ with $\operatorname{lord}\mathcal{U} = 2$, and that $\operatorname{Bd} U_{2k-1} \cap \operatorname{Bd} U_{2k} = \{x_k\}$ for any $k = 1, \dots, m$, while x_k belongs to the boundary of no other member of $\mathcal{U}$. This fact play an important role in the rest of the proof, and that is the second reason why we have not restricted this construction to isolated points only. Therefore we introduce such a space $\tilde{X}$ even in the case when no point $x_1, \dots, x_m$ is isolated!

The family $\mathcal{U}$ will not be used directly in the sequel; it only serves as a model for the behavior of the decompositions $\mathcal{V}_{i,j}$ we are going to define in the neighborhoods of the points $x_1, \dots, x_m$. Compare Figs. 1 and 2.

Let $X_1, \dots, X_{n+1}$ be fixed pairwise disjoint 0-dimensional subsets of $\tilde{X}$, such that $\tilde{X} = X_1 \cup \dots \cup X_{n+1}$.

Now we want to construct decompositions $\mathcal{V}_{i,j}$ for $i = 0, 1, 2, \ldots$ and $j = 1, 2, \ldots, n+1$, starting from decompositions $\mathcal{V}_{0,j}$, and then obtaining all other decompositions by performing an inductive construction based on Lemma 1. The decompositions $\mathcal{V}_{0,j}$ will be chosen very carefully, so that even automatic applications of Lemma 1, with no additional special control except that provided by Lemma 1, will enable us to get the embedding f from an appropriate indexing of the families $\mathcal{V}_{i,j}$, $i = 0, 1, 2, \ldots$, $j = 1, 2, \ldots, n+1$.

Therefore in the next step of our proof we shall choose appropriate families $\mathcal{V}_{0,1}, \ldots, \mathcal{V}_{0,n+1}$, which will start our inductive construction of the sequences $\mathcal{V}_{i,1}, \ldots, \mathcal{V}_{i,n+1}$, $i = 0, 1, 2, \ldots$.

The choice of $\mathcal{V}_{0,1}, \ldots, \mathcal{V}_{0,n+1}$ will depend on the indexing (which will be obtained later) of these $n+1$ families (which will in turn depend on the values of $y_1, \ldots, y_m$). Therefore we first explain how the indexing of $\mathcal{V}_{0,j}$ will be obtained, and from that it will become apparent how to define $\mathcal{V}_{0,j}$.

By I1 (see page 10) we want to index the elements of $\mathcal{V}_{0,j}$ by p-tuples of elements of Λ. Choose $p \in \mathbb{N}$, greater than the tail index of $y_{k,j}$ for any $k = 1, \ldots, m$ and for any $j = 1, \ldots, n+1$ for which $y_{k,j}$ is a rational point of $\mathcal{J}(\tau)$. If $y_{k,j}$ is an irrational point, no control is needed, therefore in the case when all coordinates of all points $y_k = (y_{k,1}, \ldots, y_{k,n+1})$ are irrational we put $p = 1$.

Then, for a fixed j, if $x_k \in \operatorname{Bd} V \cap \operatorname{Bd} V'$, for $V, V' \in \mathcal{V}_{i,j}$, in order to use (3) (see I2, too), U and V must be indexed by $(\lambda_1, \ldots, \lambda_p)$ and $(\lambda'_1, \ldots, \lambda'_p)$, with $y_{k,j} = [\lambda_1, \ldots, \lambda_p, \ldots] = [\lambda'_1, \ldots, \lambda'_p, \ldots]$ (where these two sequences are equivalent, but different). If $y_{k,j} = [\lambda_1, \ldots, \lambda_p, \ldots]$ is an irrational point, then x_k must be in the element of $\mathcal{V}_{0,j}$ indexed by $(\lambda_1, \ldots, \lambda_p)$. We'll take care of this later — at the moment all points x_k will satisfy $x_k \in \operatorname{Bd} V \cap \operatorname{Bd} V'$, for some $V, V' \in \mathcal{V}_{i,j}$. Only after the construction of the sequences $\mathcal{V}_{k,j}$ we'll modify them slightly and the modified sequences will have the required property for the case of irrationality. Another approach may have been used, with immediate different treatment of rational and irrational cases, but forcing x_k to be in the set $\operatorname{Bd} \mathcal{V}_{0,j}$ prevents it from belonging to a boundary that will be introduced at some later stage. Since it is much easier to have control over $\operatorname{Bd} \mathcal{V}_{0,j}$ only, instead of having complete control over the whole $\operatorname{Bd} \mathcal{V}_{0,j} \cup \operatorname{Bd} \mathcal{V}_{1,j} \cup \operatorname{Bd} \mathcal{V}_{0,j} \cup$, we have chosen this approach.

Fix an element $\mu \in \Lambda$, which does not belong to the set $\{\lambda_1, \ldots, \lambda_p : y_{k,j} = [\lambda_1, \ldots, \lambda_p, \ldots], k = 1, \ldots, m\}$.

In $\mathcal{V}_{0,j}$ we'll introduce an element indexed by the p-tuple $(\mu, \cdots, \mu)$. By an example we show how we may transform $(\lambda_1, \ldots, \lambda_p)$ (and $(\lambda'_1, \ldots, \lambda'_p)$ in the rational case) to $(\mu, \cdots, \mu)$ by transformations obeying behavior demanded by I2. That means that any two sets have a common point in their boundaries if and only if they are indexed either by

(A) $(\lambda_1, \ldots, \lambda_k, \mu)$, $(\lambda_1, \ldots, \lambda_k, \nu)$, $\mu \neq \nu$ (for some k), or by

(B) $(\lambda_1, \ldots, \lambda_k, \mu, \nu, \ldots, \nu)$, $(\lambda_1, \ldots, \lambda_k, \nu, \mu, \ldots, \mu)$, $\mu \neq \nu$, (for some k).

In what follows any two consecutive terms must be interpreted as indexes of two elements of $\mathcal{V}_{0,j}$ having common boundary points. Note that when exemplify the p-tuples, we simplify the notation and we write e.g. 12333 instead of $(1, 2, 3, 3, 3)$.

Suppose $(\lambda_1, \ldots, \lambda_p) = 12333$, $(\lambda'_1, \ldots, \lambda'_p) = 13222$ (note that $p = 5$ is greater than the tail index, which equals 2) and that $\mu = 4$. Then

$$12333 \sim 13222 \sim 13224 \sim 13242 \sim 13244 \sim 13422 \sim 13424 \sim 13442 \sim$$

$$13444 \sim 14333 \sim 14334 \sim 14343 \sim 14344 \sim 14433 \sim 14434 \sim 14443 \sim$$

$$14444 \sim 41111 \sim 41114 \sim 41141 \sim 41144 \sim 41411 \sim 41414 \sim 41441 \sim$$

$$41444 \sim 44111 \sim 44114 \sim 44141 \sim 44144 \sim 44411 \sim 44414 \sim 44441 \sim 44444$$

Here $\alpha \sim \beta$ means that α and β satisfy either (A) or (B) from I2.

Explain the algorithm a bit; at the beginning we have $(\lambda_1, \ldots, \lambda_p)$ and $(\lambda'_1, \ldots, \lambda'_p)$ with $(\lambda_1, \ldots, \lambda_p) \sim (\lambda'_1, \ldots, \lambda'_p)$ by our choice of p (in the irrational case we'll have only $(\lambda_1, \ldots, \lambda_p)$). Then in each step there are only two possibilities: if the last cypher is not equal 4, then we replace it by 4; if the last cypher is already 4 we perform the interweaving at the nearest place with the cypher different from 4. If there is no such place (i.e. if all the cyphers are 4), we are done.

Obviously this algorithm works for arbitrarily large p, and for any choice of cyphers. We believe that this explanation is clear and that formalization of this part of the proof may be omitted. Of course this example can be easily transformed into a formal inductive proof, but it would necessarily be lengthy and cumbersome.

So, we need enough elements of $\mathcal{V}_{0,j}$, to be able to index them by the p-tuples obtained this way.

If for a $k = 1, \ldots, m$ there is a neighborhood $V \subseteq B_X(x_k, r)$ of x_k in X with the empty boundary in X, then in the case when $y_{k,j}$ is rational we may put $(V \setminus \{x_k\}) \cup [-1, 0) \times \{k\}$ and $(0, 1] \times \{k\}$ into $\mathcal{V}_{0,j}$ and index them by $(\lambda_1, \ldots, \lambda_p)$ and $(\lambda'_1, \ldots, \lambda'_p)$. In the case when $y_{k,j}$ is irrational, we put $V \cup ([-1, 1] \times \{k\})$ into $\mathcal{V}_{0,j}$ and index it by $(\lambda_1, \ldots, \lambda_p)$. No matter how we'll decompose the rest of the space $\tilde{X}$ and how we'll index the other elements of the decomposition, the choice described above will work appropriately.

For all other k, we'll repeatedly apply the fact that for any 0-dimensional subspace Z of a metric space Y, and for any closed subset F of Y and any open neighborhood U of F, there is an open neighborhood V of F, such that $\operatorname{Cl} V \subseteq U$ and $\operatorname{Bd} V \cap Z = \emptyset$ [2]. We'll use this fact for $Y = X$.

We start from the neighborhood $N_0 = B_X(x_k, r)$ of x_k; then we get a neighborhood N_1 of x_k satisfying $x_k \in N_1 \subseteq \operatorname{Cl} N_1 \subseteq N_0$ and $(\operatorname{Bd} N_1) \cap X_j = \emptyset$. Once we have a neighborhood N_s, we get N_{s+1} satisfying $x_k \in N_{s+1} \subseteq \operatorname{Cl} N_{s+1} \subseteq N_s$ and $(\operatorname{Bd} N_{s+1}) \cap X_j = \emptyset$. We repeat this procedure until we have enough N_ss to index them with all p-tuples from $(\lambda_1, \ldots, \lambda_p)$ (and $(\lambda'_1, \ldots, \lambda'_p)$) to $(\mu, \ldots, \mu)$. See Fig. 2.

Let us illustrate this ideas on the space depicted on Fig. 1, assuming $y_{1,j}$ is rational, with $y_{1,j} = [1, 2, \ldots] = [2, 1, \ldots]$, $y_{2,j}$ is irrational, with $y_{2,j} = [3, 4, \ldots]$, and $y_{m,j} = [5, 6, \ldots] = [6, 5, \ldots]$ is rational; hence $p = 2$ may be used. Also, let $\mu = 7$. We'll have to use indices $12 \sim 21 \sim 27 \sim 72 \sim 77$, $34 \sim 37 \sim 73 \sim 77$,

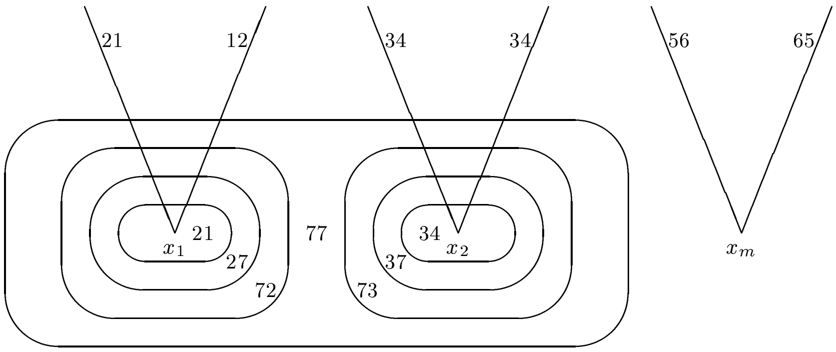

Figure 2: An example of $\mathcal{V}_{0,j}$

and $56 \sim 65$. What we get by the above construction looks like illustration on Fig. 2.

On Fig. 2 $V_{72} = N_1 \setminus \mathrm{Cl}(N_2)$, $V_{27} = N_2 \setminus \mathrm{Cl}(N_3)$, $V_{21} = (N_3 \setminus \{x_1\}) \cup ([-1,0) \times \{1\})$, $V_{12} = (0,1] \times \{1\}$, where N_1, N_2, N_3 are obtained by above procedure for the point x_1. The open sets V_{73}, V_{37}, V_{34} are obtained in a similar way from the corresponding neighborhoods N_1', N_2', N_3' of x_2, the only difference (occurring because y_2 is irrational) being that $V_{34} = N_3 \cup ([-1,1] \times \{2\})$. This is the indexing we shall use at the end; note that we'll put $(N_3' \setminus \{x_2\}) \cup ([-1,0) \times \{2\})$ and $(0,1] \times \{2\}$ into $\mathcal{V}_{0,j}$ as two different elements.

Replacing every N_s by $\mathrm{Int}\,\mathrm{Cl}\,N_s$ (and continuing to call the obtained "regularized" sets just N_s) we see that without loss of generality we may assume that $\mathrm{Bd}\,N_s = \mathrm{Bd}\,N_s \cap \mathrm{Bd}(X \setminus \mathrm{Cl}(N_s))$ (see Lemma 6 of [6]). From this additional property it follows that $\mathrm{Bd}\,N_1 = \mathrm{Bd}(X \setminus \mathrm{Cl}\,N_1) = \mathrm{Bd}(N_1 \setminus \mathrm{Cl}\,N_2) \cap \mathrm{Bd}(X \setminus \mathrm{Cl}\,N_1)$ and $\mathrm{Bd}\,N_s = \mathrm{Bd}(X \setminus \mathrm{Cl}\,N_s) = \mathrm{Bd}(N_s \setminus \mathrm{Cl}\,N_{s+1}) \cap \mathrm{Bd}(N_{s-1} \setminus \mathrm{Cl}\,N_s)$, for $s > 1$. That means that Fig. 2 presents a realistic and rather general look of $\mathcal{V}_{0,j}$, with its elements defined as the differences $N_s \setminus \mathrm{Cl}\,N_{s+1}$, for N_s not being the last (smallest) neighborhood of x_k, plus the sets $(N_t \setminus \{x_k\}) \cup [-1,0) \times \{k\}$ (if N_t is the last neighborhood), plus the sets $(0,1] \times \{k\}$, and, finally, plus the difference X minus the union of closures of all previously mentioned sets.

Now we may proceed with the inductive construction of the sequences $\mathcal{V}_{i,j}$, $i \in \mathbb{N}$, $j = 1, \ldots, n+1$. It is easily checked that the families $\mathcal{V}_{0,j}$, $j = 1, \ldots, n+1$ described above satisfy all the conditions of Lemma 1 imposed on the families $\mathcal{V}_j$ and the families $\mathcal{F}_j$ (if we substitute $\mathcal{V}_{0,j}$ for $\mathcal{V}_j$ and $\mathcal{F}_{0,j} = \{\mathrm{Bd}\,V_1 \cap \mathrm{Bd}\,V_2 : V_1, V_2 \in \mathcal{V}_{0,j}, V_1 \neq V_2, \mathrm{Bd}\,V_1 \cap \mathrm{Bd}\,V_2 \neq \emptyset\}$ for $\mathcal{F}_j$). In proving (5) it is essential that we have regularized the sets N_s [6]. The condition (6) follows from

$$\mathrm{Bd}\,\mathcal{V}_{j_1} \cap \cdots \cap \mathrm{Bd}\,\mathcal{V}_{j_k} \subseteq \left[\tilde{X} \setminus (X_{j_1} \cup \cdots \cup X_{j_k})\right] \cup \{x_1, \ldots, x_m\}$$

for each $k \in \{1,\dots,n+1\}$ and distinct $j_1,\dots,j_k \in \{1,\dots,n+1\}$, since $\tilde{X} \setminus (X_{j_1} \cup \dots \cup X_{j_k})$ is the union of $n+1-k$ 0-dimensional subsets of $\tilde{X}$.

Now let $i \geq 1$, and suppose that there are families $\mathcal{V}_{i-1,j}$ and $\mathcal{F}_{i-1,j}$, $j = 1,\dots,n+1$, satisfying conditions on $\mathcal{V}_j$ and $\mathcal{F}_j$ from Lemma 1. Let $\mathcal{W}_j^S$, $\mathcal{W}_j^B$, $\mathcal{W}_j^Q$ be the families obtained by application of Lemma 1, when we substitute $\mathcal{V}_{i-1,j}$for $\mathcal{V}_j$, $\mathcal{F}_{i-1,j}$ for $\mathcal{F}_j$, the set of all $1/i$-balls for $\mathcal{T}$, and the set of all $1/i$-balls around F, $F \in \mathcal{F}_{i-1,j}$, for $\mathcal{O}_j$. Then we define

$$\mathcal{W}_j^R = \{V \setminus \mathrm{Cl}(\mathcal{W}_j^S \cup \mathcal{W}_j^B) : V \in \mathcal{V}_{i-1,j},\ V \setminus \mathrm{Cl}(\mathcal{W}_j^S \cup \mathcal{W}_j^B) \neq \emptyset\}, \qquad (7)$$

$$\mathcal{W}_j = \mathcal{W}_j^S \cup \mathcal{W}_j^B \cup \mathcal{W}_j^R, \qquad (8)$$

$$\mathcal{V}_{i,j} = \{V \cap W : V \in \mathcal{V}_{i-1,j},\ W \in \mathcal{W}_j,\ V \cap W \neq \emptyset\}, \qquad (9)$$

$$\mathcal{F}_{i,j} = \{\mathrm{Bd}\, V_1 \cap \mathrm{Bd}\, V_2 : V_1, V_2 \in \mathcal{V}_{i-1,j},\ V_1 \neq V_2,\ \mathrm{Bd}\, V_1 \cap \mathrm{Bd}\, V_2 \neq \emptyset\}. \qquad (10)$$

This way we get the required families; among their properties is that the new families $\mathcal{V}$, $\mathcal{F}$ again satisfy the conditions of Lipscomb's lemma (therefore the induction works) and that all families appearing in the process satisfy properties D1-D14 from [6]. For details see [6, Lemma 3]. Note that all statements of [6, Lemma 3] remain true when the countability conditions are replaced by the conditions that the mentioned families are of the cardinalities $\leq \tau$. Among the consequences of the cited lemma is

$$\mathcal{F}_{i-1,j} \subseteq \mathcal{F}_{i,j}\,. \qquad (11)$$

Note that by our choice of $\mathcal{V}_{0,j}$, $j = 1,\dots,n+1$, and by (11), for any $k = 1,\dots,m$ and any $i \geq 0$, there are $V_1, V_2 \in \mathcal{V}_{i,j}$, $V_1 \neq V_2$, such that $\{x_k\} = \mathrm{Bd}\, V_1 \cap \mathrm{Bd}\, V_2 \in \mathcal{F}_{i,j}$.

We wish to construct simultaneously an indexing of the elements of the families $\mathcal{V}_{i,j}$, $i = 0, 1, \dots$, $j = 1,\dots,n+1$, satisfying the following properties:

I1 Each element of $\mathcal{V}_{i,j}$, $i \geq 0$, is indexed by an element of Λ^{p+i}.

I2 For a given $F \in \bigcup_{i=0}^{\infty} \mathcal{F}_{i,j}$, let i be the least index such that $F \in \mathcal{F}_{i,j}$ (i.e. $F \in \mathcal{F}_{i,j} \setminus \mathcal{F}_{i-1,j}$ if $i \geq 1$, or $F \in \mathcal{F}_{0,j}$ if $i = 0$). If $F = \mathrm{Bd}\, V_1 \cap \mathrm{Bd}\, V_2 \neq \emptyset$, for $V_1, V_2 \in \mathcal{V}_{i,j}$, $V_1 \neq V_2$, then V_1, V_2 are indexed either by
(A) $(\lambda_1,\dots,\lambda_k,\mu)$, $(\lambda_1,\dots,\lambda_k,\nu)$, $\mu \neq \nu$ (for some k), or by
(B) $(\lambda_1,\dots,\lambda_k,\mu,\nu,\dots,\nu)$, $(\lambda_1,\dots,\lambda_k,\nu,\mu,\dots,\mu)$, $\mu \neq \nu$, (for some k).

For any $\ell > i$, let $F = \mathrm{Bd}\, \tilde{V}_1 \cap \mathrm{Bd}\, \tilde{V}_2$, $\tilde{V}_1, \tilde{V}_2 \in \mathcal{V}_{\ell,j}$.[1] Suppose $\tilde{V}_1 \subseteq V_{(\lambda_1,\dots,\lambda_k,\mu)}$ and $\tilde{V}_2 \subseteq V_{(\lambda_1,\dots,\lambda_k,\nu)}$ (in case (A)) or $\tilde{V}_1 \subseteq V_{(\lambda_1,\dots,\lambda_k,\mu,\nu,\dots,\nu)}$ and $\tilde{V}_2 \subseteq V_{(\lambda_1,\dots,\lambda_k,\nu,\mu,\dots,\mu)}$ (in case (B)).[2] Then $\tilde{V}_1$ is indexed by the $(p+\ell)$-tuple $(\lambda_1,\dots,\lambda_k,\mu,\nu,\dots,\nu) \in \Lambda^{p+\ell}$, and similarly $\tilde{V}_2$ is indexed by the interwoven element $(\lambda_1,\dots,\lambda_k,\nu,\mu,\dots,\mu) \in \Lambda^{p+\ell}$.

[1] By (11) and (10) we know that F must be of this form.

[2] From the obvious fact that $\mathcal{V}_{i,j}$ refines $\mathcal{V}_{i-1,j}$, it follows that always either this is the case, or that the formulas obtained from these by interchanging $\tilde{V}_1$, $\tilde{V}_2$ hold true.

I3 If $V \in \mathcal{V}_{i,j}$, $i \geq 1$,[3] such that V does not contain an x_k,[4] is indexed by an index having two or more identical ciphers at the end, then there is an $F \in \mathcal{F}_{\ell,j}$, $\ell < i$, such that $F = \mathrm{Bd}\, V_1 \cap \mathrm{Bd}\, V_2 \neq \emptyset$, $V_1, V_2 \in \mathcal{V}_{i,j}$, $V_1 \neq V_2$, and either $V = V_1$ or $V = V_2$.

I4 If $V \in \mathcal{V}_{i,j}$, $i \geq 0$, is indexed by $(\lambda_1, \ldots, \lambda_{p+i})$, and if $V' \in \mathcal{V}_{k,j}$, $k > i$, is indexed by $(\mu_1, \ldots, \mu_{p+k})$, then $V' \subseteq V$ implies $(\lambda_1, \ldots, \lambda_{p+i}) = (\mu_1, \ldots, \mu_{p+i})$.

I5 If $V \in \mathcal{V}_{i,j}$, $i \geq 0$, contains a point x_k, then $y_{k,j}$ must be irrational, and V is indexed by $(\lambda_1, \ldots, \lambda_{p+i})$ if $y_{k,j} = [\lambda_1, \ldots, \lambda_{p+i}, \ldots]$.

I6 If $y_{k,j}$ is rational, and $\lambda_1, \ldots, \lambda_i, \ldots$ and $\lambda'_1, \ldots, \lambda'_i, \ldots$ are two different sequences with $y_{k,j} = [\lambda_1, \ldots, \lambda_i, \ldots] = [\lambda'_1, \ldots, \lambda'_i, \ldots]$, then for any $i \geq 0$ there are $V, V' \in \mathcal{V}_{i,j}$, indexed by $(\lambda_1, \ldots, \lambda_{p+i})$ and $(\lambda'_1, \ldots, \lambda'_{p+i})$ respectively, and such that $\{x_k\} = \mathrm{Bd}\, V \cap \mathrm{Bd}\, V'$.

But before doing this indexing, we must perform certain minor modifications of the families $\mathcal{V}_{i,j}$ (as anticipated by the Fig. 2). Namely, for any irrational point y_k, we replace the two sets $V_1, V_2 \in \mathcal{V}_{i,j}$ having the property $\mathrm{Bd}\, V_1 \cap \mathrm{Bd}\, V_2 = \{x_k\}$, by the set $V_1 \cup \{x_k\} \cup V_2$ (which is open since $\mathrm{lord}\, \mathcal{V}_{i,j} \leq 2$) and remove $\{x_k\}$ from $\mathcal{F}_{i,j}$. It is obvious that the families obtained this way (for which we continue to use the same notation) satisfy the same properties D1–D14 of [6]. It is of special importance that the small sets (i.e. the sets from the families $\mathcal{W}_j^S$ of the inductive construction) are not affected by this changes, since their properties are essential in proving that f (which is about to be obtained) is indeed an embedding.

It has already been explained how to index the families $\mathcal{V}_{0,j}$. Note (see Fig. 2; focus to 34) that for irrational points $y_{k,j}$ we have already assigned just one index to both V_1, V_2 with $V_1 \cap V_2 = \{x_k\}$, and now interpret this index as the index of $V_1 \cup \{x_k\} \cup V_2$. Clearly, the described indexing satisfies I1–I6 (those parts of it which apply). While introducing the indexing of $\mathcal{V}_{0,j}$, we have paid special attention to achieving (A) and (B) of I2 (compare our description of $\sim$ given there).

Let $\Lambda_0 \subseteq \Lambda$ be the finite set of all elements of Λ appearing as the cyphers of indices of elements of $\mathcal{V}_{0,j}$, $j = 1, \ldots, n+1$. We split $\Lambda \setminus \Lambda_0$ into countably many pairwise disjoint families of cardinality τ:

$$\Lambda \setminus \Lambda_0 = \Lambda_1 \cup \Lambda_2 \cup \Lambda_3 \cup \cdots$$

Also, let Λ'_i, $i \geq 1$, be the finite set of $(p+i)$th coordinates of $y_{k,j}$ for all k, j such that $y_{k,j}$ is irrational.

Suppose that all $\mathcal{V}_{i-1,j}$, $i \geq 1$, have been indexed according to I1–I6 and that this indexing uses only the elements of $\Lambda_0 \cup \Lambda_1 \cup \cdots \cup \Lambda_{i-1} \cup \Lambda'_1 \cup \cdots \cup \Lambda'_{i-1}$ as cyphers. According to I1 and I4, indices of elements of $\mathcal{V}_{i,j}$ are obtained

[3] Note that I3 may not be true for $i = 0$.

[4] See I5.

from certain indices of elements of $\mathcal{V}_{i-1,j}$ by juxtaposing an element of Λ at the right hand side. If an element of $\mathcal{V}_{i,j}$ has a common boundary point with an $F \in \mathcal{F}_{i-1,j}$, then its index is obtained by continuation of interweaving indices of the two elements from $\mathcal{V}_{i-1,j}$, for which F is the intersection of their boundaries (according to I1, I2, I4, and I6 ; we use (11)). That means that we need no new symbols for the cyphers to index such elements of $\mathcal{V}_{i,j}$ (i.e. we use elements of $\Lambda_0 \cup \Lambda_1 \cup \cdots \cup \Lambda_{i-1} \cup \Lambda'_1 \cup \cdots \cup \Lambda'_{i-1}$ only). There is one more case in which the new element which is juxtaposed is uniquely determined in advance: the element $V_1 \cup \{x_k\} \cup V_2$ of $\mathcal{V}_{i,j}$ containing x_k, in the case when $y_{k,j}$ is irrational, must have the first $p+i$ cyphers of $y_{k,j}$ as its index, by I5. All other members of $\mathcal{V}_{i,j}$ are indexed by indices obtained by juxtaposing elements of $\Lambda_i \setminus (\Lambda'_1 \cup \cdots \cup \Lambda'_i)$, using different elements of $\Lambda_i \setminus (\Lambda'_1 \cup \cdots \cup \Lambda'_i)$ in indices of different members of $\mathcal{V}_{i,j}$. This can be done, since the cardinality of $\mathcal{V}_{i,j}$ is $\leq \tau$, while the cardinality of $\Lambda_i \setminus (\Lambda'_1 \cup \cdots \cup \Lambda'_i)$ is exactly τ.

This completes the inductive construction of the families $\mathcal{V}_{i,j}$, $i = 0, 1, 2, \ldots$, $j = 1, \ldots, n+1$, and their indexing.

Now, in order to get an embedding $\psi : \tilde{X} \longrightarrow \mathcal{J}(\tau)$, we define the coordinate functions

$$\psi_j : \tilde{X} \longrightarrow \mathcal{J}(\tau) = \Sigma(\tau),\ j = 1, \ldots, n+1,$$

in the same way as in [6] or [8].

This means, if $x \in F$, where $F \in \mathcal{F}_{i,j}$, and i is the least index with this property (cf. I2), where $F = V_1 \cap V_2 \neq \emptyset$, $V_1, V_2 \in \mathcal{V}_{i,j}$, $V_1 \neq V_2$, and V_1 and V_2 are indexed according to I2, then $\psi_j(x)$ is defined to be

$$\psi_j(x) = [\lambda_1, \ldots, \lambda_k, \mu, \nu, \ldots, \nu, \ldots] = [\lambda_1, \ldots, \lambda_k, \nu, \mu, \ldots, \mu, \ldots].$$

Hence, in this case, $\psi_j(x)$ is rational.

If x does not belong to any $F \in \bigcup_{i=0}^{\infty} \mathcal{F}_{i,j}$, then there is a unique sequence $\lambda_1, \lambda_2, \ldots$, such that for any $i \geq 0$

$$x \in V_{\lambda_1 \ldots \lambda_{p+i}} \in \mathcal{V}_{i,j}.$$

Then we define

$$\psi_j(x) = [\lambda_1, \ldots, \lambda_k, \ldots].$$

Because of I3 and our choice of $\mathcal{O}_j$ while applying Lemma 1 (cf. D11 of [6]), $\psi_j(x)$ is irrational.

Exactly as in [6] or [8] (i.e. by proving that ψ is continuous and the family $\{\psi_1, \cdots, \psi_{n+1}\}$ separates points and closed sets) it follows that

$$\psi = (\psi_1, \cdots, \psi_{n+1}) : \tilde{X} \longrightarrow \Sigma(\tau)^{n+1}$$

is an embedding.

If $y_{k,j}$ is rational, it follows that $\psi_j(x_k) = y_{k,j}$, since p is greater than the tail index of $y_{k,j}$, and $\{x_k\} \in \mathcal{F}_{0,j}$; the indices of the two sets from $\mathcal{V}_{i,j}$ having x_k in their boundaries are obtained by continuation of the interweaving that has began already in $\mathcal{V}_{0,j}$. The resulting sequences are the two representatives of $y_{k,j}$, hence $\psi_j(x_k) = y_{k,j}$.

If $y_{k,j}$ is irrational, then x_k does not belong to any $F \in \bigcup_{i=0}^{\infty} \mathcal{F}_{i,j}$, because of the final modification we have carried out at the end of the construction of the families $\mathcal{V}_{i,j}$. Also, we were careful to juxtapose the $p+i$th cypher of $y_{k,j}$ to the index of the open set from $\mathcal{V}_{i-1,j}$ containing x_k, while indexing the open set from $\mathcal{V}_{i,j}$ containing x_k. Since the open set from $\mathcal{V}_{0,j}$ containing x_k has been indexed by the first p cyphers of $y_{k,j}$, it follows that $\psi_j(x_k) = y_{k,j}$.

Therefore, for any $k = 1, \ldots, m$ and any $j = 1, \ldots, n+1$, $\psi_j(x_k) = y_{k,j}$, i.e.

$$\psi(x_k) = y_k$$

for any k. Note that from this it follows that $\psi(x_k) \in L_n(\tau)$, since $y_k \in L_n(\tau)$.

Now, let x be any element of $\tilde{X} \setminus \{x_1, \ldots, x_m\}$.

If $\psi_j(x)$ is rational, then x belongs to some $F \in (\bigcup_{i=0}^{\infty} \mathcal{F}_{i,j}) \setminus \{\{x_1\}, \ldots \{x_m\}\}$. From the construction of $\mathcal{V}_{0,j}$ and (d) of Lemma 1, it follows that $F \cap X_j = \emptyset$. Hence $x \notin X_j$.

If all coordinates of $\psi(x)$ were rational, x would not be in the union $X_1 \cup \cdots \cup X_{n+1} = \tilde{X}$ — a contradiction. Therefore, $\psi(x)$ has at least one irrational coordinate, i.e. $\psi(x) \in L_n(\tau)$.

That proves that

$$f = \psi | X : X \longrightarrow L_n(\tau)$$

is the required embedding. ∎

Corollary 3 *For any $y_0 \in L_n(\tau)$ the pointed space $(L_n(\tau), y_0)$ is a universal object in the category of pointed metrizable spaces of dimension $\leq n$ and weight $\leq \tau$.*

Proof. ∎

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References

[1] R. Engelking. *Dimension Theory.* PWN-Polish Scientific Publishers, Warszawa and North-Holland Publishing Company, Amsterdam-Oxford-New York, 1978.

[2] R. Engelking. *General Topology.* Heldermann Verlag, Berlin, 1989.

[3] S. L. Lipscomb. *A universal one-dimensional metric space.* In *TOPO 72 - General Topology and its Applications, Second Pittsburgh Internat. Conf.*, volume 378 of *Lecture Notes in Math.* Springer-Verlag, New York, 1974, 248–257.

[4] S. L. Lipscomb. *On imbedding finite-dimensional metric spaces.* Trans. Amer. Math. Soc. 211 (1975), 143–160.

[5] S. L. Lipscomb, J. C. Perry. *Lipscomb's $L(A)$ space fractalized in Hilbert's $l^2(A)$ space.* Proc. Amer. Math. Soc. 115 (1992), 1157–1165.

[6] I. Ivanšić and U. Milutinović. *A universal separable metric space based on the triangular Sierpiński curve.* Top. Appl. 120 (2002), 237–271.

[7] I. Ivanšić and U. Milutinović. *Relative embeddability into Lipscomb's 0-dimensional universal space.* Houston J. Math. (to appear)

[8] U. Milutinović. *Completeness of the Lipscomb universal space.* Glas. Mat. Ser. III 27(47) (1992), 343–364.

[9] U. Milutinović. *Contributions to the theory of universal spaces* (Croatian). Ph.D. Thesis, University of Zagreb, Zagreb, 1993.

[10] U. Milutinović. *Approximation of maps into Lipscomb's space by embeddings.* (submitted for publication)

[11] J. Nagata. *Modern Dimension Theory.* Heldermann Verlag, Berlin, 1983.