University of Ljubljana
Institute of Mathematics, Physics and Mechanics
Department of Mathematics
Jadranska 19, 1000 Ljubljana, Slovenia

Preprint series, Vol. 41 (2003), 855

ON LINKING OF COMPACT SETS

Matjaž Željko

ISSN 1318-4865

January 8, 2003

Ljubljana, January 8, 2003

On linking of Compact sets

Matjaž Željko *

January 7, 2003

Abstract

We introduce a property $\mathcal{L}$ for a subset of a manifold which enables us to pass the geometric linking property from the manifold to this subset. We prove that cubes with handles M and N are linked if and only if subsets $X \subset \operatorname{Int} M$ and $Y \subset \operatorname{Int} N$ having property $\mathcal{L}$ are linked. We present a criterion which shows us that many of known Cantor sets explicitly given by defining sequences have this property. As an application of the property $\mathcal{L}$ we extend the theorem on rigid Cantor sets thus allowing slightly more complicated terms in their defining sequences.

Keywords: geometric linking, Cantor set, defining sequence, rigid Cantor set

AMS classification: 57M30

1. Introduction

Let $\mathbb{E}^n$ be an Euclidean n-dimensional space and $A, B \subset \mathbb{E}^n$ disjoint closed subsets. We say that A and B are (geometrically) *unlinked* if there exists $(n-1)$-dimensional sphere $S \subset \mathbb{E}^n$ which separates A and B. We say that A and B are (geometrically) *linked* if such sphere doesn't exist. One usually proves that two sets are geometricaly unlinked by explicitely constructing the separating sphere.

Suppose now we have manifolds M and N with subsets $X \subseteq M$ and $Y \subseteq N$. If X and Y are the cores of respective manifolds it is obvious that X and Y are geometrically linked if and only if M and N are geometrically linked.

We will introduce a property $\mathcal{L}$ which enables us to prove that subsets $X \subseteq M$ and $Y \subseteq N$ having this property are linked if M and N are linked.

As a corrollary we will extend the theorem on rigid Cantor sets in $\mathbb{E}^3$ allowing slightly more complicated terms in their defining sequences.

*Permanent address: Faculty of Mathematics and Physics, University of Ljubljana, Jadranska 19, 1000 Ljubljana, Slovenia; E-mail: matjaz.zeljko@fmf.uni-lj.si

2. Property $\mathcal{L}$

Let $M \subset \mathbb{E}^n$ be a compact n-manifold with boundary. We say that a closed subset $A \subset \operatorname{Int} M$ has a property $\mathcal{L}$ in M, if for every n-disk $B \subset \mathbb{E}^n \setminus A$ and every open neighbourhood $U \subset M \setminus A$ of $\operatorname{Fr} M$ there exist such n-disk $B' \subset \mathbb{E}^n$ that $B \setminus \operatorname{Int} M = B' \setminus \operatorname{Int} M$ and $B' \cap M \subset U$.

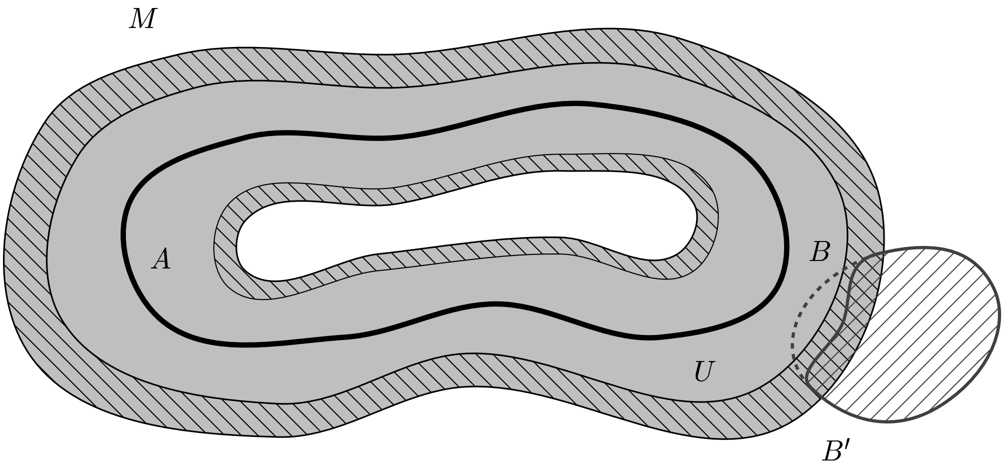


Figure 1: The set A has a property $\mathcal{L}$ in M

Assertion 1. *Let $M \subset \mathbb{E}^n$ be a manifold and $S \subset \operatorname{Int} M$ (any) core for M. Then S has a property $\mathcal{L}$ in M.*

Proof. Let $B \subset \mathbb{E}^n \setminus S$ be a n-disk and $U \subset M \setminus S$ open neighbourhood of $\operatorname{Fr} M$ in M. As S is a core of M there exists a homeomorphism $h = (h_1, h_2): M \setminus S \to \operatorname{Fr} M \times [0, 1)$ satisfying $h(x) = (x, 0)$ for $x \in \operatorname{Fr} M$. Hence there exists $\tau \in (0, 1)$ such that $h^{-1}(\operatorname{Fr} M \times [0, \tau]) \subset U$. The mapping $f: M \setminus S \to h^{-1}(\operatorname{Fr} M \times [0, \tau))$ defined by

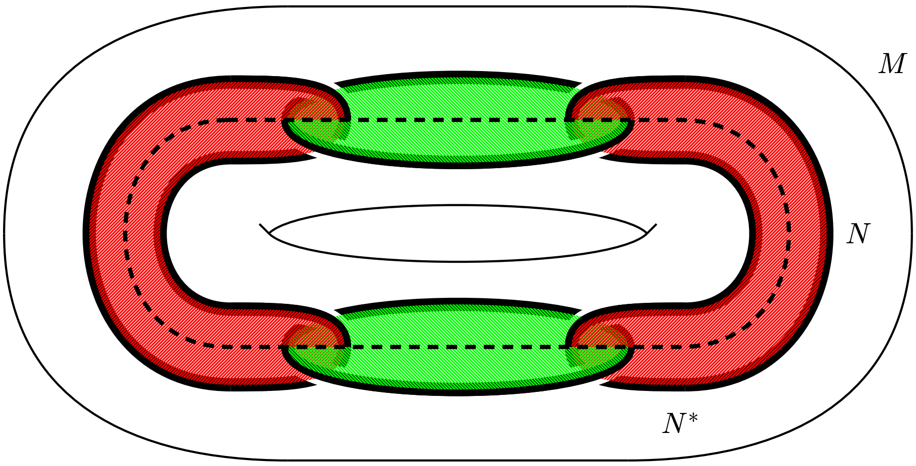
$$f(x) = h^{-1}(\, h_1(x), \tau \cdot h_2(x)\,),$$

is a homeomorphism which is identitiy on $\operatorname{Fr} M$. Finally we define $B' \cap M$ to be $f(B \cap M)$ as $f(B \cap M) \subset U$. ∎

Theorem 2. *Let $M \subset \mathbb{E}^3$ be a cube with handles and $N \subset \operatorname{Int} M$ be manifold which is finite union of cubes with handles. Let there exist a finite collection $\mathfrak{D}$ of 2-disks in $\operatorname{Int} M$ with pairwise disjoint boundaries satisfying the following conditions:*

1. *disks in $\mathfrak{D}$ intersect transversally and for every disk $D \in \mathfrak{D}$ there exists such component N' of N that* $\operatorname{Fr} D = D \cap N'$*;*
2. *no three disks from $\mathfrak{D}$ intersect;*
3. *for every two disks $D, E \in \mathfrak{D}$ the set $D \cap E$ is connected (it may be empty);*
4. *for every disk $D \in \mathfrak{D}$ the set $D \setminus \left(N \cup \bigcup_{E \in \mathfrak{D} \setminus \{D\}} E\right)$ is simply connected;*
5. *the set $M \setminus \left(N \cup \bigcup_{D \in \mathfrak{D}} D\right)$ is connected and*
6. *there exists a set $A \subset N \cup \bigcup_{D \in \mathfrak{D}} D$ having property $\mathcal{L}$ in M.*

Then N has a property $\mathcal{L}$ in M.



Proof. Let us denote $|\mathfrak{D}'| = \bigcup_{E \in \mathfrak{D}'} E$ for any subset $\mathfrak{D}' \subset \mathfrak{D}$ and $\mathfrak{D}_D = \{E \in \mathfrak{D} \setminus \{D\}; E \cap D \neq \emptyset\}$ for any $D \in \mathfrak{D}$. Let $\operatorname{Fr} \mathfrak{D} = \bigcup_{D \in \mathfrak{D}} \operatorname{Fr} D$.

We shall modify that part of the disk B which lies in M that the modified disk will not intersect A. Hence using the property $\mathcal{L}$ for A one can further modify the disk B such that $B \cap M$ lies arbitrary close to $\operatorname{Fr} M$.

Let $B \subset \mathbb{E}^3$ be a 3-disk disjoint to N. Using a small move in $\operatorname{Int} M \setminus N$ we can assume that $\operatorname{Fr} B$ intersects $|\mathfrak{D}|$ transversally. Fix an arbitrary 2-disk $D \in \mathfrak{D}$. The set $B \cap D \neq \emptyset$ is either empty or every its component is a disk with holes. As the set $D \setminus N^*$ is connected every component of $\operatorname{Int} D \cap N$ is a 2-disk.

For every $E \in \mathfrak{D}_D$ the set $D \cap E$ is an arc whose one boundary point lies in some disk in $\operatorname{Int} D \cap N$ and the other boundary point lies in $\operatorname{Fr} D$. (Both boundary points of the arc $D \cap E$ can not simultanously lie in $\operatorname{Fr} D$ as $D \setminus N^*$

is connected and similarly they can not simultanously lie in N as in this case the set $E \setminus N^*$ will not be connected.) By assumption of the theorem no three disks in $\mathfrak{D}$ intersect and hence arcs in $|\mathfrak{D}_D| \cap D$ are pairwise disjoint.

Let J be an arbitrary circle in $D \cap \operatorname{Fr} B$. The circle J bounds a 2-disk (say D_J) in $\operatorname{Int} D$. If $D_J \cap N \neq \emptyset$ then there exists a 2-disk $E \in \mathfrak{D}_D$ such that the arc $D \cap E$ has one boundary point on $N \cap D_J$ (otherwise $D \setminus N^*$ is not simply connected) and the other boundary point on $\operatorname{Fr} D$. The circle J bounds two 2-disks (say B_J and B'_J) on $\operatorname{Fr} B$ and both of them are disjoint with $\operatorname{Fr} E$. Hence the intesection number (in $\mathbb{E}^3$) of the circle $\operatorname{Fr} E$ and the 2-sphere $B_J \cup D_J$ equals to 1 which is certainly impossible. Therefore no 2-disk in $N \cap \operatorname{Int} D$ lies in any circle in $D \cap \operatorname{Fr} B$.

Let $E \in \mathfrak{D}_D$ be an arbitrary disk. Then $E \cap D$ is an arc having both boundary points outside D_J. If $E \cap D_J \neq \emptyset$ using lemma 1 one can find a 2-disk $E_J \subset D_J$ which boundary consists of two arcs: one of them lies in J, the other one lies in $E \cap D$. Using a small isotopy having its support in some small neighbourhood of E_J in M one can appropriately move B to deminish the number of arcs in the intersection of $E \cap D$ and D_J.

Hence after finitely many steps we end up with disjoint 1-spheres in $\operatorname{Fr} B \cap D$ and 2-disks in $E \in \mathfrak{D}_D$.

Now choose outermost (with respect to D) circle in $D \cap \operatorname{Fr} B$ and denote it by K. (The circle K is not necessarly unique.) The circle K bounds some 2-disk D_K. Using a twosided collar of D in M we enlarge D_K to $D_K \times [-1, 1]$. As $M \setminus N^*$ is connected we can connect points $(x, 1) \in \operatorname{Fr} D_K \times \{1\}$ and $(x, -1) \in \operatorname{Fr} D_K \times \{-1\}$ with some arc $w \subset M \setminus N^*$.

Let $W = w \times B^2$ be a small tubular neighbourhood of w in $M \setminus N^*$. Obviously $W \approx D_K \times [-1, 1]$ and using an appropriate modification of W near $\operatorname{Fr} w \times B^2$ one can obtain $W \cap B \subset D_K \times [-1, 1]$ and $\operatorname{Fr} w \times B^2 = D_K \times \{-1, 1\}$.

Hence we can divert that part of B which lies in $D_K \times [-1, 1]$ to W. (The choice of outermost component of $D \cap \operatorname{Fr} B$ was neccessary here. It is possible though that $B \cap (D_K \times [-1, 1])$ has more than one component.) We repeat the procedure for other circles in $D \cap \operatorname{Fr} B$ starting again with outermost ones.

We repeat the procedure for other 2-disks in $\mathfrak{D}$. We end up with 3-disk B satisfying $\operatorname{Fr} B \cap N^* = \emptyset$. As $A \subset N^*$ has a property $\mathcal{L}$ we can modify B accordingly. ■

We have used the folloving observation:

Lemma 1. *Let D be a 2-disk and T a nonempty finite collection of pairwise disjoint arcs in D properly embedded in D. The boundaries of arcs in T divide* $\operatorname{Fr} D$ *into collection L of circular arcs. Then there exist a disk $E \subset D$ bounded by an arc from T and an arc from L.*

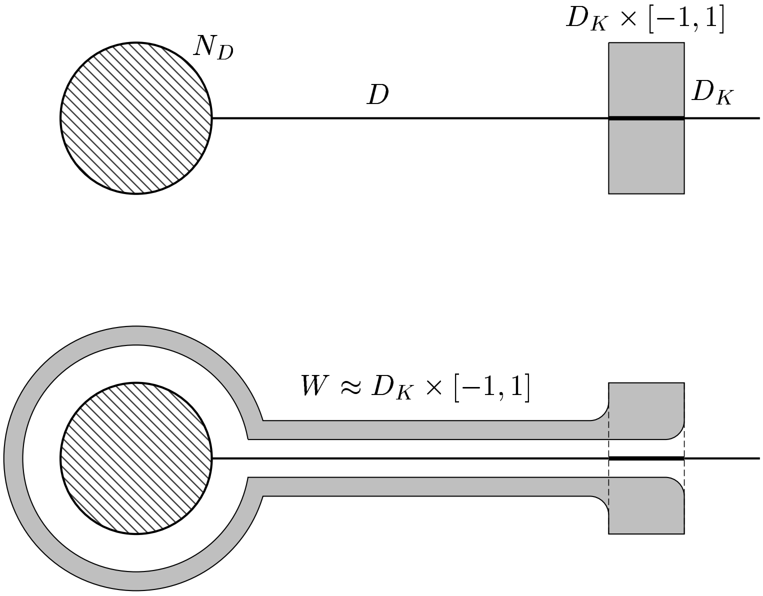


Figure 2: Modification of B near D_J

PROOF. We induct on number n of arcs in T. The case $n = 1$ is obvious. Now let there be $n+1$ arcs in T and t one of them. Using the inductive hypothesis there exist a 2-disk E' bounded by $t' \in T$ and $l \in L$ for some t' and l. If $t \cap E = \emptyset$ then $E = E'$ else $\mathrm{Fr}\, t$ splits l in three arcs. In this case the disk E is bounded by one of them and t. ∎

Theorem 3. *Let M and N be cubes with handles in $\mathbb{E}^3$ and $X \subset \mathrm{Int}\, M$ and $Y \subset \mathrm{Int}\, N$ closed subsets having property $\mathcal{L}$. Then X and Y are geometrically linked if and only if M and N are geometrically linked.*

PROOF. It is obvious that a separating sphere for M and N separates X and Y too. Now assume to the contrary that there exists a separating sphere $S \subset \mathbb{E}^3$ for X in Y while M and N are geometrically linked. Due to the symmetry we may assume that $X \subset \mathrm{Int}\, S$ and $Y \subset \mathrm{Ext}\, S$. We may also assume that S intersects $\mathrm{Fr}\, M$ and $\mathrm{Fr}\, N$ transversally. Let $B \subset \mathbb{E}^3$ be a 3-disk bounded by S.

If $B \cap N \neq \emptyset$ we can use property $\mathcal{L}$ for Y in N to replace B with B' which intersects N near $\mathrm{Fr}\, N$. Then using a small move near $\mathrm{Fr}\, N$ we push B' off N to obtain B''. Due to simplicity we denote the 3-disk B'' with B again. Then $X \subset \mathrm{Int}\, B$ and $B \cap N = \emptyset$.

If $S \cap \operatorname{Fr} M \neq \emptyset$ we embed $\mathbb{E}^3$ in $S^3 \equiv \mathbb{E}^3 \cup \{\infty\}$ naturally (one-point compactification). Choose an arbitrary point $b \in \operatorname{Fr} B$. Then there exist an arc J in $S^3 \setminus N$ connecting b and ∞ which (except b) lies in $S^3 \setminus B$.

For some small regular neighbourhood $N(J)$ of an arc J in S^3 the manifold $N(J) \cup B$ is a 3-disk disjoint to N. We note that $X \subset N(J) \cup B$.

The complement of $N(J) \cup B$ in S^3 is a 3-disk B' disjoint to X. As $\infty \in N(J) \cup B$ we use the property $\mathcal{L}$ of X to push the 3-disk B' off M. So we have obtained a 3-disk B'' whose boundary is a sphere S' separating X in Y and disjoint to $\operatorname{Fr} M \cup \operatorname{Fr} N$. Note that $Y \subset \operatorname{Int} B'$ and $X \cap B' = \emptyset$. Let us simplify the notation again and denote B'' simply by B.

As M and N are geometrically linked we have $B \subset \operatorname{Int} M$ (ie. $M \subset B$ is not possible). The manifold M is a cube with at least one handle because it is linked to N. Therefore there exists a properly embedded 2-disk D in M (ie. $\operatorname{Fr} D = D \cap \operatorname{Fr} M$) such that $\operatorname{Fr} D \not\simeq 0$ in $\mathbb{E}^3 \setminus \operatorname{Int} M$. We may assume that $D \cap B = \emptyset$. A small regular neighbourhood $N(D)$ of D in $\mathbb{E}^3$ is a 3-disk which can be pushed off M using property $\mathcal{L}$. This contradicts the fact that $\operatorname{Fr} D \not\simeq 0$ in $\mathbb{E}^3 \setminus \operatorname{Int} M$. ■

Remark. Recall that $A \subseteq M$ is geometrically central in a manifold M if for any 2-disk with holes H and any interior essential mapping $f\colon H \to M$ we have $f(H) \cap A \neq \emptyset$. In other words: if $f(H) \cap A = \emptyset$ then $f\colon H \to M$ is interior inessential and hence there exists a map $g\colon H \to \operatorname{Fr} M$ which coincides with f on $\operatorname{Fr} H$.

It is interesting to note, that one can prove a theorem similar to the theorem 2 replacing property $\mathcal{L}$ with geometric centrality. However it is not known yet whether the following linking theorem similar to the theorem 3 is valid.

Conjecture 1. *Let M and N be cubes with handles in $\mathbb{E}^3$ and $X \subset \operatorname{Int} M$ and $Y \subset \operatorname{Int} N$ closed subsets being geometrically central in M resp. N. Then X and Y are geometrically linked if and only if M and N are geometrically central.*

Definition 1. *A defining sequence $(M_i)_i \in \mathcal{D}(X)$ for a Cantor set $X \subset \mathbb{E}^3$ which consists of cubes with handles has a property $\mathcal{L}$ if for every i and every component M of M_i the manifold $M \cap M_{i+1}$ has a property $\mathcal{L}$ in M.*

Theorem 4. *Let a defining sequence $(M_i)_i \in \mathcal{D}(X)$ for a Cantor set $X \subset \mathbb{E}^3$ have property $\mathcal{L}$. Then for every i and for every component M of manifold M_i the Cantor set $X \cap M$ has property $\mathcal{L}$ in M.*

PROOF. Let $X' = X \cap M$. Let B be a 3-disk disjoint to B. First we prove that there exists a 3-disk D such that $D \setminus \operatorname{Int} M = B \setminus \operatorname{Int} M$ and $D \cap M \cap M_{i+1} = \emptyset$. Among all 3-disks D satisfying $D \setminus \operatorname{Int} M = B \setminus \operatorname{Int} M$ and $D \cap X' = \emptyset$ we choose such 3-disk that the number j, $j \geq i$, $D \cap M \cap M_j \neq \emptyset$ and $D \cap M \cap M_{j+1} = \emptyset$ is minimal. If $j > i$ we use property $\mathcal{L}$ to push D out of every component of $M \cap M_{j+1}$ which contradicts the minimality of j. Hence $j = i$ and we use property $\mathcal{L}$ to move $D \cap M$ arbitrary close to $\operatorname{Fr} M$. ■

3. Rigid Cantor sets

Definition 2. *A defining sequence $(M_i)_i \in \mathcal{D}(X)$ for the Cantor set $X \subset \mathbb{E}^3$ is* brittle *if for every component M of M_i and for every component M' of $M_{i+1} \cap M$ the following holds: if some loop in $\operatorname{Fr} M$ is contractible in M then this loop is contractible in $(M \setminus M_{i+1}) \cup M'$ as well.*

A Cantor set which is endowed with a brittle defining sequence has some nice properties. The first and the second item of the following theorem can be proved as in [2, Lemma 5.6], the last item is a slight generalization of [5, Lemma 2.1].

Theorem 5. *Let $(M_i)_{i=1}^{\infty}$ be a brittle defining sequence for a Cantor set $X \subset \mathbb{E}^3$, which consists of cubes with handles. Then:*

1. *For every nonempty subset $A \subset X$ every loop $J \subset M_1$ is contractible in $(M_1 \setminus X) \cup A$.*
2. *For every dense countable subset $A \subset X$ the set $(\mathbb{E}^3 \setminus X) \cup A$ is 1-ULC.*
3. *For every closed proper subset $A \subset X$ there exists a 3-disk $B \subset \operatorname{Int} M_1$ such that $A \subset \operatorname{Int} B$.*

Definition 3. *Let $A \subset \mathbb{E}^n$ be an arbitrary (closed) set. We say that the set A is* rigid, *if for every homeomorphism $f\colon \mathbb{E}^n \to \mathbb{E}^n$ it holds: if $f(A) = A$ then $f|_A = id_A$.*

There are many examples of rigid sets. Martin [3] has constructed a rigid 2-sphere in E^3, Böthe [1] has constructed a simply connected curve in E^3. Wright [5] has constructed rigid Cantor set in $\mathbb{E}^3$ using Antoine necklaces and has later [4] generalized construction to $\mathbb{E}^n$, $n \geq 3$.

The key part of construction [5] is lemma [5, lema 2.1]. If we substitute this lemma by theorem 3 we can take more general building blocks in the construction thus constructing many more rigid Cantor sets.

Let there be a defining sequence $(M_i)_{i=0}^{\infty} \in \mathcal{D}(X)$ for a Cantor set X. For every component M of M_i one can define a graph Γ_i^M as follows:

- The components of $M \cap M_{i+1}$ are the vertices of Γ_i^M.
- There is a connection between M' and M'' in Γ_i^M if and only if M' and M'' are geometrically linked.

We say that Γ_i^M is a *linking pattern* of X in M. Let

$$\begin{aligned} \Gamma_i &= \bigsqcup_{M \text{ component of } M_i} \Gamma_i^M, \\ \Gamma(X;(M_i)_i) &= (\Gamma_0, \Gamma_1, \Gamma_2, \ldots). \end{aligned}$$

We say that $\Gamma(X;(M_i))$ is a *linking pattern of X with respect to the defining sequence* (M_i) or simply a *linking pattern of X*.

Lemma 2. *Let Cantor set X and Y be given by defining sequences $(M_i)_{i=0}^\infty \in \mathcal{D}(X)$ and $(N_i)_{i=0}^\infty \in \mathcal{D}(Y)$ such that:*

1. *both defining sequences have property $\mathcal{L}$ (see definition 1);*
2. *both defining sequences are brittle (see definition 2);*
3. *for every component M of M_i the graph Γ_i^M is a cycle and for every component N of N_i the graph Γ_i^N is a cycle.*

If $h(X) \subset Y$ for some homeomorphisem $h: \mathbb{E}^3 \to \mathbb{E}^3$ then there exist $n \in \mathbb{N} \cup \{0\}$ and a component V of N_n such that $h(X) = V \cap Y$ and $\Gamma_0^{M_0} \approx \Gamma_n^V$.

Proof. This is essentialy lemma [5, Lemma 3.1].

Theorem 6. *Let Cantor set X be given by a defining sequence $(M_i)_{i=0}^\infty \in \mathcal{D}(X)$ such that:*

1. *defining sequence has property $\mathcal{L}$;*
2. *defining sequence is brittle;*
3. *for every component M of M_i the graph Γ_i^M is a cycle and*
4. *for every two different components M and N of M_i the sequences $\Gamma(M \cap X)$ and $\Gamma(N \cap X)$ are different.*

Then X is a rigid Cantor set.

Proof. This is essentially [5, Theorem 3.2].

As it was implicitly proven in 2 the Cantor set which was used as building block in the construction was unsplittable (ie. no two its points can by separated by a 2-sphere in its complement). Hence by the theorem 6 we get a rigid

Cantor set which complement has nontrivial fundamental group. So it is not possible to replace the building blocks in 2 with such ones with simply connected complement as these are completly splittable (ie. every two its points can by separated by a sphere in its complement).

In order to prove the following conjecture one has to find a different approach.

Conjecture 2. *There exists a rigid Cantor set in* $\mathbb{E}^3$ *which complement has trivial fundamental group.*

4. Acknowledgements

This work constitutes part of the author's doctoral thesis prepared under direction by Professor Dušan Repovš at the University of Ljubljana, Slovenia. Research was partially supported by the Ministry of Education, Science and Sport of the Republic of Slovenia research program No. 0101-509.

References

[1] H. G. Böthe, *Eine fixierte Kurve in* $\mathbb{E}^3$, General Topology and Its Relations to Modern Analysis and Algebra. II (Proc. Second Prague Topological Symposium 1966), Academia, Prague, 1967, 68–73.

[2] R. J. Daverman, *On the scarcity of tame disks in certain wild cells*, Fund. Math. **79** (1973), 63–77.

[3] J. M. Martin, *A rigid sphere*, Fund. Math. **59** (1966), 117–121.

[4] D. G. Wright, *Rigid Cantor sets in* $\mathbb{E}^n$, Pacific J. Math. **121** (1986), 245–456.

[5] D. G. Wright, *Rigid sets in manifolds*, Geometric and Algebraic Topology, Banach Center Publ. **18**, PWN, Warsaw 1986, 161–164.