University of Ljubljana
Institute of Mathematics, Physics and Mechanics
Department of Mathematics
Jadranska 19, 1000 Ljubljana, Slovenia

Preprint series, Vol. 41 (2003), 857

RUIN PROBABILITIES AND DECOMPOSITIONS FOR GENERAL PERTURBED RISK PROCESSES

Miljenko Huzak   Mihael Perman
Hrvoje Šikić   Zoran Vondraček

ISSN 1318-4865

January 22, 2003

Ljubljana, January 22, 2003

Ruin probabilities and decompositions for general perturbed risk processes *

Miljenko Huzak †, Mihael Perman,
Hrvoje Šikić, and Zoran Vondraček ‡

January 20, 2003

Abstract

We study a general perturbed risk process with cumulative claims modelled by a subordinator with finite expectation, and the perturbation by a spectrally negative Lévy process with zero expectation. We derive the Pollaczek-Hinchin formula for the survival probability of that risk process, and give an interpretation of the formula.

AMS 2000 Mathematics Subject Classification: Primary 60J25, Secondary 60G51, 60J75, 60K30, 91B30
Keywords and phrases: Risk theory, ruin probability, Pollaczek-Hinchin formula, subordinator, spectrally negative Lévy process

Running Title: General risk process

*This work was supported by the bilateral Croatian-Slovenian research grant.

†The research of this author is supported in part by MZT grant 0037115 of the Republic of Croatia.

‡The research of this author is supported in part by MZT grant 0037107 of the Republic of Croatia.

1 Introduction

The classical Cramér-Lundberg model assumes that the risk process $(R(t),\ t \geq 0)$ is given by $R(t) = ct - \sum_{i=1}^{N(t)} Y_i$, where $c > 0$ is the premium rate, $(Y_i,\ i \in \mathbb{N})$ an i.i.d. sequence of nonnegative random variables modelling individual claims, and $(N(t),\ t \geq 0)$ a homogeneous Poisson process of rate $\lambda > 0$, independent of $(Y_i,\ i \in \mathbb{N})$. Hence the cumulative claim process is modelled by the compound Poisson process $\sum_{i=1}^{N(t)} Y_i$. Let F denote the distribution function of Y_i, and let $\mu = \mathbb{E}Y_i$. The central question for the model is the computation of the ruin probability in infinite time, given initial capital $x > 0$, defined by

$$\vartheta(x) := \mathbb{P}(R(t) + x < 0 \text{ for some } t > 0)\,.$$

In case $c \leq \lambda\mu$, this quantity is identically equal to one. Hence, one always assumes the net profit condition $c > \lambda\mu$, and defines the parameter $\rho := \lambda\mu/c < 1$. Instead of studying the ruin probability, one can equivalently consider the survival probability $\theta(x) := 1 - \vartheta(x)$, which is more convenient. One of the few explicit results for the survival probability is the Pollaczek-Hinchin formula:

$$\theta(x) = (1-\rho)\sum_{n=0}^{\infty} \rho^n F_I^{n*}(x)\,, \tag{1.1}$$

where $F_I(x) = (1/\mu)\int_0^x (1 - F(t))\,dt$ is the integrated tail distribution. Formula (1.1) is usually derived via renewal arguments. The resulting integro-differential equation for ϑ is solved using Laplace transforms. The explanation of the formula is given by considering the supremum of the dual process $\widehat{R}(t) := -R(t)$. By the net profit condition $\widehat{R}(t)$ drifts to $-\infty$, hence the supremum is a.s. finite, and clearly, $\theta(x) = \mathbb{P}(\sup_{0\leq t<\infty} \widehat{R}(t) \leq x)$. It is easy to see that $\sup_{0\leq t<\infty} \widehat{R}(t)$ is a sum of geometrically many i.i.d. random variables. It is not, however, quite as easy to determine the distribution of these variables. Usually fluctuation theory is used. We refer the reader to [Asm] and [RSST] for details.

In this paper we are interested in generalizations of the Cramér-Lundberg model, which lead to the same type of the Pollaczek-Hinchin formula for the survival probability, and which admit an explanation of the formula by decomposition of the supremum of the dual process in the random sum of ladder heights. Dufresne and Gerber ([DG]) considered the risk process $(R(t),\, t \geq 0)$ perturbed by a multiple of standard Brownian motion $(W(t),\, t \geq 0)$, and defined $X(t) := R(t) + \sigma W(t)$, $\sigma > 0$. Using renewal arguments, they derived the formula

$$\theta(x) = (1-\rho) \sum_{n=0}^{\infty} \rho^n (G^{(n+1)*} * F_I^{n*})(x)\,. \tag{1.2}$$

The parameter ρ and the distribution function F_I are the same as in the unperturbed model, while G is an exponential distribution function with parameter $2c/\sigma^2$. They also gave the following interpretation of the formula (1.2): Let $\sigma_1, \sigma_2, \ldots$ be the moments when a new supremum of the dual process $\widehat{X}(t) := -X(t)$ is reached by a jump of the claim process $\sum_{i=1}^{N(t)} Y_i$. Then the number of such moments has geometric distribution with parameter ρ, G is the distribution function of the supremum of $\widehat{X}(t)$ just before σ_1, and F_I is the conditional distribution of the overshoot over the previous supremum, given $\sigma_1 < \infty$.

Furrer ([Fur]) considered the process $X(t) = R(t) + Z_\alpha(t)$, where R is the classical risk process, and Z_α, an α-stable Lévy process with no positive jumps, $1 < \alpha < 2$. He used the explicit formula for the Laplace exponent of the infimum of $X(t)$ due to Zolotarev ([Zol]) to obtain formula (1.2) for the survival probability of $X(t)$. The distribution function G is explicitly identified as the Mittag-Leffler distribution given by $1 - G(x) = \sum_{n=0}^{\infty} (-cx^{\alpha-1})^n / \Gamma(1 + (\alpha-1)n)$.

Schmidli [Sch] gives a nice interpretation of G as the distribution of the supremum of the dual process $\widehat{X}$ just before the first time the process $\widehat{X}$ exceeds its previous supremum by a jump of the cumulative claim process. His setting is more general in the sense that the cumulative claim process is generated by a stationary, ergodic, marked point process.

Another possible generalization of the classical risk process is to allow a different cumulative claim process. Dufresne, Gerber and Shiu [DGS] considered the model in which the claim process was modelled by a Gamma process. Such a process has infinitely many jumps in a finite interval. Yang and Zhang ([YZ]) study this model perturbed by a Brownian motion. Using the approach in [Fur], they derive a formula of the type (1.2) with F_I replaced by an exponential integral type distribution, and G is again the exponential distribution.

In this paper we consider a general perturbed risk process $X(t) = ct - C(t) + Z(t)$ where $(C(t),\, t \geq 0)$ is a cumulative claim process, and $(Z(t),\, t \geq 0)$ a perturbation. Note that the cumulative claim process has to be increasing. Therefore, if one wants to stay in the realm of processes with stationary independent increments, the only choice for modelling $(C(t),\, t \geq 0)$ are subordinators. Hence, we assume that $(C(t),\, t \geq 0)$ is a subordinator (without drift) having finite expectation satisfying the net profit condition $c - \mathbb{E}C(1) > 0$. The perturbation is modelled by a Lévy process $(Z(t),\, t \geq 0)$ with no positive jumps, having zero expectation. The assumption that the expectation is zero is inconsequential, since $\mathbb{E}Z(1)$ can always be moved to the premium rate. In the analysis of the risk process $(X(t),\, t \geq 0)$, we will rely heavily on fluctuation theory for general Lévy processes, which is particularly explicit for processes with no positive jumps. For background on these results, we refer the reader to the book by Bertoin [Ber].

Our first result is the formula for the survival probability for the process X which is proved in Section 3:

$$\mathbb{P}(\inf_{0 \leq t < \infty} X(t) > -x) = \theta(x) = (1-\rho) \sum_{n=0}^{\infty} \rho^n (G^{(n+1)*} * H^{n*})(x)\,. \qquad (1.3)$$

We essentially follow the approach from [Fur], and obtain explicitly the parameter ρ and the distribution functions G and H appearing in the formula. It turns out that G can be identified as the distribution function of the absolute supremum of the process $(-ct - Z(t),\, t \geq 0)$, while H is related to the subordinator $C(t)$ only, and can be thought of as the integrated tail distribution of jumps. In Section 4 we give an interpretation of the formula (1.3) by decomposing the supremum of the dual process $\widehat{X}(t) := -X(t)$ into the random sums of modified ladder heights. In order to do this, we first show that the times when the new supremum of $\widehat{X}(t)$ is reached by a jump of the subordinator are discrete. Let $0 < \sigma_1 < \sigma_2 < \dots$ be those times, and let $\tilde{G}$ be the distribution function of $\widehat{S}(\sigma_1-)$, where $\widehat{S}(t) := \sup_{0\le s\le t} \widehat{X}(s)$. We show that $\widehat{S}(\sigma_1-)$ and the overshoot $\widehat{S}(\sigma_1) - \widehat{S}(\sigma_1-)$ are conditionally independent given $\sigma_1 < \infty$, and identify the conditional distribution of the overshoot with H. Using the strong Markov property at times σ_i, we rederive the formula (1.3) with $\tilde{G}$ instead of G (and the same ρ). This clearly implies that $\tilde{G} = G$ yielding the required interpretation. Our results are more general and cover the results obtained in [Fur], [YZ] and [Sch] (in Lévy case).

A transparent view on the formula (1.3) is provided by looking at the ladder height process of $\widehat{X}$. The ladder height process is obtained by time-changing $\widehat{S}(t)$ via the inverse local time at zero of the reflected process $\widehat{S}(t) - \widehat{X}(t)$. This process records only values where the new supremum is reached, and consequently, contains all the relevant information on the distribution of the supremum of $\widehat{X}(t)$. In Section 5 the results of Section 4 are reinterpreted in terms of the ladder height process.

2 Setting and notation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which all random variables will be defined. As explained in the introduction, we model the cumulative claim process by a subordinator $C = (C(t), t \ge 0)$ without a drift. Let ν be the Lévy measure of C, i.e., ν is a σ-finite measure on $(0,\infty)$ satisfying $\int_{(0,\infty)} (x \wedge 1)\, \nu(dx) < \infty$. The Laplace exponent of C is defined by

$$\Phi_C(\beta) := \int_{(0,\infty)} (1 - e^{-\beta x})\, \nu(dx)$$

so that

$$\mathbb{E}[\exp\{-\beta C(t)\}] = \exp\{-t\Phi_C(\beta)\}\,.$$

Note that

$$\mathbb{E}C(1) = \Phi_C'(0+) = \int_{(0,\infty)} x\,\nu(dx) = \int_0^\infty \nu(x,\infty)\,dx$$

where the last equality follows by partial integration. From now on we assume that $\mathbb{E}C(1) < \infty$. Let

$$H(x) := \frac{1}{\mathbb{E}C(1)} \int_0^x \nu(y,\infty)\,dy\,. \tag{2.1}$$

Then H is an absolutely continuous distribution function with density $h(x) = \nu(x,\infty)/\mathbb{E}C(1)$. We call H the integrated tail distribution. The Laplace transform of H is given by

$$\widehat{H}(\beta) := \int_0^\infty e^{-\beta x}\,H(dx) = \int_0^\infty e^{-\beta x} h(x)\,(dx) = \frac{1}{\mathbb{E}C(1)}\frac{\Phi_C(\beta)}{\beta}\,. \tag{2.2}$$

Let $\Delta C(t) = C(t) - C(t-)$. Then $(\Delta C(t),\, t \geq 0)$ is a Poisson point process with characteristic measure ν and the state space $(0,\infty) \cup \{\partial\}$, where ∂ is an isolated point. Moreover, $C(t) = \sum_{0<s\leq t} \Delta C(s)$.

We model the risk process $R = (R(t),\, t \geq 0)$ as $R(t) = ct - C(t)$ where $c > 0$ is the premium rate. Then R is a Lévy process with no positive jumps (i.e., spectrally negative Lévy process). The Laplace exponent ψ_R of R is defined by relation

$$\mathbb{E}[\exp\{\beta R(t)\}] = \exp\{t\psi_R(\beta)\}\,.$$

Clearly, $\psi_R(\beta) = c\beta - \Phi_C(\beta)$. It is important to note that $R(t)$ stays positive in a neighborhood of $t = 0$, implying that ruin (with zero initial capital) does not occur immediately. This follows from the fact that $\lim_{t\to 0} C(t)/t = 0$ ([Ber], p.84). From now on we assume that the net profit condition $c > \mathbb{E}C(1)$ holds, and let $d := c - \mathbb{E}C(1)$. It follows that $\mathbb{E}R(1) = \psi_R'(0+) = d > 0$, which implies that R drifts to $+\infty$. We also introduce the parameter $\rho := \mathbb{E}C(1)/c \in (0,1)$.

The perturbation Z of the risk process R is modelled by a spectrally negative, mean zero, Lévy process. More precisely, let Π_Z be an infinite σ-finite measure on $(-\infty,0)$ (or $\Pi_Z = 0$) such that

$$\int_{(-\infty,0)} (x^2 \wedge 1)\, \Pi_Z(dx) < \infty \text{ and } \tag{2.3}$$

$$\int_{(-\infty,-1)} |x|\, \Pi_Z(dx) < \infty\,. \tag{2.4}$$

The first condition says that Π_Z is a Lévy measure concentrated on $(-\infty,0)$, while the second ensures finite expectation of Z. The perturbation $Z = (Z(t),\, t \geq 0)$ is the spectrally negative Lévy process with the Laplace exponent defined by

$$\psi_Z(\beta) := \frac{\sigma^2}{2}\beta^2 + \int_{(-\infty,0)} (e^{\beta x} - 1 - \beta x)\, \Pi_Z(dx) \tag{2.5}$$

where $\sigma \geq 0$. Condition (2.4) implies that the integrand in (2.5) is integrable with respect to Π_Z, and also that $\mathbb{E}|Z(1)| < \infty$. Further, $\mathbb{E}Z(1) = \psi_Z'(0+) = 0$ (e.g., [Sat], p.163). Note that we allow Z to be identically zero (both $\Pi_Z = 0$ and $\sigma = 0$). However, Z cannot be compound Poisson because such processes cannot have $\mathbb{E}Z(t) = 0$. Let us point out that our setting includes the Brownian perturbation ($\sigma > 0$, $\Pi_Z = 0$), and also the perturbation by α-stable spectrally negative Lévy process for $\alpha \in (1,2)$ ($\sigma = 0$, $\Pi_Z(dx) = (1/|x|^{\alpha+1})1_{(-\infty,0)}dx$).

Finally, we define the general perturbed risk process $X = (X(t),\, t \geq 0)$ as $X(t) := R(t) + Z(t) = ct - C(t) + Z(t)$, where C and R are independent processes. The process X is a spectrally negative Lévy process with finite positive expectation $\mathbb{E}X(1) = c - \mathbb{E}C(1) = d > 0$. Therefore, $\lim_{t\to\infty} X(t) = +\infty$ a.s., i.e., X drifts to infinity. Let $\mathcal{F}^0(t) := \sigma(C(s), Z(s),\, 0 \leq s \leq t)$, and let $\mathcal{F} = (\mathcal{F}(t),\, t \geq 0)$ be the filtration obtained in the usual way by augmenting $\mathcal{F}^0(t)$. Clearly, $X(t)$ is $\mathcal{F}(t)$-measurable. The Laplace exponent ψ of X, defined by the relation

$$\mathbb{E}[\exp\{\beta X(t)\}] = \exp\{t\psi(\beta)\}$$

is, due to independence of C and Z, given by

$$\psi(\beta) = c\beta - \Phi_C(\beta) + \psi_Z(\beta)\,, \quad \beta \geq 0\,.$$

Since ψ is strictly convex and $\psi'(0+) = \mathbb{E}X(1) > 0$, ψ is strictly increasing on $[0,\infty)$, and therefore has a strictly increasing inverse $\Phi : [0,\infty) \to [0,\infty)$. Since $\psi(0) = 0$, it follows that $\Phi(0) = 0$.

In the sequel, we will be interested in the function $\theta : [0,\infty) \to [0,1]$ defined by

$$\theta(x) := \mathbb{P}(X(t) \geq -x, \text{ for all } t \geq 0)\,. \tag{2.6}$$

This function is the survival probability of the general perturbed risk process X starting with the initial capital $x \geq 0$. The initial behavior of X determines θ at zero. If there is no perturbation, i.e., if $X = R$, then, as said before, X remains positive (a.s.) for an initial period of time, and hence $\theta(0) > 0$. On the other hand, if $Z \neq 0$, then X is of unbounded variation, hence the point 0 is regular for $(-\infty, 0)$. Thus X hits the interval $(-\infty, 0)$ immediately, implying $\theta(0) = 0$.

3 Laplace transform approach

In this section we derive the Pollaczek-Hinchin formula for the survival probability using the explicit form of the Laplace transform of the absolute infimum of X. Let $I(\infty) := \inf_{0\leq s<\infty} X(s)$ and $I(t) := \inf_{0\leq s\leq t} X(s)$. The fluctuation theory for Lévy processes provides the following formula for the Laplace transform of the infimum evaluated at an independent exponential time $\tau(q)$ with parameter $q > 0$ (see [Ber], p.192):

$$\mathbb{E}[\exp\{\beta I(\tau(q))\}] = \frac{q(\Phi(q) - \beta)}{\Phi(q)(q - \psi(\beta))}\,, \quad \beta > 0\,.$$

Letting $q \downarrow 0$, and using $I(\tau(q)) \xrightarrow{\mathbb{P}} I(\infty)$, it follows that

$$\mathbb{E}[\exp\{\beta I(\infty)\}] = \psi'(0+)\frac{\beta}{\psi(\beta)} = d\frac{\beta}{\psi(\beta)}\,, \quad \beta > 0\,. \tag{3.1}$$

Let us introduce for a moment the following notation: $Y(t) = ct + Z(t)$ and $\psi_Y(\beta) = c\beta + \psi_Z(\beta)$. By the same argument as above it follows that

$$\mathbb{E}[\exp\{-\beta(-\inf_{0\le t<\infty} Y(t))\}] = \psi_Y'(0+)\frac{\beta}{\psi_Y(\beta)} = c\frac{\beta}{\psi_Y(\beta)}\,, \quad \beta > 0\,.$$

Let G denote the distribution function of $-\inf_{0\le t<\infty} Y(t) = \sup_{0\le t<\infty}(-ct - Z(t))$. Then the last formula says that

$$\widehat{G}(\beta) := \int_0^\infty e^{-\beta x}\, G(dx) = c\frac{\beta}{\psi_Y(\beta)}\,, \quad \beta > 0\,. \tag{3.2}$$

Recall formulae (2.1) and (2.2) from Section 2: $H(x) := (1/\mathbb{E}C(1)) \int_0^x \nu(y,\infty)\, dy$ and $\widehat{H}(\beta) = \Phi_C(\beta)/(\mathbb{E}C(1)\beta)$. Also recall that $\rho = \mathbb{E}C(1)/c$, hence $d/c = (c - \mathbb{E}C(1))/c = 1-\rho$. Now we compute $d\beta/\psi(\beta)$ in terms of ρ, $\widehat{G}$ and $\widehat{H}$. This idea comes from [Fur].

$$\begin{aligned}
d\frac{\beta}{\psi(\beta)} &= d\frac{1}{\frac{\psi_Y(\beta)}{\beta} - \frac{\Phi_C(\beta)}{\beta}} \\
&= d\frac{1}{\frac{c}{\widehat{G}(\beta)} - \mathbb{E}C(1)\widehat{H}(\beta)} \\
&= \frac{d}{c}\,\frac{\widehat{G}(\beta)}{1-\rho\widehat{G}(\beta)\widehat{H}(\beta)} \\
&= (1-\rho)\widehat{G}(\beta)\sum_{n=0}^\infty (\rho\widehat{G}(\beta)\widehat{H}(\beta))^n\,.
\end{aligned}$$

By inverting the Laplace transform, we obtain the following theorem.

Theorem 3.1 *The survival probability of the general perturbed risk process X is given by*

$$\theta(x) = \mathbb{P}(I(\infty) \ge -x) = (1-\rho)\sum_{n=0}^\infty \rho^n (G^{(n+1)*} * H^{n*})(x)\,, \quad x \ge 0. \tag{3.3}$$

We point out that H depends only on the subordinator C, while G depends on the premium rate c and the perturbation Z. Brownian perturbations were considered in [Fur] and α-stable ones in [DG] and [YZ]. In both cases the distribution G is given explicitly. If there is no perturbation, $Z = 0$, then $\widehat{G}(\beta) = 1$, and consequently, the distribution function G can be omitted from formula (3.3).

4 Decomposition of the supremum of $\widehat{X}$

Let $\widehat{X}(t) := -X(t) = -ct + C(t) - Z(t)$ denote the dual process of X, and let

$$\widehat{S}(t) := \sup_{0\le s\le t} \widehat{X}(s), \quad \widehat{S}(\infty) := \sup_{0\le s<\infty} \widehat{X}(s).$$

Since $\widehat{X}$ drifts to $-\infty$, $\widehat{S}(\infty) < \infty$ a.s. Introduce the following notation: $\widehat{I}(t) := \inf_{0\le s\le t} \widehat{X}(s)$ and $S(t) := \sup_{0\le s\le t} X(s)$. Clearly, $-\widehat{I}(t) = S(t)$. By a time reversal argument, $-\widehat{I}(t) \overset{d}{=} \widehat{S}(t) - \widehat{X}(t)$, and hence

$$\widehat{S}(t) - \widehat{X}(t) \overset{d}{=} S(t). \tag{4.1}$$

In this section we give a decomposition of $\widehat{X}$ at certain stopping times which, following Schmidli [Sch], we call modified ladder epochs.

Let $\mathcal{P}(\mathcal{F})$ be the predictable σ-algebra on $\mathbb{R}_+ \times \Omega$ with respect to the filtration $\mathcal{F}$ introduced in Section 2. Let $\mathcal{B}_\partial$ be the Borel σ-algebra on $(0,\infty) \cup \{\partial\}$. If $H : \mathbb{R}_+ \times \Omega \times ((0,\infty) \cup \{\partial\}) \to \mathbb{R}_+$ is a nonnegative process measurable with respect to $\mathcal{P}(\mathcal{F}) \otimes \mathcal{B}_\partial$, then the following compensation formula is valid (e.g., [RY], p.439, or [Ber], p.9):

$$\mathbb{E}\left(\sum_{0\le t<\infty} H(t,\omega,\Delta C(t,\omega))\right) = \mathbb{E}\left(\int_0^\infty dt \int_{(0,\infty)} \nu(d\epsilon) H(t,\omega,\epsilon)\right) \tag{4.2}$$

In the next theorem we compute the expected number of times the new supremum of $\widehat{X}$ is attained by a jump of a subordinator C over the previous supremum. Note that this is the case if and only if $\Delta C(t) > \widehat{S}(t-) - \widehat{X}(t-)$.

Theorem 4.1 *The following formula is valid:*

$$\mathbb{E}\left(\sum_{0\le t<\infty} 1_{\{\Delta C(t)>\widehat{S}(t-)-\widehat{X}(t-)\}}\right) = \frac{\mathbb{E}C(1)}{c-\mathbb{E}C(1)}\,. \tag{4.3}$$

Proof. We use the compensation formula with $H(t,\omega,\epsilon) := 1_{(\widehat{S}(t-,\omega)-\widehat{X}(t-,\omega),\infty)}(\epsilon)$. The left hand side in (4.2) is then precisely the left hand side in (4.3). For the right hand side in the compensation formula compute:

$$\begin{aligned} & \mathbb{E}\left(\int_0^\infty dt \int_{(0,\infty)} \nu(d\epsilon) 1_{(\widehat{S}(t-)-\widehat{X}(t-),\,\infty)}(\epsilon)\right) \qquad (4.4)\\ = \;& \mathbb{E}\left(\int_0^\infty dt\, \nu(\widehat{S}(t-)-\widehat{X}(t-),\,\infty)\right) \\ = \;& \int_0^\infty \mathbb{E}[\nu(\widehat{S}(t)-\widehat{X}(t),\infty)]\,dt \\ = \;& \int_0^\infty dt\,\mathbb{E}[\nu(S(t),\infty)] \end{aligned}$$

where the third line follows by continuity in probability of $\widehat{X}$, and the fourth line by (4.1). Clearly, the last expression is equal to the monotone limit

$$\lim_{q\to 0}\int_0^\infty e^{-qt}dt\,\mathbb{E}[\nu(S(t),\infty)] = \lim_{q\to 0}\frac{1}{q}\int_0^\infty qe^{-qt}dt\,\mathbb{E}[\nu(S(t),\infty)] \tag{4.5}$$

Let $\tau(q)$ be an exponential time with parameter q independent of C and Z, and let F denote the distribution function of $S(\tau(q))$. Then F is exponential with parameter $\Phi(q)$. It follows that

$$\begin{aligned} \int_0^\infty qe^{-qt}dt\,\mathbb{E}[\nu(S(t),\infty)] &= \mathbb{E}[\int_0^\infty qe^{-qt}dt\,\nu(S(t),\infty)] \qquad (4.6)\\ &= \mathbb{E}[\nu(S(\tau(q)),\infty)] \\ &= \int_0^\infty \nu(x,\infty)\,F(dx) \\ &= \int_0^\infty (1-e^{-\Phi(q)x})\,\nu(dx) \end{aligned}$$

where the last equation follows by partial integration. Further,

$$\lim_{q\to 0}\frac{1-e^{-\Phi(q)x}}{q}=\lim_{q\to 0}\frac{\Phi(q)}{q}x=\frac{1}{\psi'(0+)}\,x=\frac{x}{d}\,.$$

By monotone convergence theorem

$$\begin{aligned}\lim_{q\to 0}\int_0^\infty e^{-qt}dt\,\mathbb{E}[\nu(S(t),\infty)] &= \lim_{q\to 0}\int_0^\infty \frac{1-e^{-\Phi(q)x}}{q}\,\nu(dx) \qquad (4.7)\\ &= \frac{1}{d}\int_0^\infty x\,\nu(dx)\\ &= \frac{\mathbb{E}C(1)}{c-\mathbb{E}C(1)}\,.\end{aligned}$$

This proves formula (4.3). □

¿From Theorem 4.1 it follows that the epochs when a new supremum of $\widehat{X}$ is reached by a jump of C are discrete. Define

$$\sigma := \inf\{t>0:\ \Delta C(t) > \widehat{S}(t-) - \widehat{X}(t-)\}\,.$$

Then $\sigma > 0$ a.s.

For $y>0$, let $\widehat{\tau}_y := \inf\{t>0:\ \widehat{X}(t)>y\}$ be the entrance time of $\widehat{X}$ in (y,∞), and, similarly, $\tau_y := \inf\{t>0:\ X(t)>y\}$. Note that $\widehat{S}(t-)\le y$ if and only if $t\le\widehat{\tau}_y$. We need the occupation time formula for the reflected process $\widehat{S}-\widehat{X}$ before $\sigma\wedge\widehat{\tau}_y$.

Proposition 4.2 *For $x>0$ and $y>0$ the following formula is valid:*

$$\mathbb{E}\int_0^{\sigma\wedge\widehat{\tau}_y} 1_{(\widehat{S}(t)-\widehat{X}(t)\le x)}\,dt = \mathbb{P}(\sigma=\infty,\widehat{\tau}_y=\infty)\frac{x}{d}\,. \qquad (4.8)$$

Proof. We first compute the expected occupation time of $\widehat{S}-\widehat{X}$ below x:

$$\begin{aligned}\mathbb{E}\int_0^\infty 1_{(\widehat{S}(t)-\widehat{X}(t)\le x)}\,dt &= \int_0^\infty \mathbb{P}(\widehat{S}(t)-\widehat{X}(t)\le x)\,dt \qquad (4.9)\\ &= \int_0^\infty \mathbb{P}(S(t)\le x)\,dt\\ &= \mathbb{E}\tau_x\,.\end{aligned}$$

Since $(\tau_x,\, x>0)$ is a subordinator with the Laplace exponent Φ, it follows that $\mathbb{E}\tau_x = (\mathbb{E}\tau_1)x = \Phi'(0+)x = x/d$.

Now we compute the expected occupation time of $\widehat{S}-\widehat{X}$ below x after the time $\sigma\vee\widehat{\tau}_y$:

$$\begin{aligned}
& \mathbb{E}\int_0^\infty 1_{(\widehat{S}(t)-\widehat{X}(t)\le x)} 1_{(t>\sigma)} 1_{(\widehat{S}(t)>y)}\, dt \qquad (4.10)\\
= \;& \mathbb{E}\left[\int_{\sigma\vee\widehat{\tau}_y}^\infty 1_{(\widehat{S}(t)-\widehat{X}(t)\le x)}\, dt\,,\, \sigma\vee\widehat{\tau}_y<\infty\right]\\
= \;& \mathbb{P}(\sigma\vee\widehat{\tau}_y<\infty)\mathbb{E}\left[\int_{\sigma\vee\widehat{\tau}_y}^\infty 1_{(\widehat{S}(t)-\widehat{X}(t)\le x)}\, dt \mid \sigma\vee\widehat{\tau}_y<\infty\right]\\
= \;& \mathbb{P}(\sigma\vee\widehat{\tau}_y<\infty)\mathbb{E}\int_0^\infty 1_{(\widehat{S}(t)-\widehat{X}(t)\le x)}\, dt\\
= \;& \mathbb{P}(\sigma<\infty,\widehat{\tau}_y<\infty)\frac{x}{d}\,,
\end{aligned}$$

where the fourth line follows from the strong Markov property. Similarly

$$\mathbb{E}\int_0^\infty 1_{(\widehat{S}(t)-\widehat{X}(t)\le x)} 1_{(\widehat{S}(t)>y)}\, dt = \mathbb{P}(\widehat{\tau}_y<\infty)\frac{x}{d}\,. \qquad (4.11)$$

Subtracting (4.10) from (4.11), it follows

$$\mathbb{E}\int_0^\infty 1_{(\widehat{S}(t)-\widehat{X}(t)\le x)} 1_{(t\le\sigma)} 1_{(\widehat{S}(t)>y)}\, dt = \mathbb{P}(\sigma=\infty,\widehat{\tau}_y<\infty)\frac{x}{d}\,. \qquad (4.12)$$

One can prove similarly that

$$\mathbb{E}\int_0^\infty 1_{(\widehat{S}(t)-\widehat{X}(t)\le x)} 1_{(t\le\sigma)}\, dt = \mathbb{P}(\sigma=\infty)\frac{x}{d}\,. \qquad (4.13)$$

Finally, (4.8) follows by subtracting (4.12) from (4.13). □

Let $J := (\Delta C(\sigma) - (\widehat{S}(\sigma-)-\widehat{X}(\sigma-)))1_{(\sigma<\infty)}$ be the overshoot at time σ. In the next proposition we compute the preliminary version of the joint distribution of the vector $(\widehat{S}(\sigma-), J, \widehat{S}(\sigma-)-\widehat{X}(\sigma-))$ on $\{\sigma<\infty\}$.

Proposition 4.3 *For $x, y, z > 0$*

$$\mathbb{P}(\widehat{S}(\sigma-) \le y, J > x, \widehat{S}(\sigma-) - \widehat{X}(\sigma-) > z,\, \sigma < \infty) = \qquad (4.14)$$
$$= \frac{\mathbb{P}(\sigma = \infty, \widehat{\tau}_y = \infty)}{d} \int_{x+z}^{\infty} \nu(t,\infty)\, dt\,.$$

Proof. We use the compensation formula with

$$H(t,\omega,\epsilon) := 1_{(\widehat{S}(t-,\omega)\le y)} 1_{(\widehat{S}(t-,\omega)-\widehat{X}(t-,\omega)>z)} 1_{(t\le\sigma(\omega))} 1_{(x+\widehat{S}(t-,\omega)-\widehat{X}(t-,\omega),\infty)}(\epsilon)\,.$$

Then

$$\mathbb{E} \sum_{0\le t<\infty} H(t,\omega,\Delta C(t,\omega)) =$$
$$= \mathbb{P}(\widehat{S}(\sigma-) \le y, \widehat{S}(\sigma-) - \widehat{X}(\sigma-) > z, J > x,\, \sigma < \infty)\,.$$

On the other hand,

$$\begin{aligned}
& \mathbb{E}\left(\int_0^\infty dt \int_{(0,\infty)} \nu(d\epsilon) H(t,\omega,\epsilon)\right) \\
= \;& \mathbb{E}\left(\int_0^\sigma dt\, 1_{(\widehat{S}(t-)\le y)} 1_{(\widehat{S}(t-)-\widehat{X}(t-)>z)} \int_{(0,\infty)} 1_{(x+\widehat{S}(t-)-\widehat{X}(t-),\infty)}(\epsilon)\nu(d\epsilon)\right) \\
= \;& \mathbb{E}\left(\int_0^{\sigma\wedge\widehat{\tau}_y} dt\, 1_{(\widehat{S}(t)-\widehat{X}(t)>z)}\, \nu(x+\widehat{S}(t)-\widehat{X}(t),\infty)\right) \\
= \;& \frac{\mathbb{P}(\sigma=\infty,\widehat{\tau}_y=\infty)}{d} \int_0^\infty 1_{(z,\infty)}(t)\, \nu(x+t,\infty)\, dt \\
= \;& \frac{\mathbb{P}(\sigma=\infty,\widehat{\tau}_y=\infty)}{d} \int_{x+z}^\infty \nu(t,\infty)\, dt\,,
\end{aligned}$$

where the fourth line follows from Proposition 4.2. □

¿From formula (4.14) we can easily derive several useful corollaries.

Corollary 4.4 *The following formulae are valid:*

$$\mathbb{P}(\sigma < \infty) = \rho \qquad (4.15)$$
$$\mathbb{P}(J > x \mid \sigma < \infty) = \frac{1}{\mathbb{E}C(1)} \int_x^\infty \nu(t,\infty)\, dt = 1 - H(x) \qquad (4.16)$$

Proof. Let $x \to 0$, $y \to \infty$ and $z \to 0$ in (4.14). It follows that

$$\mathbb{P}(\sigma < \infty) = \frac{\mathbb{P}(\sigma = \infty)}{d} \, \mathbb{E}C(1)\,.$$

By solving for $\mathbb{P}(\sigma < \infty)$ we get (4.14). To obtain (4.15), let $y \to \infty$ and $z \to 0$ in (4.14). It follows that

$$\mathbb{P}(J > x, \sigma < \infty) = \frac{\mathbb{P}(\sigma = \infty)}{d} \int_x^\infty \nu(t,\infty)\,dt\,.$$

By conditioning,

$$\begin{aligned}\mathbb{P}(J > x \,|\, \sigma < \infty) &= \frac{1-\rho}{\rho d} \int_x^\infty \nu(t,\infty)\,dt \\ &= \frac{1}{\mathbb{E}C(1)} \int_x^\infty \nu(t,\infty)\,dt\,.\end{aligned}$$

□

In the next corollary, we interpret $\widehat{S}(\sigma-)$ as the absolute supremum $\widehat{S}(\infty)$ in case $\sigma = \infty$.

Corollary 4.5 *The event $\{\sigma < \infty\}$ and the random variable $\widehat{S}(\sigma-)$ are independent. As a consequence, the conditional distribution of $\widehat{S}(\sigma-)$ given $\sigma < \infty$ is equal to the unconditional distribution of $\widehat{S}(\sigma-)$.*

Proof. Let $x \to 0$ and $z \to 0$ in (4.14). It follows that

$$\begin{aligned}\mathbb{P}(\widehat{S}(\sigma-) \le y, \sigma < \infty) &= \mathbb{P}(\sigma = \infty, \widehat{\tau}_y = \infty)\frac{\mathbb{E}C(1)}{d} \qquad (4.17)\\ &= \mathbb{P}(\sigma = \infty, \widehat{S}(\infty) \le y)\frac{\mathbb{E}C(1)}{d}\,.\end{aligned}$$

Clearly,

$$\mathbb{P}(\widehat{S}(\sigma-) \le y, \sigma = \infty) = \mathbb{P}(\widehat{S}(\infty) \le y, \sigma = \infty)$$

Adding up,

$$\begin{aligned}\mathbb{P}(\widehat{S}(\sigma-) \le y) &= \left(\frac{\mathbb{E}C(1)}{d} + 1\right) \mathbb{P}(\widehat{S}(\infty) \le y, \sigma = \infty) \\ &= \frac{c}{d}\,\mathbb{P}(\widehat{S}(\infty) \le y, \sigma = \infty)\,.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbb{P}(\widehat{S}(\sigma-) \le y)\mathbb{P}(\sigma < \infty) &= \frac{c}{d}\,\mathbb{P}(\widehat{S}(\infty) \le y, \sigma = \infty) \cdot \frac{\mathbb{E}C(1)}{c} \\ &= \frac{\mathbb{E}C(1)}{d}\,\mathbb{P}(\widehat{S}(\infty) \le y, \sigma = \infty) \\ &= \mathbb{P}(\widehat{S}(\sigma-) \le y, \sigma < \infty)\end{aligned}$$

by (4.17). □

It follows that

$$\begin{aligned}\mathbb{P}(\sigma = \infty, \widehat{\tau}_y = \infty) &= \mathbb{P}(\sigma = \infty, \widehat{S}(\infty) \le y) \\ &= \mathbb{P}(\sigma = \infty, \widehat{S}(\sigma-) \le y) \\ &= \mathbb{P}(\sigma = \infty)\mathbb{P}(\widehat{S}(\sigma-) \le y \,|\, \sigma = \infty) \\ &= \mathbb{P}(\sigma = \infty)\mathbb{P}(\widehat{S}(\sigma-) \le y)\,.\end{aligned} \tag{4.18}$$

Let $\widetilde{G}$ denote the distribution function of $\widehat{S}(\sigma-)$. Proposition 4.3 can be now improved to

Theorem 4.6 *The joint distribution of the vector* $(\widehat{S}(\sigma-), J, \widehat{S}(\sigma-)-\widehat{X}(\sigma-))$ *on* $\{\sigma < \infty\}$ *is given by*

$$\begin{aligned}&\mathbb{P}(\widehat{S}(\sigma-) \le y, J > x, \widehat{S}(\sigma-) - \widehat{X}(\sigma-) > z,\, \sigma < \infty) \\ = \;&\mathbb{P}(\widehat{S}(\sigma-) \le y)\left(\frac{1}{\mathbb{E}C(1)}\int_{x+z}^{\infty} \nu(t,\infty)\,dt\right)\mathbb{P}(\sigma < \infty)\,.\end{aligned} \tag{4.19}$$

Moreover, $\widehat{S}(\sigma-)$ *and* J *are conditionally independent given* $\sigma < \infty$*, and*

$$\mathbb{P}(\widehat{S}(\sigma-) \le y, J > x \,|\, \sigma < \infty) = \widetilde{G}(y)(1 - H(x))\,. \tag{4.20}$$

Remark 4.7 *Formula (4.19) considerably extends the severity of ruin formula (see, e.g., [RSST], p.168).*

Define $\sigma_1 := \sigma$ and inductively $\sigma_{n+1} := \inf\{t > \sigma_n : \Delta C(t) > \widehat{S}(t-) - \widehat{X}(t-)\}$ on $\{\sigma_n < \infty\}$. Let $L_0 := \widehat{S}(\sigma_1-)$, $J_1 := \widehat{S}(\sigma_1) - \widehat{S}(\sigma_1-)$ and

$L_1 := \widehat{S}(\sigma_2-) - \widehat{S}(\sigma_1)$ on $\{\sigma_1 < \infty\}$, etc. Let also $N := \max\{n : \sigma_n < \infty\}$. By the strong Markov property of $\widehat{X}$, N has a geometric distribution with parameter $\mathbb{P}(\sigma_1 = \infty) = 1 - \rho$. Clearly,

$$\widehat{S}(\infty) = L_0 + J_1 + L_1 + \cdots + J_N + L_N \, .$$

Note that $\mathbb{P}(L_0 \le x, N = 0) = \mathbb{P}(\widehat{S}(\sigma-) \le x, \sigma = \infty) = \widetilde{G}(x)(1-\rho)$. For every $n \in \mathbb{N}$ we have by the strong Markov property at σ_n, and by equality (4.20) that

$$\mathbb{P}(L_0 + J_1 + L_1 + \cdots + J_n + L_n \le x, N = n) = (1-\rho)\rho^n (\widetilde{G}^{(n+1)*} * H^{n*})(x) \, .$$

This leads to the Pollaczek-Hinchin formula for the distribution function of $\widehat{S}(\infty)$.

Theorem 4.8 *For $x \ge 0$,*

$$\mathbb{P}(\widehat{S}(\infty) \le x) = (1-\rho) \sum_{n=0}^{\infty} \rho^n (\widetilde{G}^{(n+1)*} * H^{n*})(x) \, . \tag{4.21}$$

Since the survival probability $\theta(x) = \mathbb{P}(\widehat{S}(\infty) \le x)$, we have obtained two formulae for θ: (3.3) and (4.21). Therefore,

$$(1-\rho) \sum_{n=0}^{\infty} \rho^n (G^{(n+1)*} * H^{n*})(x) = (1-\rho) \sum_{n=0}^{\infty} \rho^n (\widetilde{G}^{(n+1)*} * H^{n*})(x) \, .$$

By computing Laplace transforms of both sides, we get that

$$\frac{(1-\rho)\widehat{G}(\beta)}{1 - \rho\, \widehat{G}(\beta)\widehat{H}(\beta)} = \frac{(1-\rho)\widehat{\widetilde{G}}(\beta)}{1 - \rho\, \widehat{\widetilde{G}}(\beta)\widehat{H}(\beta)}\, , \quad \beta > 0 \, ,$$

from which it immediately follows that $\widetilde{G} = G$. Thus we have proved the following

Corollary 4.9 *The random variables $\sup_{0 \le t < \infty}(-ct - Z(t))$ and $\sup_{0 \le t < \sigma}(-ct + C(t) - Z(t))$ have equal distribution functions.*

5 Ladder height approach

In the previous section we looked at the process at the modified ladder epochs $\sigma_1, \sigma_2, \ldots$, and essentially decomposed the process $\widehat{X}$ at these epochs. In this section we consider the ladder height process of $\widehat{X}$ obtained by time-changing the supremum process $\widehat{S}$ by the inverse local time at zero of the reflected process $\widehat{S} - \widehat{X}$.

Recall that $\widehat{S} - \widehat{X}$ is a strong Markov process, and hence it admits a local time process at 0, $\widehat{L} = (\widehat{L}(t),\, t \geq 0)$. The process $\widehat{L}$ is continuous and increasing, and increases only when $\widehat{S} - \widehat{X}$ is at zero, i.e., when $\widehat{X}$ reaches its new supremum. More precisely, the support of the measure $\widehat{L}(dt, \omega)$ is the zero set of $\widehat{S}(t,\omega) - \widehat{X}(t,\omega)$. Let $\widehat{L}^{-1}(t) := \inf\{s > 0 : \widehat{L}(s) > t\}$ be the inverse of $\widehat{L}$, the ladder time process. Define:

$$\widehat{H}(t) := \begin{cases} \widehat{S}(\widehat{L}^{-1}(t)), & \widehat{L}^{-1}(t) < \infty \\ +\infty, & \text{otherwise} \end{cases} \tag{5.1}$$

The process $((\widehat{L}^{-1}(t), \widehat{H}(t)),\, t < \widehat{L}(\infty))$ is a two dimensional subordinator killed at rate $q := \mathbb{E}\widehat{L}(\infty)$ ([Ber], p.156). In particular, $\widehat{H} = (\widehat{H}(t),\, 0 \leq t < \widehat{L}(\infty))$ is a subordinator killed at rate $q = \mathbb{E}\widehat{L}(\infty)$, and clearly,

$$\widehat{H}(\widehat{L}(\infty)-) = \sup_{0 \leq t < \widehat{L}(\infty)} \widehat{H}(t) = \sup_{0 \leq t < \infty} \widehat{S}(t) = \widehat{S}(\infty)\,. \tag{5.2}$$

Therefore, the distribution function of $\widehat{S}(\infty)$ is equal to the distribution function of $\widehat{H}(\widehat{L}(\infty)-)$.

Lemma 5.1 *The Laplace exponent $\widehat{\kappa}$ of the killed subordinator $\widehat{H} = (\widehat{H}(t),\, 0 \leq t < \widehat{L}(\infty))$ is given by the following formula:*

$$\widehat{\kappa}(\beta) = \frac{\psi(\beta)}{\beta} = d + \int_{(0,\infty)} (1 - e^{-\beta x})\, \mu(dx) + \frac{\psi_Z(\beta)}{\beta}\,, \tag{5.3}$$

where $\mu(dx) := \nu(x,\infty)\, dx$ is a finite measure on $(0,\infty)$.

Proof. The bivariate Laplace exponent $\widehat{\kappa}(\alpha, \beta)$ of $((\widehat{L}^{-1}(t), \widehat{H}(t)),\, t < \widehat{L}(\infty))$ is defined by

$$\exp\{-\widehat{\kappa}(\alpha, \beta)\} = \mathbb{E}[\exp -\{\alpha \widehat{L}^{-1}(1) + \beta \widehat{H}(1)\}]\,, \quad \alpha, \beta > 0\,.$$

The explicit formula for $\widehat{\kappa}$ comes from fluctuation theory:

$$\widehat{\kappa}(\alpha, \beta) = k \frac{\alpha - \psi(\beta)}{\Phi(\alpha) - \beta}\,, \tag{5.4}$$

where k is an unimportant constant depending on the normalization of the local time, so from now on we take $k = 1$. By letting $\alpha = 0$ in (5.4), we obtain the Laplace exponent of $\widehat{H}$:

$$\widehat{\kappa}(\beta) = \widehat{\kappa}(0, \beta) = \frac{\psi(\beta)}{\beta} = c - \frac{\Phi_C(\beta)}{\beta} + \frac{\psi_Z(\beta)}{\beta}\,, \quad \beta > 0\,. \tag{5.5}$$

Further, by partial integration we get

$$\begin{aligned} c - \frac{\Phi_C(\beta)}{\beta} &= c + \int_0^\infty (1 - e^{-\beta x})\, \nu(x, \infty)\, dx - \int_0^\infty \nu(x, \infty)\, dx \\ &= c + \int_0^\infty (1 - e^{-\beta x})\, \mu(dx) - \mathbb{E}C(1)\,. \end{aligned}$$

Together with (5.5), this gives (5.3). □

Remark 5.2 *Note that the same argument shows that the ladder height process of $(-Z(t),\, t \geq 0)$ has the Laplace exponent equal to $\psi_Z(\beta)/\beta$.*

We are going to construct a killed subordinator with the Laplace exponent given by (5.3). Let $\eta = (\eta(t),\, t \geq 0)$ be a Poisson process with parameter $\mathbb{E}C(1)$, and let $(j_1, j_2, \dots)$ be a sequence of independent random variables with distribution given by $(1/\mathbb{E}C(1))\mu(dx) = H(dx)$. The process $\gamma = (\gamma(t),\, t \geq 0)$ defined by $\gamma(t) := \sum_{n=1}^{\eta(t)} j_n$ is a compound Poisson process with the Laplace exponent $\int_{(0,\infty)} (1 - e^{-\beta x})\, \mu(dx)$. We denote the times of jumps of η by $\varsigma_1, \varsigma_2, \dots$. Let $\zeta = (\zeta(t),\, t \geq 0)$ be a subordinator with the Laplace exponent $\psi_Z(\beta)/\beta$ independent of the process

γ. By Remark 5.2, $\psi_Z(\beta)/\beta$ is the Laplace exponent of the ladder height process of $(-Z(t), t \geq 0)$. Define $\xi(t) := \gamma(t) + \zeta(t)$, and let τ be an exponential time with parameter $d = c - \mathbb{E}C(1)$ independent of γ and ζ. Then ξ killed at the exponential time τ has the Laplace exponent given by (5.3). Therefore, ξ killed at τ and $\widehat{H}$ are equally distributed. In particular, $\xi(\tau)$ has the same distribution as $\widehat{H}(\widehat{L}(\infty)-)$, which by (5.2) implies that $\xi(\tau) \stackrel{d}{=} \widehat{S}(\infty)$. We decompose the random variable $\xi(\tau)$ as follows: Let $l_0 := \xi((\varsigma_1 \wedge \tau)-)$, $l_1 := \xi((\varsigma_2 \wedge \tau)-) - \xi(\varsigma_1 \wedge \tau)$, on $\{\varsigma_1 < \tau\}$, and inductively, $l_n := \xi((\varsigma_{n+1} \wedge \tau)-) - \xi(\varsigma_n \wedge \tau)$, on $\{\varsigma_n < \tau\}$. Note that $j_n = \xi(\varsigma_n \wedge \tau) - \xi((\varsigma_n \wedge \tau)-)$ on $\{\varsigma_n < \tau\}$. If $\bar{N} := \max\{n \geq 1 : \varsigma_n < \tau\}$, then $\bar{N}$ has the geometric distribution with parameter $\mathbb{P}(\bar{N} = 0) = \mathbb{P}(\tau < \varsigma_1) = d/(\mathbb{E}C(1) + d) = 1 - \rho$. Moreover,

$$\xi(\tau) = l_0 + j_1 + l_1 + \cdots + j_{\bar{N}} + l_{\bar{N}} \,. \tag{5.6}$$

Proposition 5.3 *The random variable l_0 has distribution function G.*

Proof. Note that $l_0 = \xi((\varsigma_1 \wedge \tau)-) = \zeta((\varsigma_1 \wedge \tau)-) = \zeta(\varsigma_1 \wedge \tau)$ a.s., and that $\varsigma_1 \wedge \tau$ has exponential distribution with parameter $\mathbb{E}C(1) + d = c$. We will compute the Laplace transform of $\zeta(\varsigma_1 \wedge \tau)$:

$$\begin{aligned} \mathbb{E}[\exp\{-\beta\zeta(\varsigma_1 \wedge \tau)\}] &= \int_0^\infty c e^{-ct} \mathbb{E}[\exp\{-\beta\zeta(t)\}]\, dt \qquad (5.7) \\ &= \int_0^\infty c e^{-ct} \exp\{-(\psi_Z(\beta)/\beta)t\}\, dt \\ &= \int_0^\infty c \exp\{-(\psi_Y(\beta)/\beta)t\}\, dt \\ &= c\frac{\beta}{\psi_Y(\beta)} = \widehat{G}(\beta)\,, \end{aligned}$$

where ψ_Y was defined in Section 3. □

Proposition 5.4 *The random variable l_0 is independent of $\{\bar{N} = 0\}$. Consequently, random variables l_0 and j_1 are conditionally independent given*

$\bar{N} \geq 1$, *and*

$$\mathbb{P}(l_0 \leq y, j_1 > x \mid \bar{N} \geq 1) = G(y)(1 - H(x)) \,. \tag{5.8}$$

Proof. The first claim is proved by the following computation:

$$\begin{aligned}
\mathbb{P}(\bar{N} = 0, l_0 \leq x) &= \mathbb{P}(\tau < \varsigma_1, \zeta(\tau) \leq x) \\
&= \int_0^\infty \mathbb{E}C(1)e^{-\mathbb{E}C(1)s}\, ds \int_0^s de^{-d\,t}\mathbb{P}(\zeta(t) \leq x)\, dt \\
&= \int_0^\infty de^{-d\,t}\mathbb{P}(\zeta(t) \leq x)\, dt \int_t^\infty \mathbb{E}C(1)e^{-s\mathbb{E}C(1)}\, ds \\
&= \frac{d}{c}\int_0^\infty ce^{-c\,t}\mathbb{P}(\zeta(t) \leq x)\, dt \\
&= (1-\rho)\mathbb{P}(\zeta(\varsigma_1 \wedge \tau) \leq t) \\
&= \mathbb{P}(\bar{N} = 0)\mathbb{P}(l_0 \leq x) \,.
\end{aligned}$$

The second claim follows from $\mathbb{P}(l_0 \leq y, j_1 > x, \bar{N} \geq 1) = \mathbb{P}(l_0 \leq y, \bar{N} \geq 1)\mathbb{P}(j_1 > x) = \mathbb{P}(l_0 \leq y)\mathbb{P}(j_1 > x)\mathbb{P}(\bar{N} \geq 1)$, and Proposition 5.3. □

Proposition 5.5 *For every $n \in \mathbb{N}$,*

$$P(l_0 + j_1 + l_1 + \cdots + j_n + l_n \leq x, \bar{N} = n) = (1-\rho)\rho^n (G^{(n+1)*} * H^{n*})(x) \,. \tag{5.9}$$

Consequently, the following formula for the distribution function of $\xi(\tau)$ is valid:

$$\mathbb{P}(\xi(\tau) \leq x) = (1-\rho)\sum_{n=0}^{\infty} \rho^n (G^{(n+1)*} * H^{n*})(x) \,, \quad x \geq 0 \,. \tag{5.10}$$

Proof. Equation (5.9) follows from a straightforward computation similar to the one from the proof of Proposition 5.4. Equation (5.10) is now a consequence of (5.6) and (5.9). □

Since $\xi(\tau) \stackrel{d}{=} \widehat{S}(\infty)$ we obtained yet another proof of the Pollaczek-Hinchin formula.

References

[Asm] S. Asmussen, *Ruin Probabilities*, World Scientific, Singapore (1996)

[Ber] J. Bertoin, *Lévy Processes*, Cambridge University Press (1996)

[DG] F. Dufresne, H. U. Gerber, Risk theory for a compound Poisson process that is perturbed by diffusion, *Insurance: Math. Econ.* **10** (1991), 51–59

[DGS] F. Dufresne, H. U. Gerber, E. W. Shiu, Risk theory with Gamma process, *Astin Bull.*, **21** (1991), 177–192

[Fur] H. Furrer, Risk processes perturbed by α-stable Lévy motion, *Scand. Actuarial J.* **1998** (1998), 59–74

[RY] D. Revuz, M. Yor, *Continuous Martingales and Brownian Motion*, Springer-Verlag (1991)

[RSST] T. Rolski, H. Schmidli, V. Schmidt, J. Teugels, *Stochastic Processes for Insurance and Finance*, John Wiley & Sons (1998)

[Sat] K.-I. Sato, *Lévy Processes and Infinitely Divisible Distributions*, Cambridge University Press (1999)

[Sch] H. Schmidli, Distribution of the first ladder height of a stationary risk process perturbed by α-stable Lévy motion, *Insurance: Math. Econ.* **28** (2001), 13–20

[YZ] H. Yang, L. Zhang, Spectrally negative Lévy processes with applications in risk theory, *Adv. Applied Prob.* **33** (2001), 281–291

[Zol] V. M. Zolotarev, The first passage time of a level and behavior at infinity for a class of processes with independent increments, *Theory Prob. Appl.* **9** (1964) 653–661

Addresses of authors:

Miljenko Huzak
Department of Mathematics
University of Zagreb
Bijenička c. 30
10000 Zagreb, CROATIA
E-mail: huzak@math.hr

Mihael Perman
Institute for Mathematics, Physics and Mechanics
Jadranska 19
1000 Ljubljana, SLOVENIA
E-mail: mihael@valjhun.fmf.uni-lj.si

Hrvoje Šikić
Department of Mathematics
University of Zagreb
Bijenička c. 30
10000 Zagreb, CROATIA
E-mail: hsikic@math.hr

Zoran Vondraček
Department of Mathematics
University of Zagreb
Bijenička c. 30
10000 Zagreb, CROATIA
E-mail: vondra@math.hr