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ON BASIC EMBEDDINGS INTO
THE PLANE

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ABSTRACT. An subset $K \subset \mathbb{R}^2$ is said to be *basic* if for each function $f: K \rightarrow \mathbb{R}$ there exist functions $g, h: \mathbb{R} \rightarrow \mathbb{R}$ such that $f(x, y) = g(x) + h(y)$ for each point $(x, y) \in K$. If all the three functions in this definition are assumed to be *continuous (differentiable)*, then the embedding is C^0 -*basic* (C^1 -*basic*). This notion appeared in studies of Hilbert's 13th problem on superpositions. We prove that *if a graph is C^0 -basically embeddable in the plane, then it is C^1 -basically embeddable in the plane*. In our proof we construct an explicit C^1 -basic embedding and use the Skopenkov characterization of graphs C^0 -basically embeddable in the plane. Our result is non-trivial because the plane contains graphs which are C^0 -basic but not C^1 -basic and graphs which are C^1 -basic but not C^0 -basic (Baran-Skopenkov). We also prove that *given any integer $k \geq 0$, there is a subset of the plane which is C^r -basic for each $0 \leq r \leq k$ but not C^r -basic for each $k < r \leq \omega$* .

1. INTRODUCTION

The notion of a basic embedding appeared implicitly in the Kolmogorov-Arnold solution of Hilbert's 13th problem [4, 1, 5]. A compactum $K \subset \mathbb{R}^2$ is said to be *basic*, if for each continuous function $f: K \rightarrow \mathbb{R}$ there exist continuous functions $g, h: \mathbb{R} \rightarrow \mathbb{R}$ such that $f(x, y) = g(x) + h(y)$ for each point $(x, y) \in K$. This note is motivated by the following problem.

Problem. *Replace in the definition of a basic embedding continuous functions by smooth functions (by Lipschitz, Hölder, analytic, etc. functions).*

Find conditions on a compactum $K \subset \mathbb{R}^2$, under which K is basically embedded into the plane in the smooth sense.

Find conditions on a finite graph K , under which K is basically embeddable into the plane in the smooth sense.

For a subset K of the plane (not necessarily open) a function $f: K \rightarrow \mathbb{R}$ is said to be r -*analytic*, $0 \leq r < \infty$, if for each point $(x_0, y_0) \in K$ there exist

$$\{a_{ij}\}_{i,j=0}^r \subset \mathbb{R} \quad \text{such that} \quad a_{00} = f(x_0, y_0)$$

and

$$f(x_0 + x, y_0 + y) = \sum_{i,j=0}^r a_{ij} x^i y^j + o((|x| + |y|)^r),$$

where $(x_0 + x, y_0 + y) \in K$ and $(x, y) \rightarrow (0, 0)$. Since $\mathbb{R} \subset \mathbb{R}^2$, this definition applies to functions $\mathbb{R} \rightarrow \mathbb{R}$ as well. Note that 0-analytic is the same as continuous, 1-analytic

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for functions $\mathbb{R} \rightarrow \mathbb{R}$ is the same as differentiable and r -analytic for functions $\mathbb{R} \rightarrow \mathbb{R}$ is approximately (but not precisely) the same as C^r .

For a subset K of the plane (not necessarily open) a function $f : K \rightarrow \mathbb{R}$ is said to be *analytic* (or ω -analytic), if for each point $(x_0, y_0) \in K$ there exist

$$\{a_{ij}\}_{i,j=0}^{\infty} \subset \mathbb{R} \quad \text{such that} \quad f(x_0 + x, y_0 + y) = \sum_{i,j=0}^{\infty} a_{ij} x^i y^j$$

for $(x_0 + x, y_0 + y)$ belonging to some neighborhood of (x_0, y_0) in K .

A compactum $K \subset \mathbb{R}^2$ is said to be C^r -*basic*, $1 \leq r \leq \omega$, if for each r -analytic function $f : K \rightarrow \mathbb{R}$ there exist r -analytic functions $g, h : \mathbb{R} \rightarrow \mathbb{R}$ such that $f(x, y) = g(x) + h(y)$ for each point $(x, y) \in K$.

The main result of this paper is the following.

Theorem 1.1. *If a finite graph K is C^0 -basically embeddable into the plane, then K is C^1 -basically embeddable into the plane.*

Theorem 1.1 is non-trivial because *the plane contains graphs which are C^1 -basic but not C^0 -basic and graphs which are C^1 -basic but not C^0 -basic* (Baran-Skopenkov). Denote by $[a]$ the integer part of a . As a subset of the plane which is C^1 -basic but not C^0 -basic we can take $\{(2^{-[\frac{n+1}{2}]}, 2^{-[\frac{n}{2}]})\} \cup \{(0, 0)\}$. It is then easy to construct in the plane a graph which is C^1 -basic but not C^0 -basic. Note that a very similar subset $\{([\frac{n+1}{2}]^{-1/2}, [\frac{n}{2}]^{-1/2})\} \cup \{(0, 0)\}$ of the plane is *not C^1 -basic*. Let V be the graph of the function $y = |x|$, $x \in [-1, 1]$. As a subset of the plane which is C^0 -basic but not C^1 -basic we can take $(V - (2, 0)) \sqcup (V + (2, 0))$. Note that the subset V itself is C^1 -basic.

Example 1.2. *Given any integer $k \geq 0$, there is a subset of the plane which is C^r -basic for each $0 \leq r \leq k$ but not C^r -basic for each $k < r \leq \omega$.*

In Example 1.2 we can take the graph V_k of the function $y = |x|^k$, $x \in [-1, 1]$ for k odd, and $W_{k+1} = (V_{k+1} - (2, 0)) \sqcup (V_{k+1} + (2, 0))$ for k even.

In the proof of Theorem 1.1 we use the following result, answering the Sternfeld problem [12].

Theorem 1.3. [10, cf. 6, 7, 9, §5] *For any finite graph $K \subset \mathbb{R}^2$ the following conditions are equivalent:*

- (C) K is C^0 -basically embeddable in \mathbb{R}^2 ;
- (G) K does not contain any of the following three graphs: a circle S , a pentod P or a cross C with branched ends;
- (E) K can be embedded in R_n for some n .

Definition of the graphs R_n is given in §2. Our proof of Theorem 1.1 is based on a construction of a C^1 -basic embedding $R_n \subset \mathbb{R}^2$ (§2). We prove elementary that this embedding is also C^0 -basic, which yields an elementary proof of Theorem 1.3 as explained in §3.

2. PROOFS

Let us define inductively the graphs R_n together with an embedding $R_n \rightarrow \mathbb{R}^2$. We embed R_1 into $[-10, 10] \times [-10, 10]$ as shown in Figure 1. Then we repeat the

procedure by embedding copies of R_1 into squares A , B and C shown in Figure 1 to get R_2 . Note that the embedded R_1 into B was mirrored over ℓ to get a connected R_2 .

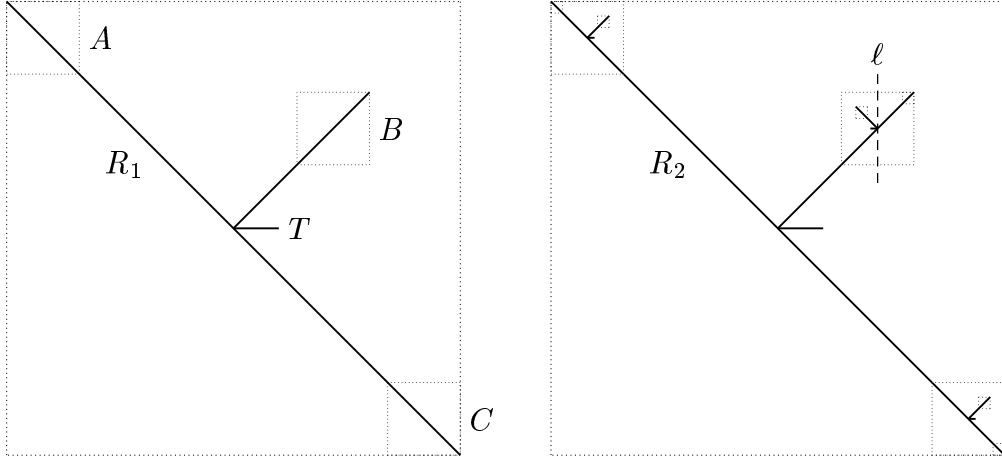


Figure 1

In general, the graph R_n is constructed by embedding R_{n-1} into appropriate small squares A , B , C attached to R_1 . The squares A , B and C have to be chosen carefully. Let $p_1: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ and $p_2: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ denote projections onto x and y axes. We require that $p_1(A)$, $p_1(B)$, $p_1(C)$, $p_1(T)$ are disjoint and $p_2(A)$, $p_2(B)$, $p_2(C)$, $p_2(T)$ are disjoint.

Proof of Theorem 1.1. The boundary in R_n of any subgraph $K \subset R_n$ consists of a finite number of points. Hence any 1-analytic mapping $K \rightarrow \mathbb{R}$ can be extended to a 1-analytic mapping $K \rightarrow \mathbb{R}$. So it suffices to prove that R_n is C^1 -basic. We prove this by induction. Given a mapping $f: R_n \rightarrow \mathbb{R}$ we shall find functions $g, h: \mathbb{R} \rightarrow \mathbb{R}$ such that $f(x, y) = g(x) + h(y)$. Then we shall show that we can obtain g and h to be 1-analytic (i.e. differentiable), when f is 1-analytic.

Put $h(0) = 0$ and define $g(x) = f(x, 0)$ for every $x \in [0, 2]$. Extend g to a function $g: [0, 10] \rightarrow \mathbb{R}$.

Note that for every $y \in [0, 6]$ there exist an unique $x_y = |y| \in [0, 10]$ such that $(x_y, y) \in R_1$. Therefore using g and f for $x \in [0, 10]$ we can define $h: [-10, 6] \rightarrow \mathbb{R}$ as $h(y) = f(|y|, y) - g(|y|)$. Extend h to $h: [-10, 10] \rightarrow \mathbb{R}$.

Note that for every $x \in [-10, 0]$ there exists a unique $y_x = -x$ such that $(x, y_x) \in R_1$. Therefore using h we can define $g: [-10, 0] \rightarrow \mathbb{R}$ as $g(x) = f(x, -x) - h(-x)$. Finally, we extend g and h to $g, h: \mathbb{R} \rightarrow \mathbb{R}$.

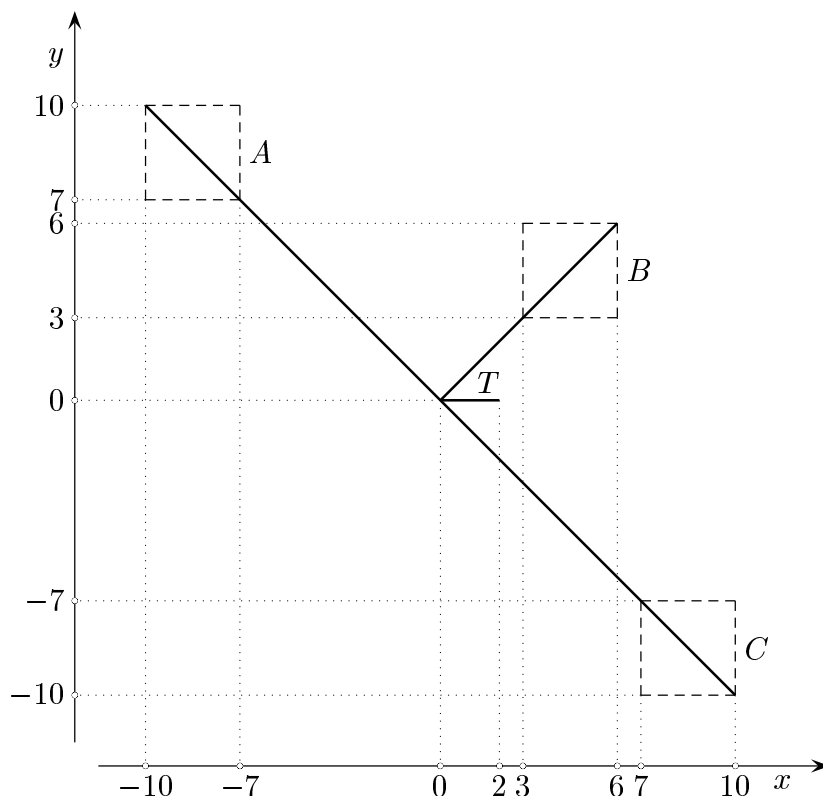


Figure 2

Now let $f: R_n \rightarrow \mathbb{R}$, $n > 1$, be given. We put $h(0) = 0$ and define $g(x) = f(x, 0)$ for every $x \in [0, 2]$. As R_n is constructed by embedding R_{n-1} into appropriate small squares A, B, C attached to R_1 , by inductive hypothesis there exist functions $g', h': \mathbb{R} \rightarrow \mathbb{R}$ such that $f(x, y) = g'(x) + h'(y)$ on $(x, y) \in (A \cup B \cup C) \cap R_n$. Hence we can extend g smoothly onto $[0, 10]$ so that $g = g'$ on $p_1(B \cup C)$. Using functions g and f for $x \in [0, 10]$ we can define $h: [-10, 6] \rightarrow \mathbb{R}$ as $h(y) = f(|y|, y) - g(|y|)$. Then we extend h onto $[-10, 10]$ so that $h = h'$ on $[7, 10]$. Using h we finally define $g: [-10, 0] \rightarrow \mathbb{R}$ as $g(x) = f(x, -x) - h(-x)$.

For $n = 1$, if f is 1-analytic, then it is clear that at each step the constructed functions g and h are differentiable except maybe at 0. So all the extensions can be chosen to be differentiable. Since f is 1-analytic at $(0, 0)$, it follows that there exist $a, b \in \mathbb{R}$ such that

$$f(x, y) = f(0, 0) + ax + by + o(|x| + |y|), \quad \text{where } (x, y) \in R_1 \quad \text{and} \quad (x, y) \rightarrow (0, 0).$$

We may assume that $f(0, 0) = g(0) = h(0) = 0$. Then according to the structure of R_1 one can write

$$\left\{ \begin{array}{l} f(x, x) = g(x) + h(x) \\ f(x, -x) = g(x) + h(-x) \\ f(x, 0) = g(x) \\ f(-x, x) = g(-x) + h(x) \end{array} \right. , \quad \text{so} \quad \left\{ \begin{array}{l} g(x) = f(x, 0) \\ h(x) = f(x, x) - f(x, 0) \\ h(-x) = f(x, -x) - f(x, 0) \\ g(-x) = f(-x, x) - f(x, x) + f(x, 0) \end{array} \right.$$

for small $x \geq 0$. Hence

$$g(x) = ax + o(x) \quad \text{and} \quad g(-x) = -ax + bx - ax - bx + ax + o(x) = -ax + o(x)$$

when $x \rightarrow +0$. So g is differentiable at 0. Also,

$$h(x) = ax + bx - ax + o(x) = bx + o(x) \quad \text{and} \quad h(-x) = ax - bx - ax + o(x) = -bx + o(x)$$

when $x \rightarrow +0$. So h is differentiable at 0.

Hence for $n > 1$, if f is 1-analytic, then it is clear that at each step the constructed functions g and h are differentiable everywhere. So all the extensions can be chosen to be differentiable and thus the resulting functions are differentiable. \square

An elementary proof of $(R) \Rightarrow (C)$ in Theorem 1.3. Analogously to the proof of Theorem 1.1 above. The reduction from K to R_n follows also by the Tietze-Uryhson Extension Theorem. We construct g and h from f as above. From the construction it is clear that at each step the constructed functions g and h are continuous. So all the extensions can be chosen to be continuous and thus the resulting functions are continuous. \square

Note that for each function $f : R_1 \rightarrow \mathbb{R}$ the functions $g, h : \mathbb{R} \rightarrow \mathbb{R}$ such that $f(x, y) = g(x) + h(y)$ are *uniquely* defined by f in a neighborhood of 0. Hence *any* such functions g and h are 0- or 1-analytic in a neighborhood of 0, if f is 0- or 1-analytic. Surprisingly, this is false for r -analytic functions with $1 < r \leq \omega$: the subset $R_1 \subset \mathbb{R}^2$ is C^1 -basic but not C^r -basic for each $1 < r \leq \omega$. This is proved analogously to Example 1.2 for k odd.

Proof of Example 1.2 for k odd. First we prove that $V = V_1$ is C^1 -basic. Take a 1-analytic function $f : V \rightarrow \mathbb{R}$. Since f is 1-analytic at $(0, 0)$, it follows that there exist $a, b \in \mathbb{R}$ such that

$$f(x, |x|) = f(0, 0) + ax + b|x| + o(|x| + |x|), \quad \text{where } x \rightarrow 0.$$

Take $h(y) = by$ and $g(x) = f(x, |x|) - h(|x|)$. Clearly, h is 1-analytic (i.e. differentiable) and g is 1-analytic outside 0. Since $g(x) = f(0, 0) + ax + o(x)$ when $x \rightarrow 0$, it follows that g is 1-analytic also at 0.

Now we prove that V_k is C^r -basic for each $0 \leq r \leq k$. Take an r -analytic function $f : V_k \rightarrow \mathbb{R}$. Since f is r -analytic at $(0, 0)$, it follows that there exist $\{a_{ij}\}_{i,j=0}^r \subset \mathbb{R}$ such that

$$a_{00} = f(0, 0) \quad \text{and} \quad f(x, |x|^k) = \sum_{i,j=0}^r a_{ij} x^i |x|^{kj} + o((|x| + |x|^r)^r), \quad \text{where } x \rightarrow 0.$$

Since

$$o((|x| + |x|^r)^r) = o_1(x^r), \quad \text{we have} \quad f(x, |x|^k) = a_{00} + a_{01}|x|^k + a_{10}x + \cdots + a_{r0}x^r + o_2(x^r).$$

Take $h(y) = a_{01}y$ and $g(x) = f(x, |x|^k) - h(|x|^k)$. Clearly, h is r -analytic and g is r -analytic outside 0. We also have $g(x) = a_{00} + a_{10}x + \cdots + a_{r0}x^r + o_2(x^r)$ when $x \rightarrow 0$. So g is r -analytic also at 0.

Next we prove that $V = V_1$ is not C^r -basic for each $1 < r \leq \omega$. Define an analytic function $f : V \rightarrow \mathbb{R}$ by $f(x, y) = xy$, where $y = |x|$. If V is C^r -basic for some $r \geq 2$, then there are r -analytic functions

$$g, h : \mathbb{R} \rightarrow \mathbb{R} \quad \text{such that} \quad f(x, |x|) = x|x| = g(x) + h(|x|) \quad \text{for each } x \in [0, 1].$$

Hence $g(x) - g(-x) = 2x^2$. But this is impossible because g is 2-analytic, hence

$$g(x) = g(0) + ax + bx^2 + o(x^2) \quad \text{and so} \quad g(-x) = g(0) - ax + bx^2 + o(x^2) \quad \text{for } x \rightarrow +0.$$

At last we prove that V_k is not C^r -basic for k odd and each $k < r \leq \omega$. Define an analytic function $f : V_k \rightarrow \mathbb{R}$ by $f(x, y) = xy$, where $y = |x|^k$. If V is C^r -basic for some $r > k$, then there are r -analytic functions

$$g, h : \mathbb{R} \rightarrow \mathbb{R} \quad \text{such that} \quad f(x, |x|^k) = x|x|^k = g(x) + h(|x|^k) \quad \text{for each } x \in [0, 1].$$

Hence $g(x) - g(-x) = 2x|x|^k$. But this is impossible for k odd because g is $(k+1)$ -analytic, hence

$$g(x) = g_0 + g_1x + \dots + g_{k+1}x^{k+1} + o(x^{k+1}) \quad \text{and so} \quad g(-x) = g_0 - g_1x + \dots + g_{k+1}x^{k+1} + o(x^{k+1})$$

for $x \rightarrow +0$. \square

Note that a function $f(x, y)$ on the graph V is 1-analytic if and only if $p(t) = f(t, |t|)$ is differentiable on $[-1; 0]$ and on $[0; 1]$.

Proof of Example 1.2 for k even. Let us prove that W_{k+1} is C^r -basic for each $0 \leq r \leq k$. Given an r -analytic function $f : W_{k+1} \rightarrow \mathbb{R}$, take functions $h(y) = 0$ and $g(x) = f(x, |x - 2 \operatorname{sign} x|^{k+1})$. Clearly, h is r -analytic and $f(x, y) = g(x) + h(y)$ for each $(x, y) \in W_{k+1}$. Since the function $p(t) = |t|^{k+1}$ is k -analytic and $r \leq k$, it follows that g is r -analytic.

Let us prove that W_{k+1} is not C^r -basic for k even and each $k < r \leq \infty$. Define an analytic function $f : W_{k+1} \rightarrow \mathbb{R}$ by $f(x, y) = y \operatorname{sign} x$. If W_{k+1} is C^r -basic, then there are r -analytic functions g and h such that $f(x, y) = g(x) + h(y)$. For

$$x \in [-1, 1] \quad \text{we have} \quad g(\pm 2 + x) + h(|x|^{k+1}) = f(\pm 2 + x, |x|^{k+1}) = \pm |x|^{k+1}.$$

Since g is $(k+1)$ -analytic and $k+1$ is odd, it follows that $\frac{dg}{dx}|_{x=0} = +1$ and $\frac{dg}{dx}|_{x=0} = -1$, which is a contradiction. \square

3. THE STERNFELD CRITERION

The proof of Theorem 1.3 in [10] was based on the solution of the Arnold problem [2]: find conditions on a compactum $K \subset \mathbb{R}^2$, under which K is C-basic. This problem was solved by Sternfeld [11, 12] (who was apparently unaware of [2]). In order to formulate the Sternfeld criterion, let us introduce some definitions. Let p_1 and p_2 be projections onto the coordinate axes in \mathbb{R}^2 . For $Z \subset \mathbb{R}^2$ let

$$E(Z) = \{z \in Z : |Z \cap p_1^{-1}(p_1(z))| \geq 2 \text{ and } |Z \cap p_2^{-1}(p_2(z))| \geq 2\}.$$

Set $E^2(Z) = E(E(Z))$, $E^3(Z) = E(E(E(Z)))$, etc. An ordered sequence $\{a_1, \dots, a_n\} \subset \mathbb{R}^2$ is called an *array*, if for each i we have $p_1(a_i) = p_1(a_{i+1})$ for i even and $p_2(a_i) = p_2(a_{i+1})$ for i odd ($a_i \neq a_{i+1}$, but it is not required that all the points of an array should be distinct).

Theorem 3.1. [11, 12] *For any compactum $K \subset \mathbb{R}^2$ the following conditions are equivalent:*

- (B) *the embedding $K \subset \mathbb{R}^2$ is basic;*
- (E) *$E^n(K) = \emptyset$ for some n ;*
- (A) *K does not contain any array of n points for some n .*

In this paper we prove Theorem 3.1 following [12] (we believe our exposition is clearer). One can see that the proof of Theorem 3.1 is non-elementary in a sense that it used the Banach Inverse Operator Theorem.

The proof of $(E) \Leftrightarrow (G)$ in Theorem 1.3 is elementary, cf. [3]. The proof of $(C) \Rightarrow (G)$ in Theorem 1.3 is elementary modulo the implication $(B) \Rightarrow (A)$ of Theorem 3.1 [10]. The latter implication has an elementary proof by [8]. The proof of $(R) \Rightarrow (C)$ in Theorem 1.3 used the non-elementary implication $(E) \Rightarrow (B)$ of Theorem 3.1 [10]. In this paper we give an elementary proof of $(R) \Rightarrow (C)$ in Theorem 1.3, which yields an elementary proof of the whole Theorem 1.3.

The Sternfeld proof of Theorem 3.1. First we prove the easy assertion $(A) \Rightarrow (E)$. Suppose to the contrary that $E^n(K) \neq \emptyset$. Take a point $a_0 \in E^n(K)$. Then there exist points $a_{-1}, a_1 \in E^{n-1}(K)$ such that $p_1(a_{-1}) = p_1(a_0)$ and $p_2(a_1) = p_2(a_0)$. Analogously, there exist points $a_{-2}, a_2 \in E^{n-2}(K)$ such that $\{a_{-2}, a_{-1}, a_0, a_1, a_2\}$ is an array. Analogously we construct an array of $2n + 1$ points in K .

The proof of $(E) \Rightarrow (\Phi) \Rightarrow (A)$ is based on a reformulation of (B) terms of linear operators in functional spaces. Denote by $C(X)$ the space of continuous functions on X with the norm $|f| = \sup\{|f(x)| : x \in X\}$. For a subset $K \subset I^2$ define the *linear superposition operator*

$$\varphi: C(I) \oplus C(I) \rightarrow C(K) \quad \text{by} \quad \varphi(g, h)(x, y) = g(x) + h(y).$$

Clearly, the embedding $K \subset I^2$ is basic if and only if $\varphi = \varphi_K$ is epimorphic. Denote by $C^*(X)$ the space of bounded linear functionals on $C(X)$ with the norm $|\mu| = \sup\{|\mu(f)| : f \in C(X), |f| = 1\}$. For a subset $K \subset I^2$ define the *dual linear superposition operator*

$$\varphi^*: C^*(K) \rightarrow C^*(I) \oplus C^*(I) \quad \text{by} \quad \varphi^* \mu(g, h) = (\mu(g \circ p_1), \mu(h \circ p_2)).$$

Since $|\varphi^* \mu| \leq 2|\mu|$, it follows that φ^* is bounded. By duality, φ_K is epimorphic if and only if $\varphi^* = \varphi_K^*$ is monomorphic. By the Banach Inverse Operator Theorem, φ^* is monomorphic if and only if

(Φ) *there exist $\varepsilon > 0$ such that $|\varphi^* \mu| > \varepsilon|\mu|$ for each $\mu \in C^*(K)$*

(because this condition ensures that $\text{im } \varphi^*$ is closed). Thus $(B) \Leftrightarrow (\Phi)$. So it remains to prove $(E) \Rightarrow (\Phi) \Rightarrow (A)$.

First we prove $(\Phi) \Rightarrow (A)$. If (A) is false, then for each n there exists an array $\{a_1, \dots, a_n\} \subset K$. Define a linear functional $\mu \in C^*(K)$ by $\mu(f) = \sum_{i=1}^n (-1)^i f(a_i)$. Then $|\mu| = n$ and $|\varphi^* \mu| \leq 4$. Hence (Φ) is false.

Now we prove $(E) \Rightarrow (\Phi)$. We use the fact that $C^*(X)$ is the space of σ -additive regular real valued Borel measures (in the sequel - simply 'measures') on X . We have

$$\varphi^* \mu = (\mu_x, \mu_y), \quad \text{where} \quad \mu_x(U) = \mu(p_1^{-1}U) \quad \text{and} \quad \mu_y(U) = \mu(p_2^{-1}U).$$

If $\mu = \mu^+ - \mu^-$ is the decomposition of a measure μ to its positive and negative parts, then $|\mu| = \bar{\mu}(X)$, where $\bar{\mu} = \mu^+ + \mu^-$ is the absolute value of μ . Let D_x (D_y) be the set

of points of K which are not shadowed by some other point of K in x - (y -) direction. Take any measure μ on K of the norm 1.

If

$$E(K) = \emptyset, \quad \text{then} \quad D_x \cup D_y = K, \quad \text{so} \quad 1 = \bar{\mu}(K) \leq \bar{\mu}(D_x) + \bar{\mu}(D_y).$$

Therefore without loss of generality, $\bar{\mu}(D_x) \geq 1/2$. Since p_1 is injective over D_x , it follows that $|\mu_x| \geq 1/2$, thus (Φ) holds.

If

$$E(E(K)) = \emptyset, \quad \text{then} \quad D_x \cup D_y = K - E(K), \quad \text{so} \quad E(D_x \cup D_y) = \emptyset.$$

Therefore in the case when $\bar{\mu}(E(K)) < 3/4$ we have $\bar{\mu}(D_x \cup D_y) > 1/4$ and without loss of generality $\bar{\mu}(D_x) > 1/8$. Then as above $|\mu_x| > 1/8$, thus (Φ) holds. In the case when $\bar{\mu}(E(K)) \geq 3/4$ we have $\bar{\mu}(K - E(K)) \leq 1/4$. By the case $E(K) = \emptyset$ above without loss of generality $\bar{\mu}_x(p_1(E(K))) \geq \bar{\mu}(E(K))/2$. Hence $|\mu_x| \geq \frac{1}{2} \cdot \frac{3}{4} - \frac{1}{4} = \frac{1}{8}$, thus (Φ) holds. The case of arbitrary n is proved analogously. \square

We remark that not only some linear relation on $\text{im } \varphi_K$ can force it to be strictly less than $C(K)$. Or, in other words, φ_K^* can be injective but not monomorphic. If an embedding $K \subset \mathbb{R}^2$ is basic, then we can prove that φ^* is monomorphic without use of φ as follows. Define a linear operator

$$\Psi: C^*(I) \oplus C^*(I) \rightarrow C^*(K) \quad \text{by} \quad \Psi(\mu_x, \mu_y)(f) = \mu_x(g) + \mu_y(h),$$

where $g, h \in C(I)$ are such that $g(0) = 0$ and $f(x, y) = g(x) + h(y)$ for $(x, y) \in K$. Clearly, $\Psi\Phi = \text{id}$ and Ψ is bounded, hence Φ is monomorphic.

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