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Rational Canonical Forms and Efficient Representations of Hypergeometric Terms

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Abstract

We propose four multiplicative canonical forms that exhibit the shift structure of a given rational function. These forms in particular allow one to represent a hypergeometric term efficiently. Each of these representations is optimal in some sense.

1 Introduction

Let K be a field of characteristic zero. Representations of a rational function $R \in K(x)$ in the form

$$R(x) \;=\; F(x) \cdot \frac{V(x+1)}{V(x)} \tag{1}$$

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where $F, V \in K(x)$ satisfy some specific conditions, play a substantial role in various computer algebra algorithms operating on *hypergeometric terms*. Recall that a sequence $T(n)$ of elements of K defined for all integers $n \geq n_0$ is a hypergeometric term if there are polynomials $p, q \in K[x]$ such that $q(n)T(n+1) = p(n)T(n)$ for all $n \geq n_0$. If $T(n)$ is eventually nonzero then the rational function p/q is unique, and is called the *certificate* of T.

The main part of

(a) Gosper's algorithm for hypergeometric indefinite summation [3],

(b) algorithm `Hyper` for finding hypergeometric solutions of linear difference equations with polynomial coefficients [7],

(c) the algorithm for finding a minimal multiplicative decomposition of a hypergeometric term [2], and

(d) the algorithm for finding a minimal additive decomposition of a hypergeometric term [2]

starts with the certificate of a hypergeometric term. Each algorithm then proceeds by representing this certificate in the form (1). One can also add to the above list Zeilberger's algorithm [8], and the algorithm to compute the hypergeometric dispersion [1].

Algorithm (c) can be used to construct an economic representation of a hypergeometric term $T(n)$. Using the certificate R of $T(n)$, we can write

$$T(n) \;=\; c \prod_{k=n_0}^{n-1} R(k), \qquad (2)$$

where c is determined from some initial conditions. Let $F(x)$ in (1) be written as $r(x)/s(x)$ where $r, s \in K[x]$ and $\gcd(r(x), s(x+k)) = 1$ for all $k \in \mathbb{Z}$. Then (1) is a rational normal form (RNF) of $R(x)$, and $F(x), V(x)$ are the *kernel* and the *shell* of this RNF, respectively. By using any RNF of R, we can rewrite (2) in the form

$$T(n) \;=\; c\,V(n) \prod_{k=n_0}^{n-1} F(k) \qquad (3)$$

where both the numerator and the denominator of F are of minimal possible degrees [2].

It was shown in [2] that a rational function can have several different RNF's. In Sections 3–4.4, we distinguish four rational canonical forms (RCF's) in the set of all RNF's. Each of these four RCF's minimizes the shell in one sense or another: RCF_1 and RCF_2 minimize the degree of the denominator and of the numerator of the shell, respectively. RCF_1^* and RCF_2^* both minimize the sum of the degrees of the numerator and of the denominator of the shell, and under this condition, also minimize the degree of the denominator and of the numerator of the shell, respectively. By using these canonical forms in the problem of representing a hypergeometric term $T(n)$ economically, we can minimize V in (3) (recall that F is minimized by any RNF of R). As a consequence, we can rewrite (3) in the "optimal" form

$$\alpha^n \, V(n) \, Q(n), \tag{4}$$

where $\alpha \in K$, and $Q(n)$ is a product of Gamma-function values (if $K = \mathbb{C}$) or Pochhammer symbols (i.e., rising factorial powers) and their reciprocals. Additionally,

- $Q(n)$ has the minimal possible number of factors,
- $V(n)$ is a rational function which is minimal in one sense or another, depending on the particular RCF chosen to represent the certificate of $T(n)$.

Economic representations of hypergeometric terms are useful in the output routines of algorithms which return hypergeometric terms, but compute their certificates first and need to construct the terms themselves before outputting them. Examples of such algorithms include (a) and (b) above. Other important problems where these representations can be used to advantage are simplification of hypergeometric terms, and investigation of asymptotics of hypergeometric terms.

The algorithms for constructing the four RCF's of a rational function, and the four economic representations of a hypergeometric term have been implemented in Maple, and are available from

`http://scg.uwaterloo.ca/~hqle/code/RNF/RNF.html`

2 Preliminaries

In this section we give definitions of basic notions, and formulate some necessary results from [2].

Throughout the paper, K is a field of characteristic zero, $\mathbb{Z}$ and $\mathbb{N}$ respectively denote the set of integers and nonnegative integers, E denotes the shift operator acting both on rational functions by $ER(x) = R(x+1)$, and on sequences by $ET(n) = T(n+1)$. For $p, q \in K[x]$, we write $p \perp q$ to indicate that p and q are coprime. We denote the leading coefficient of p by $lc(p)$. For every rational function $R \in K(x)$, its *numerator* $\operatorname{num} R$ and *denominator* $\operatorname{den} R$ are uniquely determined by requiring that $\operatorname{num} R, \operatorname{den} R \in K[x]$, $R = \operatorname{num} R/\operatorname{den} R$, $\operatorname{num} R \perp \operatorname{den} R$, and $lc(\operatorname{den} R) = 1$. The *leading coefficient* of R is $lc(R) = lc(\operatorname{num} R)$, and R is *monic* if $lc(R) = 1$.

2.1 PNF, RNF and their strict versions

A rational function $R \in K(x)$ is *shift-reduced* if $\operatorname{num} R \perp E^k \operatorname{den} R$ for all $k \in \mathbb{Z}$. Irreducible polynomials $p, q \in K[x]$ are *shift-equivalent* if $p \mid E^k q$ for some $k \in \mathbb{Z}$. A rational function $R \in K(x)$ is *shift-homogeneous* if all irreducible factors of $\operatorname{num} R$ and $\operatorname{den} R$ belong to the same shift-equivalence class. By grouping together shift-equivalent irreducible monic factors of its numerator and denominator, every rational function $R(x) \in K(x)$ can be written in the form

$$R(x) = z\, R_1(x)\, R_2(x) \, \cdots \, R_k(x) \tag{5}$$

where $z \in K$, $k \geq 0$, each R_i is a monic shift-homogeneous rational function, and $R_i\, R_j$ is not shift-homogeneous whenever $i \neq j$. We call (5) a *shift-homogeneous factorization* of R.

Definition 1 *Let $R \in K(x)$. If there are $z \in K$, and monic polynomials $a, b, c \in K[x]$ such that*

(i) $R = z \cdot \dfrac{a}{b} \cdot \dfrac{Ec}{c}$,

(ii) *$a \perp E^k b$ for all $k \in \mathbb{N}$,*

then (z, a, b, c) is a polynomial normal form (PNF) *of R. If, in addition,*

(iii) *$a \perp c$ and $b \perp Ec$,*

then (z, a, b, c) is a strict PNF *of R.*

Every nonzero rational function has a unique strict PNF. For a proof of this, and for an algorithm to compute it, see [7] or [8]. We denote the strict PNF of $R \in K(x) \setminus \{0\}$ by $\operatorname{SPNF}(R)$.

Definition 2 *Let $R \in K(x)$. If there are $z \in K$, and monic polynomials $r, s, u, v \in K[x]$ such that*

(i) $R = F \cdot \frac{EV}{V}$ *where* $F = z \cdot \frac{r}{s}$, $V = \frac{u}{v}$ *and* $u \perp v$,

(ii) $r \perp E^k s$ *for all* $k \in \mathbb{Z}$ *(i.e., F is shift-reduced),*

then (z, r, s, u, v) *is a* rational normal form (RNF) *of R. If in addition,*

(iii) $r \perp u \cdot Ev$ *and* $s \perp Eu \cdot v$,

then (z, r, s, u, v) *is a* strict RNF *of R.*

Every nonzero rational function has a strict RNF. For a proof of this, and for an algorithm to compute it, see [2].

Definition 3 *The rational functions $F = zr/s$ and $V = u/v$ are called, respectively, the* kernel *and the* shell *of the RNF (z, r, s, u, v).*

For notational convenience, an RNF of a rational function R is sometimes written in the short form (F, V) instead of in the long form (z, r, s, u, v).

2.2 Minimizing the kernel

Theorem 1 *Let $\varphi = (z, r, s, u, v)$ be any RNF of $R \in K(x) \setminus \{0\}$. Then*

(i) *z is unique;*

(ii) *if R is shift-homogeneous then $r = 1$ or $s = 1$;*

(iii) *the degrees of the polynomials r and s are unique, and have minimal possible values in the sense that if*

$$R(x) = \frac{p(x)}{q(x)} \frac{EG(x)}{G(x)}$$

where $p, q \in K[x]$ and $G \in K(x)$, then $\deg r \leq \deg p$ and $\deg s \leq \deg q$;

(iv) *given $F = zr/s$, the RNF of R is uniquely determined;*

(v) $\varphi^{-1} = (1/z, s, r, v, u)$ *is an RNF of $1/R$. If φ is strict then so is φ^{-1}.*

For a proof, see [2].

Proposition 1 *Let $R \in K(x) \setminus \{0\}$. A strict RNF (z, r, s, u, v) of R is uniquely determined by either u or v.*

Proof: Let $(z_1, r_1, s_1, u_1, v_1)$, $(z_2, r_2, s_2, u_2, v_2)$ be two strict RNF's of R. This implies

$$z_1 \frac{r_1}{s_1} \frac{Eu_1}{u_1} \frac{v_1}{Ev_1} = z_2 \frac{r_2}{s_2} \frac{Eu_2}{u_2} \frac{v_2}{Ev_2}.$$

If $v_1 = v_2$ then (z_1, r_1, s_1, u_1) and (z_2, r_2, s_2, u_2) are both strict PNF's of $R_1 = R \cdot (Ev_1/v_1)$. Similarly, if $u_1 = u_2$ then $(1/z_1, s_1, r_1, v_1)$ and $(1/z_2, s_2, r_2, v_2)$ are both strict PNF's of $R_2 = (1/R)\,(Eu_1/u_1)$. Since the strict PNF of a rational function is unique, we have proved the assertion. □

Example 1 Let $R \in K(x) \setminus \{0\}$. While the strict PNF of R is unique, R can have infinitely many distinct PNF's. For instance, for any irreducible monic $p \in K[x]$, the four-tuple $(1, p, Ep, p)$ is a PNF of $R(x) = 1$. Likewise, some rational functions R can have infinitely many distinct RNF's. For instance, for any monic $p \in K[x]$ and $k \in \mathbb{N}$, the five-tuple $(1, E^k p, 1, 1, p\,Ep \cdots E^{k-1}p)$ is an RNF of $R(x) = p(x)$.

It is easy to see that for every $R \in K(x) \setminus \{0\}$, there are only finitely many *strict* RNF's which can be found as follows:

1. Factor R into maximal shift-homogeneous factors of the form (5), and work with each factor R_i, $1 \le i \le k$, separately.

2. Consider the monic shift-homogeneous factor R_i. Set $n_i = \deg \operatorname{num} R_i$, $m_i = \deg \operatorname{den} R_i$. If $n_i \ge m_i$, check all possible choices of a monic factor $r_{i,j}$ of $\operatorname{num} R_i$ of degree $n_i - m_i$ to see if there exists a strict RNF φ_j of R_i of the form $\varphi_j = (1, r_{i,j}, 1, u_{i,j}, v_{i,j})$. The procedure is analogous for the case where $n_i < m_i$.

Property (iii) in Theorem 1 shows the minimality of the kernel of any RNF. An interesting question is how to compute an RNF not only with the minimal kernel, but also with a minimal shell (in some sense).

3 Minimizing the shell: $\deg \operatorname{num} V$ or $\deg \operatorname{den} V$

3.1 Definition and properties of RCF_1 and RCF_2

Among all possible RNF's of R we distinguish two (not necessarily distinct) forms which are called the *first* and the *second rational canonical forms* (RCF_1 and RCF_2) of R.

Definition 4 *Let $R \in K(x) \setminus \{0\}$. A strict RNF (z, r_1, s_1, u_1, v_1) of R is the* first rational canonical form (RCF_1) *of R if $v_1 \,|\, v$ for every RNF (z, r, s, u, v) of R. A strict RNF (z, r_2, s_2, u_2, v_2) of R is the* second rational canonical form (RCF_2) *of R if $u_2 \,|\, u$ for every RNF (z, r, s, u, v) of R.*

Theorem 2 *Every $R \in K(x) \setminus \{0\}$ has a unique RCF_1 and a unique RCF_2.*

Proof: By Definition 4, any two RCF_1's of R have the same v, hence by Proposition 1 they are equal. Similarly, any two RCF_2's of R have the same u, hence they are equal. This proves uniqueness of RCF_1 and RCF_2 (and justifies our use of the word "canonical"). Their existence is established constructively by Algorithms RCF_1 and RCF_2, respectively, in Section 3.2. □

We denote the unique RCF_1 and RCF_2 of $R \in K(x) \setminus \{0\}$ by $\mathrm{RCF}_1(R)$ and $\mathrm{RCF}_2(R)$, respectively.

From Definition 4 it follows that $\mathrm{RCF}_1(R)$ (resp. $\mathrm{RCF}_2(R)$) guarantees minimality of the denominator (resp. of the numerator) of the shell among all RNF's of R. Furthermore, it also guarantees minimality of its numerator (resp. of its denominator) among all those RNF's of R that have the same (i.e., minimal) degree of the denominator (resp. of the numerator) as $\mathrm{RCF}_1(R)$ (resp. $\mathrm{RCF}_2(R)$):

Proposition 2 *Let $RCF_i(R) = (z, r_i, s_i, u_i, v_i)$, $i \in \{1, 2\}$. Let (z, r, s, u, v) be an RNF of R.*

(i) *If $\deg v = \deg v_1$ then $v_1 = v$ and $u_1 \,|\, u$.*

(ii) *If $\deg u = \deg u_2$ then $u_2 = u$ and $v_2 \,|\, v$.*

Proof: (i) Let $\deg v = \deg v_1$. By definition of RCF_1 we have $v_1 \,|\, v$, hence $v_1 = v$. Then

$$\frac{r_1}{s_1}\frac{Eu_1}{u_1} = \frac{r}{s}\frac{Eu}{u}.$$

As $\mathrm{RCF}_1(R)$ is strict and r/s is shift-reduced, [8, Lemma 5.3.1] implies that $u_1 \,|\, u$. – The proof of (ii) is analogous. □

However, as shown by the next proposition, the price for absolute minimality of the denominator (resp. of the numerator) of the shell in RCF_1 (resp. in RCF_2) is maximality of its numerator (resp. of its denominator) among all *strict* RNF's of R.

Proposition 3 *Let* $RCF_i(R) = (z, r_i, s_i, u_i, v_i)$, $i \in \{1, 2\}$. *If* (z, r, s, u, v) *is a strict RNF of* R *then* $u \mid u_1$ *and* $v \mid v_2$.

Proof: By definition of RCF_1, there is $w \in K[x]$ such that $v = v_1 w$. Then

$$\frac{r_1}{s_1} \frac{E(u_1 w)}{(u_1 w)} = \frac{r}{s} \frac{Eu}{u}.$$

As (z, r, s, u, v) is strict and r_1/s_1 is shift-reduced, [8, Lemma 5.3.1] implies that $u \mid u_1 w$. From $u \perp v$ it follows that $u \perp w$, so $u \mid u_1$ as desired. – The proof that $v \mid v_2$ is analogous. □

Corollary 1 *If* $RCF_1(R) = RCF_2(R)$ *then this is the only strict RNF of* R.

Proof: Let $\varphi = (z, r, s, u, v)$ be any strict RNF of R. Write $\mathrm{RCF}_1(R) = \mathrm{RCF}_2(R) = (z, r_1, s_1, u_1, v_1)$. Then $v_1 \mid v$ by Definition 4 and $v \mid v_1$ by Proposition 3, hence $v = v_1$. By Proposition 1, $\varphi = \mathrm{RCF}_1(R) = \mathrm{RCF}_2(R)$. □

3.2 Existence and computation of $\mathbf{RCF_1}$ and $\mathbf{RCF_2}$

In this section we prove the existence of RCF_1 and RCF_2 by giving algorithms to construct them.

`Algorithm` $\mathbf{RCF_1}$
`input:` $R \in K(x) \setminus \{0\}$
`output:` $\mathrm{RCF}_1(R)$

$(z, a, b, c) := \mathrm{SPNF}(R)$;
$(1, a_1, b_1, c_1) := \mathrm{SPNF}(b/a)$;
$g := \gcd(c, c_1)$; (take g monic)
$d := c/g$; $d_1 := c_1/g$;
return (z, b_1, a_1, d, d_1).

`Algorithm` $\mathbf{RCF_2}$
`input:` $R \in K(x) \setminus \{0\}$
`output:` $\mathrm{RCF}_2(R)$

$(z, r, s, u, v) := \mathrm{RCF}_1(1/R)$;
return $(1/z, s, r, v, u)$.

Now we proceed to prove correctness of these algorithms.

Lemma 1 *Let (z,a,b,c) be the strict PNF of $R \in K(x) \setminus \{0\}$, and let (z,r,s,u,v) be an RNF of R such that $r \perp u$ and $s \perp Eu$. Then $u \,|\, c$.*

Proof: We have

$$R \;=\; z\,\frac{a}{b}\,\frac{Ec}{c} \;=\; z\,\frac{r}{s}\,\frac{E(u/v)}{(u/v)}. \tag{6}$$

Set

$$R_1 = \frac{a}{b}\,\frac{E(cv)}{(cv)}.$$

It follows from (6) that

$$R_1 \;=\; \frac{1}{z}\,\frac{Ev}{v}\,R \;=\; \frac{r}{s}\,\frac{Eu}{u}.$$

As $r \perp u$, $s \perp Eu$ and $\gcd(a, E^k b) = 1$ for all $k \in \mathbb{N}$, it follows from [8, Lemma 5.3.1] that $u \,|\, c\,v$. Hence $u \,|\, c$. □

Lemma 2 *Let (z,r,s,u,v) be any RNF of $R \in K(x) \setminus \{0\}$. Then there is an RNF (z,r',s',u',v) of R such that $r' \perp u'$ and $s' \perp Eu'$.*

Proof: Let $\mathcal{R}$ be the set of all pairs of monic polynomials (ρ,τ) such that ρ/s is shift-reduced and $\rho\,E\tau/\tau = r\,Eu/u$. The set $\mathcal{R}$ contains (r,u), so $\mathcal{R} \neq \emptyset$. Let $(r',u_1) \in \mathcal{R}$ be such that $\deg u_1$ is minimal among all pairs in $\mathcal{R}$. Then

$$r'\,\frac{Eu_1}{u_1} \;=\; r\,\frac{Eu}{u}. \tag{7}$$

Denote $g = \gcd(r',u_1)$, $r_2 = r'/g$ and $u_2 = u_1/g$. Then $\deg u_2 \le \deg u_1$. As $r_2 \,|\, r'$, $Eg \,|\, Er'$, and r'/s is shift-reduced, so is $r_2\,Eg/s$. As $r_2\,Eg\,Eu_2/u_2 = r'\,Eu_1/u_1 = r\,Eu/u$, it follows that $(r_2\,Eg, u_2) \in \mathcal{R}$. By definition of u_1 we have $\deg u_1 \le \deg u_2$, so $\deg u_1 = \deg u_2$ and $\deg g = 0$. Hence $r' \perp u_1$.

Let $\mathcal{S}$ denote the set of all pairs of monic polynomials (σ,τ) such that r'/σ is shift-reduced and $(1/\sigma)\,E\tau/\tau = (1/s)\,Eu_1/u_1$. The set $\mathcal{S}$ contains (s,u_1), so $\mathcal{S} \neq \emptyset$. Let $(s',u') \in \mathcal{S}$ be such that $\deg u'$ is minimal among all pairs in $\mathcal{S}$. Then

$$\frac{1}{s'}\,\frac{Eu'}{u'} \;=\; \frac{1}{s}\,\frac{Eu_1}{u_1}. \tag{8}$$

Denote $h = \gcd(s',Eu')$, $s_2 = s'/h$ and $u_2 = u'/E^{-1}h$. Then $\deg u_2 \le \deg u'$. As $s_2 \,|\, s'$, $E^{-1}h \,|\, E^{-1}s'$, and r'/s' is shift-reduced, so is $r'/(s_2E^{-1}h)$. As $(1/(s_2\,E^{-1}h))\,Eu_2/u_2 = (1/s')\,Eu'/u' = (1/s)\,Eu_1/u_1$, it follows that $(s_2\,E^{-1}h, u_2) \in \mathcal{S}$. By definition of u' we have $\deg u' \le \deg u_2$, so

$\deg u_2 = \deg u'$ and $\deg h = 0$. Hence $s' \perp Eu'$. Together with (8) and [8, Lemma 5.3.1] this implies that $u' \mid u_1$, and so $r' \perp u'$. Finally, from (7) and (8) we have

$$\frac{r'}{s'}\frac{Eu'}{u'} = \frac{r'}{s}\frac{Eu_1}{u_1} = \frac{r}{s}\frac{Eu}{u},$$

so (z, r', s', u', v) is an RNF of R with all required properties. □

Theorem 3 *Algorithms RCF_1 and RCF_2 are correct.*

Proof: Let $z, a, b, c, a_1, b_1, c_1, g, d, d_1$ be as in Algorithm RCF_1. We claim that $\varphi_1 = (z, b_1, a_1, d, d_1)$ equals $\mathrm{RCF}_1(R)$. It follows from (the proof of) [2, Theorem 1] that φ_1 is a strict RNF of R. We need to show that if $\varphi = (z, r, s, u, v)$ is any RNF of R then $d_1 \mid v$. By Lemma 2, there is an RNF (z, r', s', u', v) of R such that $r' \perp u'$ and $s' \perp Eu'$. By Lemma 1, we have $u' \mid c$, so $c_2 := c\,v/u'$ is a polynomial and

$$\frac{a_1}{b_1}\frac{Ec_1}{c_1} = \frac{b}{a} = z\,\frac{1}{R}\frac{Ec}{c} = \frac{s'}{r'}\frac{Ec_2}{c_2}.$$

As $a_1 \perp c_1$, $b_1 \perp Ec_1$ and s'/r' is shift-reduced, [8, Lemma 5.3.1] implies that $c_1 \mid c_2$. Let $q_1 = c_2/c_1 \in K[x]$. Then

$$d_1 q_1 u' = \frac{c_1}{g}\,q_1 u' = \frac{c_2}{g}\,u' = \frac{c\,v}{g} = d\,v.$$

As $d_1 \perp d$, it follows that $d_1 \mid v$ which proves the claim.

Let $\varphi_2 = (1/z, s, r, v, u)$ be the output of Algorithm RCF_2. We claim that φ_2 equals $\mathrm{RCF}_2(R)$. By Theorem 1 (v), φ_2 is a strict RNF of R. Let $\varphi = (1/z, s', r', v', u')$ be any strict RNF of R. Then by Theorem 1 (v), (z, r', s', u', v') is a strict RNF of $1/R$. Since $(z, r, s, u, v) = \mathrm{RCF}_1(1/R)$, it follows that $v \mid v'$, proving the claim. □

Example 2 Consider the rational function

$$R = \frac{n\,(n+2)(n-4+\sqrt{2})(n-3+\sqrt{2})(n+2+\sqrt{2})(n+11+\sqrt{2})}{(n-3)(n-2)^2(n+6)(n+12)(n-1+\sqrt{2})(n+1+\sqrt{2})}.$$

Following Algorithm RCF_1, the strict PNF (z, a, b, c) of R is

$$\Big(1,\ (n-4+\sqrt{2})(n-3+\sqrt{2}),\ (n-3)(n+6)(n+12),$$
$$(n-2)^2(n-1)^2 n\,(n+1)(n-1+\sqrt{2})(n+\sqrt{2})(n+1+\sqrt{2})^2$$
$$(n+2+\sqrt{2})(n+3+\sqrt{2})(n+4+\sqrt{2})(n+5+\sqrt{2})(n+6+\sqrt{2})$$
$$(n+7+\sqrt{2})(n+8+\sqrt{2})(n+9+\sqrt{2})(n+10+\sqrt{2})\Big),$$

and the strict PNF (s, a_1, b_1, c_1) of b/a is

$$\left(1,\ (n-3)(n+6)(n+12),\ (n-4+\sqrt{2})(n-3+\sqrt{2}),\ 1\right).$$

Since $\gcd(c, c_1) = 1$, the RCF_1 $(z,\ r_1,\ s_1,\ u_1,\ v_1)$ of R is

$$\begin{aligned}\Big(1,\ &(n-4+\sqrt{2})(n-3+\sqrt{2}),\ (n-3)(n+6)(n+12),\\ &(n-2)^2(n-1)^2 n\,(n+1)(n-1+\sqrt{2})(n+\sqrt{2})(n+1+\sqrt{2})^2\\ &(n+2+\sqrt{2})(n+3+\sqrt{2})(n+4+\sqrt{2})(n+5+\sqrt{2})(n+6+\sqrt{2})\\ &(n+7+\sqrt{2})(n+8+\sqrt{2})(n+9+\sqrt{2})(n+10+\sqrt{2}),\ 1\Big).\end{aligned}$$

Following Algorithm RCF_2, the RCF_2 $(z,\ r_2,\ s_2,\ u_2,\ v_2)$ of R is

$$\begin{aligned}\Big(1,\ &(n+2+\sqrt{2})(n+11+\sqrt{2}),\ (n-3)(n-2)^2,\ 1,\\ &n\,(n+1)(n+2)^2(n+3)^2(n+4)^2(n+5)^2(n+6)\\ &(n+7)(n+8)(n+9)(n+10)(n+11)(n-4+\sqrt{2})\\ &(n-3+\sqrt{2})^2(n-2+\sqrt{2})^2(n-1+\sqrt{2})(n+\sqrt{2})\Big).\end{aligned}$$

Notice that $\deg r_1 = \deg r_2$, $\deg s_1 = \deg s_2$, $u_2 \mid u_1$, $v_1 \mid v_2$, as expected.

4 Minimizing the shell: total degree

4.1 The multiplicative structure of the shell

Let (5) be the shift-homogeneous factorization of a rational function R. Denote by $\mathrm{RNF}_x(R)$ the set of all RNF's of R. Then, obviously, there exists a one-to-one correspondence between $\mathrm{RNF}_x(R)$ and $\mathrm{RNF}_x(R_1) \times \cdots \times \mathrm{RNF}_x(R_k)$:

$$(F, V) \leftrightarrow ((F_1, V_1), \ldots, (F_k, V_k))$$

where $F = z\,F_1 \cdots F_k$, $V = V_1 \cdots V_k$, (F, V) is an RNF of R, and (F_i, V_i) is an RNF of R_i for $1 \le i \le k$. Therefore we can limit our attention to a monic shift-homogeneous rational function of the form

$$\frac{p(x+a_1)p(x+a_2)\cdots p(x+a_m)}{p(x+b_1)p(x+b_2)\cdots p(x+b_n)} \tag{9}$$

where $p(x)$ is an irreducible polynomial while $a_1 \le a_2 \le \cdots \le a_m$ and $b_1 \le b_2 \le \cdots \le b_n$ are nonnegative integers. If $m = n$ then it follows from

Theorem 1 (ii) and (iv) that (9) has a unique RNF (z, r, s, u, v) such that $z = r = s = 1$. It is easy to see that

$$\begin{aligned} \deg u + \deg v &= (\deg p) \sum_{k=1}^{m} |a_k - b_k|, \\ \deg u - \deg v &= (\deg p) \sum_{k=1}^{m} (a_k - b_k). \end{aligned} \tag{10}$$

Otherwise $m \neq n$. W.l.g. we can assume that $m < n$. Then it is evident that any RNF of R can be defined by an injection

$$f : \{1, 2, \dots, m\} \to \{1, 2, \dots, n\}, \quad f(1) < f(2) < \dots < f(m). \tag{11}$$

Similarly to (10) we obtain the following theorem.

Theorem 4 *Let R be written in the form (9), let f be an injection of the form (11), and let (F, V) be the corresponding* RNF *of R. Then*

$$\deg \operatorname{num} V + \deg \operatorname{den} V = (\deg p) \sum_{k=1}^{m} |a_k - b_{f(k)}|,$$

$$\deg \operatorname{num} V - \deg \operatorname{den} V = (\deg p) \sum_{k=1}^{m} (a_k - b_{f(k)}).$$

In general, not all RNF's induced by injections of the form (11) are strict. As customary, we denote the range of f by $\operatorname{rng}(f) = \{f(1), f(2), \dots, f(m)\}$.

Lemma 3 *An injection f of the form (11) induces a strict* RNF *of R iff for any $j \notin \operatorname{rng}(f)$ the number of k such that $b_{f(k)} < b_j$ is equal to the number of k such that $a_k < b_j$.*

Proof: Suppose that $j \notin \operatorname{rng}(f)$, and $l = \max\{k : f(k) < j\}$. Let V_1, V_2 be such that

$$\frac{p(x + a_1) \cdots p(x + a_l)}{p(x + b_{f(1)}) \cdots p(x + b_{f(l)})} = \frac{EV_1}{V_1}, \quad \frac{p(x + a_{l+1}) \cdots p(x + a_m)}{p(x + b_{f(l+1)}) \cdots p(x + b_{f(m)})} = \frac{EV_2}{V_2}.$$

Then $p(x + b_j)$ does not divide $\operatorname{den} V_1$ because $b_j > b_{f(l)}$, and $p(x + b_j)$ divides $\operatorname{num} EV_1$ iff $b_j \le a_l$. The case of V_2 is treated similarly. □

4.2 RNF*: forms with minimal total degree of the shell

Definition 5 *Let $R \in K(x) \setminus \{0\}$. An RNF (z, r, s, u, v) of R is an* RNF* *of R if* $\deg u + \deg v$ *is minimal among all RNF's of R.*

The problem of finding an RNF* is equivalent to the problem of finding an injection f of the form (11) such that the sum

$$\sum_{k=1}^{m} |a_k - b_{f(k)}| \tag{12}$$

is minimal.

Proposition 4 *Any such injection gives rise to a strict* RNF* *of R.*

Proof: Suppose that an injection f minimizes the sum (12), but does not induce a strict RNF. Then by Lemma 3 there exists $j \notin \text{rng}(f)$ such that the number N_b of k such that $b_{f(k)} < b_j$ is not equal to the number N_a of k such that $a_k < b_j$. W.l.g. assume that $N_a < N_b$. Set $t = \max\{k : b_{f(k)} < b_j\}$. Then $a_t > b_j > b_{f(t)}$, $b_{f(t+1)} > b_j$, and $|a_t - b_j| < |a_t - b_{f(t)}|$; thereby

$$\sum_{k=1}^{m} |a_k - b_{f(k)}| > \sum_{k=1}^{t-1} |a_k - b_{f(k)}| + |a_t - b_j| + \sum_{k=t+1}^{m} |a_k - b_{f(k)}|.$$

This means that the injection f' defined by $f'(t) = j$, $f'(k) = f(k)$ for $k \neq t$, produces a smaller sum (12) than f. A contradiction.

Due to the minimality of the sum (12), it is evident that such an injection gives rise to an RNF* of R. □

Example 3 Consider the rational function R in Example 2. R can be written as $R_1 \cdot R_2$ where R_1, R_2 each is a monic shift-homogeneous rational function:

$$R_1 = \frac{n\,(n+2)}{(n-3)(n-2)^2(n+6)(n+12)},$$
$$R_2 = \frac{(n-4+\sqrt{2})(n-3+\sqrt{2})(n+2+\sqrt{2})(n+11+\sqrt{2})}{(n-1+\sqrt{2})(n+1+\sqrt{2})}.$$

For the monic shift-homogeneous factor R_1, there exist two injections f_1, f_2 such that the sum $\sum_{k=1}^{2} |a_k - b_{f_j(k)}|$ is minimal for $1 \leq j \leq 2$.
For the injection f_1:

$$R_1 = \frac{1}{n-3} \cdot \boxed{\frac{n}{n-2}} \cdot \boxed{\frac{n+2}{n-2}} \cdot \frac{1}{n+6} \cdot \frac{1}{n+12},$$

the corresponding RNF* (z, r_1, s_1, u_1, v_1) is

$$(1, 1, (n-3)(n+6)(n+12), (n-2)^2(n-1)^2 n\ (n+1), 1).$$

For the injection f_2:

$$R_1 = \frac{1}{n-3} \cdot \boxed{\frac{n}{n-2}} \cdot \frac{1}{n-2} \cdot \boxed{\frac{n+2}{n+6}} \cdot \frac{1}{n+12},$$

the corresponding RNF* (z, r_2, s_2, u_2, v_2) is

$$(1, 1, (n-3)(n-2)(n+12), (n-1)(n-2), (n+2)(n+3)(n+4)(n+5)).$$

For the monic shift-homogeneous factor R_2, there exists one injection f such that the sum $\sum_{k=1}^{2} |a_k - b_{f(k)}|$ is minimal:

$$R_2 = (n-4+\sqrt{2}) \cdot \boxed{\frac{n-3+\sqrt{2}}{n-1+\sqrt{2}}} \cdot \boxed{\frac{n+2+\sqrt{2}}{n+1+\sqrt{2}}} \cdot (n+11+\sqrt{2}),$$

and the corresponding RNF* $(z_3, r_3, s_3, u_3, v_3)$ is

$$\left(1, (n-4+\sqrt{2})(n+11+\sqrt{2}), 1, (n+1+\sqrt{2}), (n-2+\sqrt{2})(n-3+\sqrt{2})\right).$$

As the result, the two RNF*'s $(z, r_1^*, s_1^*, u_1^*, v_1^*)$ and $(z, r_2^*, s_2^*, u_2^*, v_2^*)$ respectively of R are

$$\begin{aligned}&\Big(1, (n-4+\sqrt{2})(n+11+\sqrt{2}), (n-3)(n+6)(n+12),\\ &\quad (n-2)^2(n-1)^2 n\ (n+1)(n+1+\sqrt{2}), (n-3+\sqrt{2})(n-2+\sqrt{2})\Big),\end{aligned}$$

and

$$\begin{aligned}&\Big(1, (n-4+\sqrt{2})(n+11+\sqrt{2}), (n-3)(n-2)(n+12), (n-1)(n-2)\\ &\quad (n+1+\sqrt{2}), (n+2)(n+3)(n+4)(n+5)(n-3+\sqrt{2})(n-2+\sqrt{2})\Big).\end{aligned}$$

Note that the total degree of the shell in both RNF*'s is 9, while it is 19 for $\mathrm{RCF}_1(R)$, and 23 for $\mathrm{RCF}_2(R)$ (see Example 2).

4.3 Reduction to a linear programming problem

The problem of computing an injection f such that the sum in (12) is minimal can be reduced to a well-known combinatorial problem which can be solved by linear programming techniques. This is the Minimum Weighted Bipartite Matching Problem (MWBM): Given a complete bipartite graph $K_{m,n}$ (where $m \leq n$) with rational weights on the edges, find a matching (i.e., a set of pairwise nonadjacent edges) of size m which has minimum total weight. It is well known ([5]; see also [6]) that MWBM can be solved efficiently (in time polynomial in $\max\{m,n\}$, i.e., avoiding exhaustive search).

To reduce the problem of constructing an RNF* to MWBM (which is also known as the Assignment Problem), construct a complete bipartite graph with vertex sets $\{u_1, u_2, \ldots, u_m\}$ and $\{v_1, v_2, \ldots, v_n\}$ where all the u_j's and v_k's are pairwise distinct, and let the weight on the edge connecting u_j with v_k be $|a_j - b_k|$. This special case of MWBM can be solved even in linear time [4].

4.4 Definition and properties of RCF$_1^*$ and RCF$_2^*$

There may exist several RNF's (z, r, s, u, v) of R with minimum value of $\deg u + \deg v$ (we have called them RNF*'s). Among all such forms, we can again distinguish two forms which minimize $\deg v$ or $\deg u$ (i.e., maximize or minimize $\deg u - \deg v$, respectively). We denote them by RCF$_1^*$ and RCF$_2^*$, respectively.

Theorem 5 *Both* RCF$_1^*(R)$ *and* RCF$_2^*(R)$ *are uniquely defined strict* RNF**'s of* R*.*

Proof: The strictness of RCF$_1^*$ and RCF$_2^*$ follows from Proposition 4. Let us prove, for example, uniqueness of RCF$_1^*(R)$. Suppose that R is of the form (9), and that there are two injections f, f' which induce two different RCF$_1^*$'s of R. Let l, $1 \leq l \leq n$, be such that $f(l) > f'(l)$, and therefore $b_{f(l)} > b_{f'(l)}$.

(a) $a_l - b_{f(l)} < 0$ and $a_l - b_{f'(l)} < 0$. The injection f'' such that if $k \neq l$ then $f''(k) = f(k)$ while $f''(l) = f'(l)$ produces a smaller sum (12) than the one produced by f.

(b) $a_l - b_{f(l)} < 0$ and $a_l - b_{f'(l)} > 0$. Consider two cases:

(b1) $|a_l - b_{f(l)}| \neq |a_l - b_{f'(l)}|$. Similarly to (a), it is possible to decrease the sum produced by f.

(b2) $|a_l - b_{f(l)}| = |a_l - b_{f'(l)}|$. By changing f as described in (a), we get f'' which does not change the sum, but decreases $\deg v$.

□

Example 4 For the rational function R in Example 3, the computed RNF* $(z, r_1^*, s_1^*, u_1^*, v_1^*)$ is the RCF$_1^*$ of R, and the computed RNF* $(z, r_2^*, s_2^*, u_2^*, v_2^*)$ is the RCF$_2^*$ of R ($\deg u_1^* = 7$, $\deg v_1^* = 2$, $\deg u_2^* = 3$, $\deg v_2^* = 6$).

4.5 Computation of RCF$_1^*$ and RCF$_2^*$

Suppose again that $m < n$ in (12). Computation of RCF$_1^*(R)$ is a special choice of an injection f or, equivalently, of m factors $p(x + b_{f(1)})p(x + b_{f(2)}) \cdots p(x+b_{f(m)})$ of the denominator of (9). If we wish to obtain RCF$_1^*(R)$ then we should find an RNF* that maximizes the sum $\sum_{k=1}^{m}(a_k - b_{f(k)})$ or, equivalently, minimizes the sum $b_{f(1)} + \cdots + b_{f(m)}$. For this purpose, we add $n-m$ new vertices $u_{m+1}, \ldots, u_n$ to the vertex set $\{u_1, \ldots, u_m\}$, and connect each of them with each of $v_1, \ldots, v_n$. Set

$$N = b_1 + \cdots + b_n + 1. \tag{13}$$

Let the weight w_{jk} of the edge $[u_j, v_k]$ be equal to $|a_j - b_k|$ if $j \leq m$, and to $1 - b_k/N$ otherwise. When MWBM is solved, any vertex u_j, $j \leq m$, is connected with a unique vertex v_k. This gives an injection $f : \{1, 2, \ldots, m\} \to \{1, 2, \ldots, n\}$.

Lemma 4 *The algorithm described above constructs an injection f such that the sum (12) is minimal. Additionally, among all injections that minimize this sum, the constructed injection minimizes the sum $b_{f(1)}+\cdots+b_{f(m)}$.*

Proof: It is easy to see that if we set

$$w_{\sigma\tau} = 1 - \varepsilon_\tau,\ \varepsilon_\tau \geq 0,\ \sigma = m+1, \ldots, n,\ \tau = 1, \ldots, n,$$

and $\varepsilon_1 + \cdots + \varepsilon_n < 1$, then any solution of MWBM minimizes the sum of the corresponding integer weights (and, thereby, gives us an RNF$^*(R)$), and under this condition, maximizes the sum of those ε_τ for which the solution of MWBM contains an edge $[u_\sigma, v_\tau]$ with $\sigma > m$. This means that if we define ε_k as b_k/N then we obtain an injection f that gives an RNF* with minimal $(b_{f(1)} + \cdots + b_{f(m)})/N$ or, equivalently, with minimal $b_{f(1)} + \cdots + b_{f(m)}$. □

Theorem 6 *The algorithm described above constructs an injection f which induces* RCF$_1^*$ *of (9).*

Proof: The claim follows immediately from Lemma 4. □

Note that in order to construct RCF_2^* we have to maximize the sum $b_{f(1)} + \cdots + b_{f(m)}$. To attain this goal, set w_{jk} to $|a_j - b_k|$ if $j \le m$, and b_k/N otherwise.

We conclude this section by giving detailed descriptions of the algorithms to compute RCF_1^* and RCF_2^*. First, we present Algorithm mshRCF$_1^*$ for computing RCF_1^* of a monic shift-homogeneous rational function R of the form (9). Let $\mathrm{MWBM}(m, w)$ be an algorithm for solving the Minimum Weighted Bipartite Matching Problem on the balanced complete bipartite graph $K_{m,m}$ with the $m \times m$ weight matrix w. The output of $\mathrm{MWBM}(m, w)$ is the injection f such that the sum (12) is minimal. Then Algorithm mshRCF$_1^*$ can be described as follows.

`Algorithm` **mshRCF$_1^*$**
`input:` a monic shift-homogeneous rational function R of the form (9)
`output:` $\mathrm{RCF}_1^*(R)$

```
if m < n then
    N := b_1 + ··· + b_n + 1;
    for s from 1 to n do
        for r from 1 to m do
            w_rs := |a_r − b_s|;
        od;
        for r from m + 1 to n do
            w_rs := 1 − b_s/N;
        od;
    od;
    f := MWBM(n, w);
    (1, 1, 1, u, v) := RCF_1 ( ∏_{k=1}^{m} p(x + a_k)/p(x + b_{f(k)}) );
    return (1, 1, ∏_{k∈{1,...,n}\rng(f)} p(x + b_k), u, v).
else
    M := a_1 + ··· + a_m + 1;
    for r from 1 to m do
        for s from 1 to n do
            w_rs := |a_r − b_s|;
        od;
        for s from n + 1 to m do
            w_rs := a_r/M;
```

od;
od;
$f :=$ MWBM(m, w);
$(1,1,1,u,v) := \mathrm{RCF}_1\left(\prod_{k=1}^{n} p(x+a_{f(k)})/p(x+b_k)\right)$;
return $(1, \prod_{k\in\{1,\ldots,m\}\setminus \mathrm{rng}(f)} p(x+a_k), 1, u, v)$.
fi.

For $R \in K(x)$, let the output of the function SHF(R) be the shift-homogeneous factorization in the form (5) of R. The following algorithms compute RCF_1^* and RCF_2^* of R, respectively.

`Algorithm` $\mathbf{RCF_1^*}$
`input:` $R \in K(x) \setminus \{0\}$
`output:` $\mathrm{RCF}_1^*(R)$

$(z, R_1, R_2, \ldots, R_k) := \mathrm{SHF}(R)$;
for i from 1 to k do
$(1, r_i, s_i, u_i, v_i) := \mathrm{mshRCF}_1^*(R_i)$;
od;
return $\left(z, \prod_{i=1}^{k} r_i, \prod_{i=1}^{k} s_i, \prod_{i=1}^{k} u_i, \prod_{i=1}^{k} v_i\right)$.

`Algorithm` $\mathbf{RCF_2^*}$
`input:` $R \in K(x) \setminus \{0\}$
`output:` $\mathrm{RCF}_2^*(R)$

$(z, r, s, u, v) := \mathrm{RCF}_1^*(1/R)$;
return $(1/z, s, r, v, u)$.

5 Representing hypergeometric terms efficiently

A hypergeometric term $T(n)$ is usually represented as

$$\alpha^n P(n), \tag{14}$$

where $\alpha \in K$ and $P(n)$ is a product of Gamma-function values (if $K = \mathbb{C}$), or Pochhammer symbols (i.e., rising factorial powers) and their reciprocals.

Such representation can be simplified: we can replace (14) by

$$\alpha^n V(n) Q(n), \tag{15}$$

where $V(n)$ is a rational function, and $Q(n)$ is a product that looks like $P(n)$, but has the minimal possible number of factors. This can be achieved by using any RNF of the certificate of $T(n)$ ($V(n)$ is the shell of this RNF). If we use any of the rational canonical forms of the certificate of T as discussed in Sections 3 and 4, then we can additionally minimize $V(n)$ in one sense or another.

5.1 Efficient multiplicative decompositions using RCF's and RCF*'s

Definition 6 *Let $T(n)$ be a hypergeometric term. A* multiplicative decomposition *of T is a triple (F, W, n_0) where $F, W \in K(x)$ and $n_0 \in \mathbb{Z}$ are such that for all integers $n \geq n_0$:*

(i) *T is defined at n, F has neither a pole nor a zero at n, W has no pole at n,*

(ii) *$T(n)$ can be written as*

$$T(n) = W(n) \prod_{k=n_0}^{n-1} F(k). \tag{16}$$

This decomposition is minimal *if for any multiplicative decomposition (G, W, n_1) of T we have* $\deg \operatorname{num} F \leq \deg \operatorname{num} G$ *and* $\deg \operatorname{den} F \leq \deg \operatorname{den} G$.

Let $T(n)$ be a hypergeometric term with the certificate $R \in K(x)$. Let $n_0 \in \mathbb{Z}$ be such that $T(n)$ is defined for all integers $n \geq n_0$, and R has neither a pole nor a zero at n. It is easy to check that the triple $(R, T(n_0), n_0)$ is a multiplicative decomposition of T. Let (F, V) be an RNF of R. Set $W(n) = V(n)T(n_0)/V(n_0)$. Then it follows from Definition 6 and Theorem 1 that the multiplicative decomposition (F, W, n_0) is minimal.

Let (F, V) be one of the four RCF's of R as discussed in Sections 3 and 4, and the hypergeometric term $T(n)$ be written in the form (16). Then in addition to the property that the numerator and the denominator of the kernel F are of minimal possible degrees, the shell V is also minimal in some sense. That is, if we use RCF_1, then $\operatorname{den} V$ is of minimal degree; if we use

RCF_2, then $\operatorname{num} V$ is of minimal degree; if we use RCF_1^* or RCF_2^*, then $\deg\operatorname{num} V + \deg\operatorname{den} V$ is minimal, and under this condition, $\deg\operatorname{den} V$ is minimal for RCF_1^*, and $\deg\operatorname{num} V$ is minimal for RCF_2^*. In this case the representation of $T(n)$ of the form (16) is called an *efficient multiplicative decomposition* of T, denoted by $\mathrm{EMD}(T)$.

For a hypergeometric term $T(n)$, let R be the certificate of T, denoted by $\mathrm{cer}(T)$. Let $\mathrm{RCF}[1] \equiv \mathrm{RCF}_1$, $\mathrm{RCF}[2] \equiv \mathrm{RCF}_2$, $\mathrm{RCF}[3] \equiv \mathrm{RCF}_1^*$, and $\mathrm{RCF}[4] \equiv \mathrm{RCF}_2^*$. The following is a description of the algorithm to construct an efficient multiplicative decomposition of $T(n)$.

`Algorithm` **EMD[i]**
`input:` a hypergeometric term $T(n)$, $i \in \{1, 2, 3, 4\}$
`output:` an efficient multiplicative decomposition $W(n)\prod_{k=n_0}^{n-1} F(k)$ of $T(n)$ where:
If $i = 1$ then $\deg\operatorname{den} W$ is minimal.
If $i = 2$ then $\deg\operatorname{num} W$ is minimal.
If $i = 3$ then $\deg\operatorname{num} W + \deg\operatorname{den} W$ is minimal, and $\deg\operatorname{den} W$ is minimal.
If $i = 4$ then $\deg\operatorname{num} W + \deg\operatorname{den} W$ is minimal, and $\deg\operatorname{num} W$ is minimal;

$R := \mathrm{cer}(T)$;
$(F, V) := \mathrm{RCF}[i](R)$;
let $n_0 \in \mathbb{Z}$ be such that $T(n)$ is defined for all integers $n \geq n_0$,
and R has neither a pole nor a zero at n;
$W := V(n)T(n_0)/V(n_0)$;
return $W(n)\prod_{k=n_0}^{n-1} F(k)$.

5.2 Gamma-function values and Pochhammer symbols

Using Pochhammer symbol we can write

$$\prod_{k=n_0}^{n-1} (k - c) = (n_0 - c)_{n-n_0}$$

for any $c \in K$, $n_0 \in \mathbb{Z}$. If $K = \mathbb{C}$, then similarly

$$\prod_{k=n_0}^{n-1} (k - c) = \frac{\Gamma(n - c)}{\Gamma(n_0 - c)}.$$

Conversely, each expression

$$(-c)_n, \text{ or } \Gamma(n - c) \tag{17}$$

can be represented in the form $\delta\prod_{k=n_0}^{n-1}(k-c)$, where δ is a constant. Suppose that a hypergeometric term $T(n)$ is represented in an efficient multiplicative decomposition proposed in Subsection 5.1 as

$$\alpha^n V(n) \prod_{k=n_0}^{n-1} F(n) \tag{18}$$

where $\alpha \in K$, and $F(n)$ is a monic rational function. If we factorize the numerator and the denominator of F over $\overline{K}$ into linear factors, then by the above reasoning we can represent $T(n)$ in the form (15) with minimized $V(n)$ and with $Q(n)$ having the minimal possible number of factors of the form (17). Such a form is called an *efficient representation* of T.

Example 5 Consider the hypergeometric term

$$T = 24\prod_{k=1}^{n-1}\frac{1}{2}\,\frac{(3\,k^2+6\,k+4)\,(2\,k+3)\,(4\,k+5)\,(k+1)\,(4\,k+3)}{k\,(4\,k-1)\,(2\,k-1)\,(4\,k-3)\,(2\,k+5)\,(k+2)\,(3\,k^2+1)}$$

A multiplicative decomposition

$$T(n) = T(n_0)\prod_{k=n_0}^{n-1} R(k) \tag{19}$$

where the product in (19) is expressed in terms of a product of Gamma-function values in (17) is:

$$T_1 = 1536\sqrt{\pi}\left(\frac{1}{4}\right)^n \frac{p}{q} \tag{20}$$

where

$$\begin{aligned} p &= \Gamma\left(n+\frac{3}{4}\right)\Gamma(n+1)\,\Gamma\left(n+\frac{5}{4}\right)\Gamma\left(n+\frac{3}{2}\right)\times\\ &\quad\ \Gamma\left(n+1-\frac{\sqrt{3}}{3}i\right)\Gamma\left(n+1+\frac{\sqrt{3}}{3}i\right),\\ q &= \Gamma\left(n-\frac{3}{4}\right)\Gamma\left(n-\frac{1}{2}\right)\Gamma\left(n-\frac{1}{4}\right)\Gamma(n)\,\Gamma(n+2)\,\Gamma\left(n+\frac{5}{2}\right)\times\\ &\quad\ \Gamma\left(n-\frac{\sqrt{3}}{3}i\right)\Gamma\left(n+\frac{\sqrt{3}}{3}i\right).\end{aligned}$$

The four efficient representations of the hypergeometric term T_1 in (20) which are based on the four RCF's of its certificate are:

$$1536\,\sqrt{\pi}\,\left(\frac{1}{4}\right)^n \frac{\left(n^2+\frac{1}{3}\right)\left(n-\frac{3}{4}\right)\left(n-\frac{1}{2}\right)\left(n-\frac{1}{4}\right)n\left(n+\frac{1}{4}\right)\left(n+\frac{1}{2}\right)}{\Gamma(n+2)\,\Gamma\left(n+\frac{5}{2}\right)},$$

$$1536\,\sqrt{\pi}\,\left(\frac{1}{4}\right)^n \frac{\left(n^2+\frac{1}{3}\right)\left(n-\frac{3}{4}\right)\left(n-\frac{1}{4}\right)\left(n+\frac{1}{4}\right)}{(n+1)\left(n+\frac{3}{2}\right)\Gamma\left(n-\frac{1}{2}\right)\Gamma(n)},$$

$$1536\,\sqrt{\pi}\,\left(\frac{1}{4}\right)^n \frac{\left(n^2+\frac{1}{3}\right)\left(n-\frac{3}{4}\right)\left(n-\frac{1}{4}\right)n\left(n+\frac{1}{4}\right)}{\left(n+\frac{3}{2}\right)\Gamma\left(n-\frac{1}{2}\right)\Gamma(n+2)},$$

$$1536\,\sqrt{\pi}\,\left(\frac{1}{4}\right)^n \frac{\left(n^2+\frac{1}{3}\right)\left(n-\frac{3}{4}\right)\left(n-\frac{1}{4}\right)\left(n+\frac{1}{4}\right)}{(n+1)\left(n+\frac{3}{2}\right)\Gamma\left(n-\frac{1}{2}\right)\Gamma(n)}.$$

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