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ON UNIQUENESS OF
CARTESIAN PRODUCTS OF
SURFACES WITH BOUNDARY

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Abstract. It is known that if one of the factors of a decomposition of a manifold into Cartesian product is an interval then the decomposition is not unique. We prove that the decomposition of a 4-manifold (possibly with boundary) into 2-dimensional factors is unique, provided that the factors are not products of 1-manifolds.¹

1 Introduction

In 1945 Borsuk [B2] showed that any connected compact n -dimensional manifold without boundary has at most one decomposition into a Cartesian product of factors of dimension ≤ 2 . If we consider Cartesian products of higher-dimensional manifolds then such uniqueness property does not hold (see Theorem 11.5 in [C-R] and [K-R]). Even if we consider the classical Ulam problem [U] of uniqueness of Cartesian squares, one can find counterexamples for 3-manifolds (cf. [K-S]).

The uniqueness of the decomposition into Cartesian products fails if the factors are 2-manifolds with boundary. A torus with a hole and a disk with two holes are not homeomorphic, however, their Cartesian products with the interval $I=[0,1]$ are homeomorphic.

Similarly, the product of a Möbius band with a hole and the interval I is homeomorphic to the product of a Klein bottle with a hole and the interval I . All 2-manifolds in the examples above can be constructed by identifying two pairs of disjoint arcs in the boundary of a disk. After multiplication by the interval I , the order of identified arcs on the boundaries of disks becomes inessential. If 3-manifold or more general 3-polyhedron has two different decompositions into Cartesian product then one of the factors in these decompositions must be an interval (see [R2]).

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The uniqueness property holds for Cartesian squares (cf. [F]) and Cartesian powers (cf. [R3]) of 2-manifolds with boundary. The uniqueness (up to permutation of factors) of a Cartesian product of circles and intervals is obvious. We have the uniqueness of decomposition into a finite Cartesian product of 1-polyhedra (cf. [B1]) and 1-dimensional locally connected continua (cf. [C]). A Cartesian product of 1-polyhedra does not have another decomposition into a Cartesian product of polyhedra of dimension ≤ 2 (cf. [R4]). Before we begin to consider uniqueness of Cartesian products of connected 2-manifolds with boundary we need some preliminaries.

Definition 1.1 *Let X be a compact connected 2-manifold with non-empty boundary. We associate to X the following number:*

$$s(X) = \text{rank}H_1(X) - \text{rank}H_1(\partial X) + 1.$$

Lemma 1.1 *Let X, Y, X' , and Y' be any compact connected 2-manifolds with non-empty boundary and suppose that the Cartesian products $X \times Y$ and $X' \times Y'$ are homeomorphic. Then*

$$s(X)s(Y) = s(X')s(Y').$$

Proof. We use an argument similar to the one in [R3], Theorem 2.1. We consider the map

$$i_* : H_2(X \times Y) \rightarrow H_2(X \times Y, \partial(X \times Y)),$$

which is induced by the inclusion of the pair $(X \times Y, \emptyset)$. The image of this map is generated by all products $\zeta_1 \otimes \zeta_2$ such that $\zeta_1 \in H_1(X)$ and $\zeta_2 \in H_1(Y)$, such that $j_{k*}(\zeta_k) \neq 0$, for $k = 1, 2$, where

$$j_{1*} : H_1(X) \rightarrow H_1(X, \partial X) \text{ and } j_{2*} : H_1(Y) \rightarrow H_1(Y, \partial Y)$$

are given by inclusions. The number $s(X)$ is equal to $\text{rank im } j_{1*}$ and the number $s(Y)$ is equal to $\text{rank im } j_{2*}$. So $s(X)s(Y)$ is equal to $\text{rank im } i_*$. Hence if $X \times Y$ and $X' \times Y'$ are homeomorphic it follows that $s(X)s(Y) = s(X')s(Y')$.

Lemma 1.2 *Let X, Y, X' , and Y' be any compact connected 2-manifolds with non-empty boundary and suppose that the Cartesian products $X \times Y$ and $X' \times Y'$ are homeomorphic. Then with respect to the order of the factors we have:*

- (i) $H_1(X) = H_1(X')$ and $H_1(Y) = H_1(Y')$
- (ii) $H_1(X, \partial X) = H_1(X', \partial X')$ and $H_1(Y, \partial Y) = H_1(Y', \partial Y')$.

Proof. Let $H_1(X) = Z^x$, $H_1(Y) = Z^y$, $H_1(X') = Z^{x'}$ and $H_1(Y') = Z^{y'}$. By the Künneth formula we conclude that:

$$\begin{aligned} Z^{xy} &\cong H_2(X \times Y) \cong H_2(X' \times Y') \cong Z^{x'y'} \text{ and} \\ Z^{x+y} &\cong H_1(X \times Y) \cong H_1(X' \times Y') \cong Z^{x'+y'}. \end{aligned}$$

Hence, $x = x'$ and $y = y'$ or $x = y'$ and $y = x'$. We can assume that the first case holds. This completes the proof of (i).

If X is orientable then $H_1(X, \partial X) = Z^x$. If it is not then $H_1(X, \partial X) = Z^{x-1} \oplus Z_2$. Similarly, for Y, X' , and Y' . By the relative Künneth formula,

$$H_2(X \times Y, \partial(X \times Y)) = Z^{xy - o_2x - o_1y + o_1o_2} \oplus Z_2^{o_2x + o_1y - o_1o_2},$$

where $o_1 = 1$ if X is nonorientable and $o_1 = 0$ if X is orientable, and $o_2 = 1$ if Y is nonorientable and $o_2 = 0$ if Y is orientable. Similarly for X' and Y' . Hence $xy - o_2x - o_1y + o_1o_2 = xy - o'_2x - o'_1y + o'_1o'_2$. So, if $x > 1$ and $y > 1$ then $H_1(X, \partial X) = H_1(X', \partial X')$ and $H_1(Y, \partial Y) = H_1(Y', \partial Y')$.

If $x = 0$ then X and X' are homeomorphic to the disk. Therefore Y and Y' are both orientable or both nonorientable, and their relative first homology groups are the same.

If $x = 1$ then X can be the annulus $A = S^1 \times I$ or the Möbius band M . Similarly for X' .

If X is an annulus then $H_2(X \times Y, \partial(X \times Y)) = Z \otimes H_1(Y, \partial Y) = H_1(Y, \partial Y)$. If X' is a Möbius band then $H_2(X' \times Y', \partial(X' \times Y')) = Z_2 \otimes H_1(Y', \partial Y')$. These groups can be isomorphic only if $H_1(Y, \partial Y) = Z_2$ and if $H_1(Y', \partial Y')$ is equal to Z or Z_2 . The spaces $A \times M$ and $M \times M$ are not homeomorphic by Lemma 1.1; by definition $s(A) = 0$, and $s(M) = 1$, so $s(A)s(M) \neq s(M)s(M)$.

We start the consideration of the Cartesian products of connected 2-manifolds with boundary by presenting the case where one of the factors is not prime. In this paper a *prime* manifold is a manifold which is not a nontrivial Cartesian product. There exist three nonprime surfaces: $I \times I, I \times S^1$, and $S^1 \times S^1$. We have the following:

Proposition 1.1 *Let X and Y be any compact 2-manifolds, possibly with boundary, and suppose that the Cartesian products $X \times Y$ and $X' \times Y'$ are homeomorphic. If X is prime and Y is a product of two 1-manifolds, then X' is also a prime 2-manifold and Y' is a product of two 1-manifolds (up to a permutation of X' and Y'). In both cases, Y and Y' are homeomorphic. Furthermore, if X and X' are not homeomorphic, then Y and Y' are homeomorphic either to I^2 or to $S^1 \times I$.*

Proof. By Kosiński's theorem [K], all 2-dimensional Cartesian factors of a polyhedron are polyhedra, so X' and Y' are 2-manifolds, possibly with boundary. If $\partial X = \emptyset$ and $Y = S^1 \times S^1$ then we have the uniqueness by a classical result of Borsuk [B2].

If $\partial X = \emptyset$ and $Y = I \times S^1$, then one of the factors X', Y' , say X' has an empty boundary, because $H_3(X \times Y; Z_2) = H_3(X' \times Y'; Z_2) \neq 0$. Since

$\partial(X \times Y) = X \times \partial Y = X' \times \partial Y' = \partial(X' \times Y')$, the surfaces X and X' are homeomorphic. Hence, comparing the homology groups we obtain that Y' is an annulus, also.

Now, let $\partial X = \emptyset$ and $Y = I^2$. If X is non-orientable then $0 = H_2(X) = H_2(X \times Y) = H_2(X' \times Y')$, so one of the factors X', Y' is a disk. The second factor is homeomorphic to X . If X is orientable, $\partial X' \neq \emptyset$ and $\partial Y' \neq \emptyset$ then $Z = H_2(X) = H_1(X') \otimes H_1(Y')$. Therefore X' and Y' are homeomorphic to $S^1 \times I$ and X is a torus. If $\partial X' = \emptyset$ then the boundaries $\partial(X \times Y)$ and $\partial(X' \times Y')$ are homeomorphic, so X and X' are homeomorphic and Y' is a disk.

If $\partial X \neq \emptyset$ and $Y = S^1 \times S^1$, then $Y' = S^1 \times S^1$ because $\partial(X \times Y)$ is a disjoint union of the sets homeomorphic to $S^1 \times S^1 \times S^1$. Hence X and X' are homeomorphic by a special case of Theorem 2 [R4]. If Y is homeomorphic to a disk or to an annulus and $\partial X \neq \emptyset$, then by Lemma 1.2, Y' is also homeomorphic to a disk or to an annulus.

2 The Main Result

The following is the main result of our paper:

Theorem 2.1 *Any connected 4-dimensional manifold, possibly with boundary, has at most one decomposition into Cartesian products of prime 2-manifolds, possibly with boundary.*

The techniques which were used in a similar lemma in [R1] are not strong enough for our purpose. We shall use the Splitting Theorem in the proof of our theorem above (see [J-S], [Jo]) – for investigation of the boundaries of the manifolds $X \times Y$ and $X' \times Y'$. So we use this theorem in the case when ∂M is empty.

In [J-S],[Jo] manifolds are orientable, so we must also assume that the manifold M is orientable. We denote by $\sigma_W(M)$ the 3-manifold obtained by splitting M along W . Similarly we define the 2-manifold $\sigma_{\partial W}(\partial M)$, which can be naturally identified with a submanifold of the boundary of $\sigma_W(M)$.

Theorem 2.2 (Splitting Theorem [J-S] p.157) *Let M be any compact, orientable, sufficiently-large, irreducible and boundary-irreducible 3-manifold. Then there exists a two-sided, incompressible 2-manifold, W properly embedded in M , unique up to ambient isotopy, having the following three properties:*

- (a) *The components of W are annuli and tori, and none of them is boundary-parallel in M ;*
- (b) *Each component of $(\sigma_W(M), \sigma_{\partial W}(\partial M))$ is either a Seifert pair or a simple pair; and*
- (c) *W is minimal with respect to inclusion among all two-sided 2-manifolds in M having properties (a) and (b).*

Proof. of Theorem 2.1. If both surfaces X and Y are without boundary, the uniqueness holds by Borsuk's theorem [B2] .

If $\partial X = \emptyset$ and $\partial Y \neq \emptyset$ then $\partial(X \times Y) = X \times \partial Y$. Since $Y \neq I^2$, like in the proof of Proposition 1.1, one of the factors X', Y' , say X' has an empty boundary, because $H_3(X \times Y; Z_2) = H_3(X' \times Y'; Z_2) \neq 0$ and $\partial Y' \neq \emptyset$. So, $\partial(X' \times Y') = X' \times \partial Y'$. Therefore X and X' are homeomorphic and the numbers of the components of the boundaries ∂Y and $\partial Y'$ are the same. Looking at the homology and relative homology groups we obtain that the surfaces Y and Y' are also homeomorphic.

Now we consider the case when ∂X and ∂Y are nonempty. Again by Lemma 1.2, the first Betti numbers of X and X' are the same and the first Betti numbers of Y and Y' , are also the same. The coincidence of the first relative homology groups implies that the orientability of X and Y agree with the orientability of X' and Y' , respectively . We consider three cases.

In the *first case*, X and Y are orientable, $M = \partial(X \times Y)$, $W = \partial X \times \partial Y$. Since by assumption, X and Y are not homeomorphic to I^2 or $S^1 \times I$, the manifolds M and W satisfy the hypotheses of the Splitting Theorem. Since the boundary of M is empty, the manifold W is a disjoint union of tori.

For somebody who is familiar with 3-manifolds the irreducibility of M is a simple exercise, but for the reader's convenience we outline a proof. If S is a 2-sphere contained in M we can assume that it is in a general position with W , so the intersection $S \cap W$ is a disjoint union of closed curves. Some of them bound innermost disks in S . Such a disk lies in one of components of $\sigma_W(M)$. The boundaries of the components are incompressible (II.2.4 [J-S]), so the boundary of the disk bound a disk in W . The components of $\sigma_W(M)$ are irreducible (II.2.3 [J-S]), so the union of our two disk bounds a ball. Via this ball we isotope parts of S into the adjacent component of $\sigma_W(M)$ eliminating one closed curve of $S \cap W$. We repeat this operation as many times as S lies in one component and it bounds a ball.

We will show that W is minimal. Assume that $V = W \setminus (S_1 \times S_2)$ where $S_1 \times S_2$ is a component of W also gives a splitting in the sense of Theorem 2.2. According to V , we have $U = (X \times S_2) \cup (S_1 \times Y)$ as a component of $\sigma_V(M)$. It must be either a Seifert pair or a simple pair. The set U is not a simple pair because the incompressible torus $S_1 \times S_2$ is not boundary-parallel in U (see [J-S], p.154).

The fundamental group of U is infinite, so by Corollary 8.3 in [G] or VI.11.a in [J], the manifold U is a Seifert manifold if and only if its fundamental group has a normal cyclic infinite subgroup. Let an element α of $\pi_1(U)$ be a generator of this subgroup. By Seifert-van Kampen theorem $\pi_1(U)$ is a sum with amalgamation of the groups $\pi_1(X \times S_2)$ and $\pi_1(S_1 \times Y)$. The natural projections map the element α onto elements of the centers of $\pi_1(X \times S_2)$ and $\pi_1(S_1 \times Y)$. So, if $\pi_1(X)$ and $\pi_1(Y)$ have more than one generator, it is impossible.

The same holds for X' and Y' , where $M' = \partial(X' \times Y')$, $W' = \partial X' \times \partial Y'$. The components of $\sigma_{W'}(M')$ are homeomorphic to spaces $X \times S^1$ and $S^1 \times Y$. Because the manifolds M and M' are homeomorphic and W is unique up to

ambient isotopy, the components of $\sigma_W(M)$ and the components of $\sigma_{W'}(M')$ are homeomorphic. The components of $\sigma_{W'}(M')$ are homeomorphic to spaces $X' \times S^1$ and $S^1 \times Y'$, so the manifolds X and Y are homeomorphic to X' and Y' .

In the *second case* only one manifold is orientable. Let X be non-orientable and Y be orientable. We consider the oriented double covers \tilde{X} and \tilde{X}' of X and X' . The manifolds $\tilde{X} \times Y$, and $\tilde{X}' \times Y'$ are orientable double covers of the homeomorphic manifolds $X \times Y$ and $X' \times Y'$, so our manifolds are homeomorphic.

If X is the Möbius band, then X' is also nonorientable and $H_1(X) = H_1(X') = \mathbb{Z}$, by Lemma 1.2, so X' is the Möbius band, too.

If X is not the Möbius band, then as before, we have homeomorphy either according to $\tilde{X} \approx \tilde{X}'$ and $Y \approx Y'$ or according to $\tilde{X} \approx Y'$ and $Y \approx \tilde{X}'$ by the Splitting theorem. In the first case X and X' are also homeomorphic. In the second case if $H_1(X) = \mathbb{Z}^x$ then $H_1(Y) = \mathbb{Z}^{2x-1}$. Putting $s(X') = s(X) + a$, $s(Y') = s(Y) + b$, $s(\tilde{X}) = 2(s(X) - 1)$ and $s(\tilde{X}') = 2(s(X') - 1)$ to the equations

$$\begin{aligned} s(X)s(Y) &= s(X')s(Y') \\ s(\tilde{X})s(Y) &= s(\tilde{X}')s(Y') \end{aligned}$$

we obtain $s(Y) = s(Y')$, so Y and Y' are homeomorphic. Then

$$\tilde{X} \approx Y' \approx Y \approx \tilde{X}' ,$$

so $X \approx X'$ also.

If X and X' are Möbius bands then we use Lemma 1.1. We have that $s(X)s(Y) = s(X')s(Y')$. Hence $s(Y) = s(Y')$, because $s(X) = s(X') = 1$. Since $H_1(Y) = H_1(Y')$ and $s(Y) = s(Y')$, they have the same number of components of their boundaries, so they are homeomorphic.

In the *third case* both surfaces X and Y are nonorientable. We cannot use exactly the same argument, but we make a similar consideration. First, we know by Lemma 1.2 that both surfaces X' and Y' are also nonorientable. We consider the manifolds $X \times S_i$ where S_i are components of ∂Y , and $S_j \times Y$ where S_j are components of ∂X .

Next, we take the oriented double covers \tilde{X} and \tilde{Y} of X and Y . The manifolds $\tilde{X} \times S_i$ and $S_j \times \tilde{Y}$ are the oriented double covers of $X \times S_i$ and $S_j \times Y$. Each of the tori $S_j \times S_i$ is covered by tori $S'_j \times S_i$ and $S''_j \times S_i$ in $\tilde{X} \times S_i$ and is covered by tori $S_j \times S'_i$ and $S_j \times S''_i$ in $S_j \times \tilde{Y}$.

By identifying $S'_j \times S_i$ with $S_j \times S'_i$ and $S''_j \times S_i$ with $S_j \times S''_i$, we obtain the oriented double cover M of $\partial(X \times Y)$. It is not essential which circles we denoted by S'_i, S'_j and S''_i, S''_j because in every case we obtain the unique the oriented double cover of $\partial(\tilde{X} \times \tilde{Y})$.

Analogously, we construct the oriented double cover M' of $\partial(X' \times Y')$. Of course M and M' are homeomorphic. If the manifolds X and Y are not the Möbius bands then we solve the problem by the Splitting Theorem.

If X is a Möbius band then we solve the problem using Lemma 1.1, like in the second case. \square

We also include the following new related result:

Theorem 2.3 *Let X_1, \dots, X_n and Y_1, \dots, Y_n be any surfaces with nonempty boundary and suppose that their Cartesian products $X_1 \times \dots \times X_n$ and $Y_1 \times \dots \times Y_n$ are homeomorphic. Then there exists a one-to-one correspondence between them (assume X_i corresponds to Y_i) such that $\text{rank}H_1(X_i) = \text{rank}H_1(Y_i)$ and if*

$$s(X_i) = \text{rank}H_1(X_i) - \text{rank}H_1(\partial X_i) + 1$$

for $i = 1, 2, \dots, n$ then

$$s(X_1)s(X_2)\dots s(X_n) = s(Y_1)s(Y_2)\dots s(Y_n).$$

Proof. Let $H_1(X_i) = Z^{n_i}$ and $H_1(Y_i) = Z^{m_i}$. We can conclude from the Künneth formula that

$$\begin{aligned} H_1(X_1 \times \dots \times X_n) &= Z^{\sum_{i=1}^n n_i}; \\ H_2(X_1 \times \dots \times X_n) &= Z^{\sum_{i_1 \neq i_2} n_{i_1} n_{i_2}}; \text{ and} \\ &\dots \\ H_n(X_1 \times \dots \times X_n) &= Z^{n_1 \dots n_n}. \end{aligned}$$

We obtain similar formulae for the product $Y_1 \times \dots \times Y_n$. Because $\text{rank}H_i(X_1 \times \dots \times X_n) = \text{rank}H_i(Y_1 \times \dots \times Y_n)$ we can conclude that $n_i = m_i$ for $i = 1, 2, \dots, n$. This follows from the fact that the ranks of the homology groups above are the coefficients of the polynomials $\prod_{i=1}^n (x - n_i)$ and $\prod_{i=1}^n (x - m_i)$. The polynomials are equal, so the numbers n_i and m_i are the same.

We obtain the equality $s(X_1)s(X_2)\dots s(X_n) = s(Y_1)s(Y_2)\dots s(Y_n)$ like in the previous proof. \square

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